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LUBRICATION MODEL FOR THE FLOW DRIVEN BY HIGH SURFACE TENSION

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Abstract In this paper the flow of a thin fluid film, driven by a surface tension, down an inclined plane, is considered. By using the Navier-Stokes equations for thin film flow, the continuity equation, the no-slip condition and the boundary conditions, we obtain the horizontal velocity, the vertical velocity and the governing equation of the film height. In general, the introduction of surface tension into standard lubrication theory leads to a nonlinear parabolic equation.

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1. INTRODUCTION

The lubrication approximation of the Navier-Stokes equations (NSE) has been used to describe a multitude of situations. Our attention has been focussed on the situations where surface tension plays an important role, such as: rain running down a window, the evolution of drying paint layers or the spreading of a fluid drop on a surface.

Research on lubrication equations with non-negligible surface tension appears into two distinct situations. Firstly, there are the physical studies where the work is directly motivated by a specific problem. In this case, after deriving the equation for film thickness, the mathematical treatment is generally limited to asymptotic or numerical methods. Secondly, there are the mathematical studies which are not directly motivated by a specific problem but delve into the lubrication equation in greater detail.

In this paper we study the flow of a thin fluid layer lies on a plane which is at an angle α to the horizontal. The flow is driven by a surface tension. In this case the flow is governed by the nonlinear degenerate parabolic equation. Implicit in the derivation of this equation is the assumption that surface tension and gravity effects are of the same order.

2. MAIN RESULTS

Consider the flow of a thin-layer, of an incompressible Newtonian fluid with constant density ρ and dynamic viscosity μ, down a plane inclined at an angle α to the horizontal (fig.1). The flow is driven simultaneously by gravity and a surface tension gradient $\Sigma = \frac{\partial \gamma}{\partial x}$. The velocity is $\overrightarrow{u} = u(x,z,t)\overrightarrow{i} + w(x,z,t)\overrightarrow{k}$.

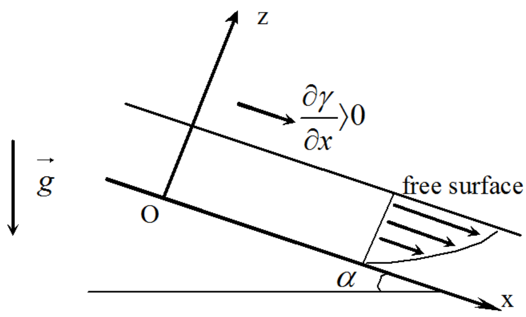

Fig. 1.

In the thin-film approximation the NSE reduce to [1], [4]:

$$0 = -\frac{1}{\rho} p_x + \nu u_{zz} + g \sin \alpha, \tag{1}$$

$$0 = -\frac{1}{\rho} p_z - g \cos \alpha, \tag{2}$$

where p is the pressure in the fluid, g is the gravitational acceleration. Here $z = h(x,t)$ is the unknown equation of the free surface and $\nu = \frac{\mu}{\rho}$ is the coefficient of kinematic viscosity. Here the subscripts denote differentiation with respect to the corresponding variable. The motion of the fluid is governed by equations (1)-(2) and some initial and boundary conditions.

The non-slip conditions must be satisfied:

$$\vec{u} = 0, \quad \text{at} \quad z = 0. \tag{3}$$

On the free surface $z = h(x,t)$ the condition that the normal stress be equal to the atmospheric pressure p_0 reduces to

$$p = p_0, \qquad \text{at} \quad z = h(x,t). \tag{4}$$

The condition for the tangential stress (Marangoni effect) at the free surface [2] is

$$\mu u_z = \Sigma, \qquad \text{at} \quad z = h(x,t). \tag{5}$$

and the kinematic boundary condition [3]

$$w = h_t + uh_x, \qquad \text{at} \quad z = h(x,t). \tag{6}$$

The continuity equation is

$$u_x + w_z = 0. \tag{7}$$

From equation (2) and condition (5), we obtain for fluid pressure

$$p = -\rho \cdot g \cdot cos\alpha \cdot z + C_1.$$

For $z = h$ we have

$$C_1 = p_0 + \rho \cdot g \cdot cos\alpha \cdot h,$$

and thus

$$p = -\rho \cdot g \cdot cos\alpha \cdot (z - h) + p_0. \tag{8}$$

By integrating (1) we get for u

$$u_z = \frac{1}{\rho \cdot \nu} \cdot p_x \cdot z - \frac{1}{\nu} \cdot g \cdot sin\alpha \cdot z + C_2.$$

The boundary condition (5) implies for $z = h$

$$C_2 = \frac{1}{\mu}\Sigma + \frac{1}{\nu} \cdot (g \cdot sin\alpha - \frac{1}{\rho} \cdot p_x)h$$

and

$$u_z = \frac{1}{\nu} \cdot (g \cdot sin\alpha - \frac{1}{\rho} \cdot p_x)(h - z) + \frac{1}{\mu}\Sigma.$$

By integrating the last relation we obtain

$$u = \frac{1}{\nu} \cdot (g \cdot sin\alpha - \frac{1}{\rho} p_x)(hz - \frac{z^2}{2}) + \frac{1}{\mu}\Sigma z + C_3$$

and by using the condition (3) the horizontal velocity is

$$u = \frac{\rho g}{2\mu}\left(-\sin\alpha + h_x \cdot \cos\alpha\right) z^2 + \left(\frac{\Sigma}{\mu} + \frac{\rho g}{\mu} h \sin\alpha - \frac{\rho g}{\mu} \cdot h_x \cdot h \cos\alpha\right) z. \tag{9}$$

This may be used in the continuity equation (7) to determine w

$$w_z = -u_x,$$

$$w(h) = -\int_0^z u_x \cdot dz$$

and by using for $z = 0$ the relation (3) we have for the vertical velocity

$$w = -\frac{\rho g}{6\mu} \cdot h_{xx} \cdot z^3 \cdot \cos\alpha + \frac{\rho g}{2\mu} \cos\alpha \left[h \cdot h_{xx} + (h_x)^2 \right] z^2 - \frac{\rho g}{2\mu} \cdot h_x \cdot z^2 \cdot \sin\alpha. \quad (10)$$

This expression together with the kinematic condition (6) leads to the governing equation for film height $h(x,t)$

$$\frac{\rho g}{3\mu} \cos\alpha \left(h^3 \cdot h_x \right)_x = h_t + \frac{\Sigma}{\mu} \cdot h \cdot h_x + \frac{\rho g}{\mu} \cdot \sin\alpha \cdot h^2 \cdot h_x. \quad (11)$$

If the thin fluid layer lies on a plane which is at an angle α to the horizontal then a surface tension driven flow is governed by the nonlinear degenerate parabolic equation (11).

Appropriate form of equation (11) have been used to model fluid flows in a number of physical situations such as coating, draining of foams and the movement of contact lenses.

3. CONCLUSIONS

In the lubrication approximation we have considered the flow of a thin layer on an inclined plane. The flow is driven simultaneously by gravity and a gradient of surface tension. This gradient implies a non-zero tangential stress boundary condition (Marangoni effect). We have estimated the response of the fluid to such a stress.

We have determined the horizontal velocity, the vertical velocity and the governing equation for the film height, which is a nonlinear degenerate parabolic equation. This equation can be linearized [5] and in some special cases we can obtain implicit solutions of the linearized system.

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