ROMAI J., **5**, 2(2009), 59–70

A SEVEN EQUATION MODEL FOR RELATIVISTIC TWO FLUID FLOWS-I

Sebastiano Giambò, Serena Giambò

Department of Mathematics, University of Messina, Italy

giambo@dipmat.unime.it, sgiambo@unime.it

Abstract An interface-capturing method is used to describe relativistic two-fluid flows. The conservation equations for the particle number of each fluid and for the total momentum-energy tensor of the mixture are the starting point of this approach. A model for relativistic two-fluid flow without friction and heat conduction and differential equations, plus additional algebraic relations, consistent with this model, are derived. The weak discontinuities propagating in this relativistic two-fluid system are examined and the expressions for the speeds of propagation are obtained.

Keywords: general relativity, relativistic fluid dynamics, two-fluid mixtures, nonlinear waves.

2000 MSC: 83C99, 80A10, 80A17, 76T99, 74J30.

1. INTRODUCTION

There are many topics in General Relativity where matter is represented as a mixture of two fluids. In fact, some astrophysical and cosmological situations need to be described by an energy tensor consisting of the sum of two or more perfect fluids. For most of the history of the universe, the dominant matter content is a mixture of matter and radiation [1]-[12]; other examples are a null fluid with string fluid [13], or a radiation fluid in addiction to a string fluid [14]-[17]. It was also shown [18]-[22] that an anisotropic relativistic fluid can be consistently described by two-perfect-fluid components and inflationary models have been deduced as mixtures of two relativistic fluids [23], [24]. Moreover, the acoustic modes [25] and the wave fronts [26], [27] have been studied in some of the cases quoted above.

The purpose of this paper is to build up a relativistic formulation of some recent results on the classical dynamics of a mixture of two perfect fluids based on the papers of H. Guillard and A. Murrone [28]-[30]. The model presented here is a two-phase flow model, in which the entire flow domain is filled with a mixture of the two fluids. However, in this underlying two-phase model, the fluids are not mixed on the molecular level: the mixture consists of very small elements of the two pure fluids, arranged in an irregular pattern. So the fluid is a mixture in the macroscopic sense.

Moreover, both fluids are assumed to be present everywhere in the flow domain and the interfaces between the two fluids are considered as gradual transitions from fluid 1 to fluid 2. In this way, the concept of interface between the two fluids disappears from the model [28]-[32].

Each fluid still has its own particle number density, r_k, its specific internal energy, ϵ_k, and its energy density ρ_k, [33], [34]:

$$\rho_k = r_k (1 + \epsilon_k) \ , \qquad k = 1, 2. \tag{1}$$

In what follows, the units are such that the velocity of light is unitary: $c = 1$. Conversely, a single pressure, p, and a single four-velocity, u^α, are assumed for the two fluids. Here, u^α is the fluid unit four-vector defined to be future-pointing

$$g_{\alpha\beta} u^\alpha u^\beta = 1, \tag{2}$$

where $g_{\alpha\beta}$ are the covariant components of Lorentz metric tensor with signature $+, -, -, -$.

In this paper we derive the complete system of governing differential equations and we determine the propagation speed of weak discontinuity wavefronts in this relativistic two-fluid model.

The paper is organized as follows. Section 2 starts with a description of the relativistic mixture and the derivation of the flow equations. Section 3 analyzes the source term appearing in the flow equations. In Section 4, the evolution equation for the pressure is derived in order to perform the closure of the system of differential equations obtained in Section 2. In Section 5, the propagation of weak discontinuities admitted by the model under consideration are examined and the expressions for their speeds of propagation are obtained. Section 6 concerns a special case, that may be physically relevant, in which the expression of the velocity is the relativistic version of the Wallis formula [39].

2. RELATIVISTIC FLOW MODEL

The standard equations for simple relativistic fluid flow hold for the two-fluid model. The total energy-momentum conservation is

$$\nabla_\alpha T^{\alpha\beta} = 0 \ , \tag{3}$$

where the stress energy tensor is given by

$$T^{\alpha\beta} = r f u^\alpha u^\beta - p g^{\alpha\beta}, \tag{4}$$

being r the total particle number density, f the relativistic total specific enthalpy

$$f = 1 + h = 1 + \epsilon + \frac{p}{r} \ , \tag{5}$$

with $h = \epsilon + \frac{p}{r}$ the "classical" specific enthalpy, of the mixture, ϵ and p denoting the total energy density and pressure, respectively.

Moreover, the balance law for the total particle number is

$$\nabla_\alpha \left(r u^\alpha \right) = 0 \ . \tag{6}$$

The projection of equation (3) along u^α is

$$u_\beta \nabla_\alpha T^{\alpha\beta} \equiv u^\alpha \partial_\alpha \rho + (\rho + p) \nabla_\alpha u^\alpha = 0, \tag{7}$$

being $\rho = r(1+\epsilon)$ the total energy density, whereas the spatial projection of equation (3) is

$$\gamma^\lambda_\beta \nabla_\alpha T^{\alpha\beta} \equiv r f u^\alpha \nabla_\alpha u^\lambda - \gamma^{\alpha\lambda} \partial_\alpha p = 0, \tag{8}$$

where $\gamma^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ is the projection tensor onto the three-space orthogonal to u^α (the rest space of an observer moving with four-velocity u^α).

However, we have to determine suitable expressions for the bulk quantities r, ϵ, ρ and f.

First, the volume fraction X and the mass fraction Y,

$$Y = \frac{X r_1}{r} \tag{9}$$

of fluid 1 are chosen as field variables. The variables X and Y allow to define any bulk quantity; the particle number density r, the specific internal energy ϵ, the energy density ρ and the relativistic specific enthalpy f are defined as:

$$\begin{cases} r = X_1 r_1 + X_2 r_2 \ , \\ \epsilon = Y_1 \epsilon_1 + Y_2 \epsilon_2 \ , \\ f = Y_1 f_1 + Y_2 f_2 \ , \\ \rho = X_1 \rho_1 + X_2 \rho_2 \ , \\ r f = X_1 r_1 f_1 + X_2 r_2 f_2 \ , \end{cases} \tag{10}$$

with

$$\begin{cases} X_1 = X \ , \qquad X_2 = 1 - X \ , \\ Y_1 = Y \ , \qquad Y_2 = 1 - Y \ . \end{cases} \tag{11}$$

Thus, for regular solutions, the mathematical study of the model can be performed using the following set of seven independent field variables u^α, r, p, X, Y. The governing system (6)-(8) is a set of five equations for seven variables. Thus, two more equations are to be determined in order to close the system.

Since all the bulk equations have already been used, the only option is to take into account quantities characterizing one of the two fluids.

The first equation to be considered is, of course, the balance law for the particle number density for fluid 1. Using the partial density $X_1 r_1$, the corresponding equation is

$$\nabla_\alpha \left(X r_1 u^\alpha \right) = 0. \tag{12}$$

From the conservation equation (12) (written using the relation $X r_1 = Y r$)

$$\nabla_\alpha \left(Y r u^\alpha \right) = 0 \ ,$$

taking into account equation (6), we obtain the following evolution law for the variable Y

$$u^\alpha \partial_\alpha Y = 0 \ . \tag{13}$$

Observe that equation (12), together with equation (6), implies the following balance law for the particle number density of fluid 2

$$\nabla_\alpha \left(X_2 r_2 u^\alpha \right) = 0 \ , \tag{14}$$

where, as already said, $X_2 = 1 - X$.

At this point, it is clear that the only option in order to get one more equation, and then the closure of the governing system, is to determine the balance equation for the energy-momentum tensor of fluid 1

$$T_1^{\alpha\beta} = \left(\rho_1 + p \right) u^\alpha u^\beta - p g^{\alpha\beta}. \tag{15}$$

Since exchanges of energy and momentum between the fluid components are allowed, there will be no local energy-momentum conservation for each fluid component separately. Then, the equation for the energy-momentum tensors of each of the two fluids, $T_k^{\alpha\beta}$, $k = 1, 2$, has the following form

$$\nabla_\alpha \left(X_k T_k^{\alpha\beta} \right) = F_k^\beta \ , \qquad k = 1, 2 \ , \tag{16}$$

where F_k^β represents the loss and source term in the separate balance. Now, since the total energy-momentum tensor is conserved, according to (3), and taking into account the expression (15) of $T_1^{\alpha\beta}$ and the relation

$$T^{\alpha\beta} = X_1 T_1^{\alpha\beta} + X_2 T_2^{\alpha\beta} \tag{17}$$

it is easily shown that

$$F_1^\beta = -F_2^\beta = F^\beta \ .$$

3. DERIVATION OF THE SOURCE TERM

This section is devoted to handling the source term F^β in equations (16).

The projection along u^α and the spatial projection of equation (16) for fluid 1 are, respectively,

$$X u^\alpha \partial_\alpha \rho_1 + \rho_1 u^\alpha \partial_\alpha X + X\left(\rho_1 + p\right) \nabla_\alpha u^\alpha = u_\alpha F^\alpha \tag{18}$$

and

$$X\left\{\left(\rho_1 + p\right) u^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta}\partial_\alpha p\right\} - p\gamma^{\alpha\beta}\partial_\alpha X = \gamma^\beta_\alpha F^\alpha \ . \tag{19}$$

Equation (18), taking into account equations (1) and (12), yields the following equation

$$X r_1 u^\alpha \left(\partial_\alpha \epsilon_1 + p\,\partial_\alpha \frac{1}{r_1}\right) = p\, u^\alpha \partial_\alpha X + u_\alpha F^\alpha \ . \tag{20}$$

We assume the following axiom: the entropy S_k of each fluid component is a function of the energy ϵ_k and the specific volume $1/r_k$

$$S_k = S_k\left(\epsilon_k, r_k\right) \ , \qquad k = 1, 2 \ . \tag{21}$$

By thermodynamic arguments, the derivatives of entropy can be related to some observable variables. Thus, we can write

$$\left\{ \begin{aligned} &\left(\frac{\partial S_k}{\partial \epsilon_k}\right)_{r_k} = \frac{1}{T_k} \ , \\ &\left(\frac{\partial S_k}{\partial r_k}\right)_{\epsilon_k} = -\frac{p}{r_k^2 T_k} \ , \end{aligned} \right. \tag{22}$$

where T_k is the temperature of fluid component k. From equation (**??**), it follows that

$$T_k dS_k = d\epsilon_k + p d\frac{1}{r_k} \tag{23}$$

and then

$$T_k u^\alpha \partial_\alpha S_k = u^\alpha \left(\partial_\alpha \epsilon_k + p\,\partial_\alpha \frac{1}{r_k}\right) \ . \tag{24}$$

We now also suppose that the entropy S_k is conserved along the flow lines

$$u^\alpha \partial_\alpha S_k = 0 \ , \quad (k = 1, 2) \ .$$

Thus, from equation (24) we can deduce that

$$u^\alpha \left(\partial_\alpha \epsilon_k + p\,\partial_\alpha \frac{1}{r_k}\right) = 0 \tag{25}$$

and equation (20) allows to write the following relation involving F_α

$$u_\alpha F^\alpha = -p\, u^\alpha \partial_\alpha X \ . \tag{26}$$

Next, using equations (19) and (8), we obtain

$$X\left(r_1 f_1 - r f\right) \gamma^{\alpha\beta}\partial_\alpha p - p r f \gamma^{\alpha\beta}\partial_\alpha X = r f \gamma^\beta_\alpha F^\alpha \ . \tag{27}$$

Now, introducing the relativistic enthalpy concentration

$$\chi = \frac{f_1}{f} Y \ ,$$

we have

$$r_1 f_1 = \frac{\chi}{X} r f \ , \tag{28}$$

thus, from (27) we deduce

$$\gamma^{\alpha\beta} F_\alpha = \left(\chi - X\right) \gamma^{\alpha\beta}\partial_\alpha p - p \gamma^{\alpha\beta}\partial_\alpha X \ . \tag{29}$$

Therefore, using equations (26) and (29), the source term F_β can now be computed as

$$F_\beta = \left(\chi - X\right) \gamma^\alpha_\beta \partial_\alpha p - p \partial_\beta X \ , \tag{30}$$

which represents the relativistic formulation of the classical expression of the source terms obtained by Wackers and Koren [31], [32].

4. PRESSURE EQUATION

The derivation of a pressure equation is rather involved, as it requires the two energy equations (7) and (18). From equation (7), because $\rho = r(1+\epsilon)$, we deduce that

$$r^2 u^\alpha \partial_\alpha \epsilon + \rho u^\alpha \partial_\alpha r + r^2 f \nabla_\alpha u^\alpha = 0 \ . \tag{31}$$

The total specific internal energy can be expressed in terms of variables r, p, X and Y by an equation of state. For this analysis, we use equations of state of the most general form, writing it as

$$\epsilon_1 = \epsilon_1\left(r_1, p\right) \ , \qquad \epsilon_2 = \epsilon_2\left(r_2, p\right) \ , \tag{32}$$

with

$$r_1 = \frac{Y}{X} r \ , \qquad r_2 = \frac{1-Y}{1-X} r \ . \tag{33}$$

Substituting (32) in equation $(10)_2$, the bulk specific internal energy ϵ can be written as

$$\epsilon\left(r, p, X, Y\right) = Y \epsilon_1\left(r_1, p\right) + \left(1 - Y\right) \epsilon_2\left(r_2, p\right) \ . \tag{34}$$

Now, thanks to this last expression (34) of ϵ, and using equations (13) and (6), the following form of the bulk energy equation (31) is deduced

$$r\frac{\partial\epsilon}{\partial p}u^\alpha\partial_\alpha p + r\frac{\partial\epsilon}{\partial X}u^\alpha\partial_\alpha X + \left(p - r^2\frac{\partial\epsilon}{\partial r}\right)\nabla_\alpha u^\alpha = 0\ . \tag{35}$$

Conservation of energy for fluid 1 (equation (20)), with equation (26) and (12), becomes

$$r_1 u^\alpha\partial_\alpha\epsilon_1 + \frac{p}{X}u^\alpha\partial_\alpha X + p\nabla_\alpha u^\alpha = 0\ , \tag{36}$$

and, using $(33)_1$ and taking into account the equation of state $(32)_1$, equation (36) writes as

$$r_1\frac{\partial\epsilon_1}{\partial p}u^\alpha\partial_\alpha p + \left(\frac{p}{X} + r_1\frac{\partial\epsilon_1}{\partial X}\right)u^\alpha\partial_\alpha X + \left(p - rr_1\frac{\partial\epsilon_1}{\partial r}\right)\nabla_\alpha u^\alpha = 0\ . \tag{37}$$

From equations (35) and (37), we are able to deduce the following evolution equations for the pressure p and the volume fraction α, respectively

$$\begin{cases} u^\alpha\partial_\alpha p + \omega\nabla_\alpha u^\alpha = 0\ , \\ \\ u^\alpha\partial_\alpha X + \xi\nabla_\alpha u^\alpha = 0\ , \end{cases} \tag{38}$$

where ω and ξ are defined by

$$\begin{cases} \omega = \dfrac{\left(p - r^2\frac{\partial\epsilon}{\partial r}\right)\left(r_1\frac{\partial\epsilon_1}{\partial X} + \frac{p}{X}\right) - r\frac{\partial\epsilon}{\partial X}\left(p - r_1 r\frac{\partial\epsilon_1}{\partial r}\right)}{r\frac{\partial\epsilon}{\partial p}\left(r_1\frac{\partial\epsilon_1}{\partial X} + \frac{p}{X}\right) - rr_1\frac{\partial\epsilon}{\partial X}\frac{\partial\epsilon_1}{\partial p}}\ , \\ \\ \xi = \dfrac{r\frac{\partial\epsilon}{\partial p}\left(p - rr_1\frac{\partial\epsilon_1}{\partial r}\right) - r_1\frac{\partial\epsilon_1}{\partial p}\left(p - r^2\frac{\partial\epsilon}{\partial r}\right)}{r\frac{\partial\epsilon}{\partial p}\left(r_1\frac{\partial\epsilon_1}{\partial X} + \frac{p}{X}\right) - rr_1\frac{\partial\epsilon}{\partial X}\frac{\partial\epsilon_1}{\partial p}}\ . \end{cases} \tag{39}$$

To end this section, we note that the complete system of governing differential equations may be written in term of variables $(u^\alpha,\ r,\ p,\ X,\ Y)$ as

$$\begin{cases} u^\alpha\partial_\alpha r = -r\nabla_\alpha u^\alpha\ , \\ \\ rfu^\alpha\nabla_\alpha u^\beta = \gamma^{\alpha\beta}\partial_\alpha p\ , \\ \\ u^\alpha\partial_\alpha p = -\omega\nabla_\alpha u^\alpha\ , \\ \\ u^\alpha\partial_\alpha X = -\xi\nabla_\alpha u^\alpha\ , \\ \\ u^\alpha\partial_\alpha Y = 0\ . \end{cases} \tag{40}$$

5. DISCONTINUITIES

In a domain Ω of space-time V_4, let Σ be a regular hypersurface, not generated by the flow lines, being $\varphi(x^\alpha) = 0$ its local equation. We set $L_\alpha = \partial_\alpha \varphi$. As it will be clear below, the hypersurface Σ is a space-like one, i.e. $L^\alpha L_\alpha < 0$. In the following, N_α will denote the normalized vector

$$N_\alpha = \frac{L_\alpha}{\sqrt{-L^\beta L_\beta}} \ , \qquad N_\alpha N^\alpha = -1 \ .$$

We consider a particular class of solutions of system (40) namely, weak discontinuity waves Σ, on which the field variables u^α, r, p, X, Y are continuous, but, conversely, jump discontinuities may occur in their normal derivatives. In this case, if Q denotes any of these fields, then there exists [33],[38] the distribution δQ, with support Σ, such that

$$\overline{\delta}\left[\nabla_\alpha Q\right] = N_\alpha \delta Q \ ,$$

where $\overline{\delta}$ is the measure of Dirac defined by φ with Σ as support, square brackets denote the discontinuity, δ being an operator of infinitesimal discontinuity; δ behaves like a derivative insofar as algebraic manipulations are concerned.

Then, from the system (40), we obtain the following linear homogeneous system in the distributions $N_\alpha \delta u^\alpha$, δr, δp, δX and δY

$$\begin{cases} L\delta r + rN_\alpha \delta u^\alpha = 0 \ , \\ rfL\delta u^\alpha - \gamma^{\alpha\beta} N_\beta \delta p = 0 \ , \\ L\delta p + \omega N_\alpha \delta u^\alpha = 0 \ , \\ L\delta X + \xi N_\alpha \delta u^\alpha = 0 \ , \\ L\delta Y = 0 \ , \end{cases} \tag{41}$$

where $L = u^\alpha N_\alpha$. Moreover, from the unitary character of u^α we have

$$u_\alpha \delta u^\alpha = 0 \ . \tag{42}$$

Now, we want to investigate the normal speeds of propagation of the various waves with respect to an observer moving with the mixture velocity u^α. The normal speed λ_Σ of propagation of the wave front Σ, described by a timelike word line having tangent vector field u^α, that is with respect to the time direction u^α, is given by [33]-[38]

$$\lambda_\Sigma^2 = \frac{L^2}{1 + L^2} \ . \tag{43}$$

The local causality condition, i.e. the requirement that the characteristic hypersurface Σ has to be timelike or null (or equivalently that the normal N_α be spacelike or null, that is $g^{\alpha\beta}N_\alpha N_\beta \leq 0$), is equivalent to the condition $0 \leq \lambda_\Sigma^2 \leq 1$.

From the above equations (41), we first obtain the solution $L = 0$, which represents a wave moving with the mixture. For the corresponding discontinuities we find

$$N_\alpha \delta u^\alpha = 0 \ , \qquad \delta p = 0 \ . \tag{44}$$

Since the coefficients characterizing the discontinuities exhibit five degrees of freedom, then system (41) admits five independent eigenvectors corresponding to $L = 0$ in the space of the field variables.

From now on we suppose $L \neq 0$. Equation $(41)_2$, multiplied by N_β, give us

$$rfLN_\alpha \delta u^\alpha - l^2 \delta p = 0 \ , \tag{45}$$

where $l^2 = 1 + L^2$.

As a consequence, $(41)_3$ and (45) represent a linear homogeneous system in the two scalar distributions $N_\alpha \delta u^\alpha$ and δp, which may have non trivial solutions only if the determinant of the coefficients vanishes. Therefore, we find the equation

$$\mathcal{H} \equiv rfL^2 - \omega l^2 = 0 \ , \tag{46}$$

which corresponds to the hydrodynamical waves propagating in such a two-fluid system. Their speeds of propagation are given by

$$\lambda_\Sigma^2 = \frac{\omega}{rf} \tag{47}$$

and the condition $0 < \frac{\omega}{rf} \leq 1$ ensures their spatial orientation.

The associated discontinuities can be written in terms of $\psi = n_\alpha \delta u^\alpha$ as follows

$$\begin{cases} \delta u^\alpha = -\psi n^\alpha \ , \\ \delta r = -r\dfrac{l}{L}\psi \ , \\ \delta p = -\omega\dfrac{l}{L}\psi \ , \\ \delta X = -\xi\dfrac{l}{L}\psi \ , \\ \delta Y = 0 \ , \end{cases} \tag{48}$$

where n^α is the unitary space-like four-vector defined by

$$n_\alpha = \frac{1}{l}\left(N_\alpha - Lu_\alpha\right) \ . \tag{49}$$

Observe that if the above condition characterizing the space-like orientations of the surface is verified, then the governing equations represent a (not strictly) hyperbolic system. In fact, all velocities (eigenvalues) are real, and there is a complete set of eigenvectors in the space of field variables, i.e. seven independent eigenvectors (5 from $L = 0$ and 2 from $\mathcal{H} = 0$), for the seven independent field variables u^α, r, p, X and Y.

6. APPLICATION

Now, we want to examine the application of the preceding method in order to determine weakly discontinuous solutions in the case of a mixture of two fluids of cosmological interest. To this end, we assume that each fluid ($k = 1, 2$) satisfy the equation of state of perfect gases:

$$\epsilon_k = \frac{1}{\gamma_k - 1}\frac{X_k}{Y_k}\frac{p}{r} , \qquad k = 1, 2, \tag{50}$$

where γ_k is the ratio of the specific heat capacities at constant pressure and volume of the k-th fluid. Using (50), $(10)_2$ writes as

$$\epsilon = \left(\frac{X}{\gamma_1 - 1} + \frac{1 - X}{\gamma_2 - 1}\right)\frac{p}{r} . \tag{51}$$

Then, (39) can be written, respectively, in the following form

$$\begin{cases} \omega = \dfrac{\gamma_1\gamma_2}{X\gamma_2 + (1 - X)\,\gamma_1} p , \\[2ex] \xi = X\,(1 - X)\,\dfrac{\gamma_1 - \gamma_2}{X\gamma_2 + (1 - X)\,\gamma_1} . \end{cases} \tag{52}$$

Replacing this last expression $(46)_1$ of ω into the equation (47), we get

$$\lambda_\Sigma^2 = \frac{1}{rf}\frac{\gamma_1\gamma_2}{X_1\gamma_2 + X_2\gamma_1} p . \tag{53}$$

Recalling that the normal speeds of propagation of hydrodynamical waves, λ_k, of each fluid k is given by

$$\lambda_k^2 = \gamma_k \frac{p}{r_k f_k} , \qquad k = 1, 2, \tag{54}$$

equation (53) can be rewritten under the form

$$\frac{1}{rf\lambda_\Sigma^2} = \frac{X_1}{r_1 f_1 \lambda_1^2} + \frac{X_2}{r_2 f_2 \lambda_2^2} . \tag{55}$$

Equation (55) represents the relativistic generalization of the formula due to Wallis [39], allowing to express the speed of acoustic modes for a two-fluid system as combination of the individual speeds of acoustic modes in each species.

References

[1] P. C. Vaidya, K. B. Shah, *A relativistic model for a shell of flowing radiation in a homogeneous universe*, Progr. Theor. Phys., **24** (1960), 111-125.

[2] C. B. G. McIntosh, *A cosmological model with both radiation and matter*, Nature, **215** (1967), 36-37 .

[3] C. B. G. McIntosh, *Cosmological models containing both radiation and matter*, Nature, **216** (1967), 1297-1298.

[4] K. C. Jacobs, *Friedmann cosmological model with both radiation and matter*, Nature, **215** (1967), 1156-1157.

[5] A. A. Coley, B. O. J. Tupper, *Viscous fluid collapse*, Phys. Rev., **D 29** (1984), 2701-2704.

[6] A. A. Coley, D. J. McManus, *New slant on tilted cosmology*, Phys. Rev., **D 54** (1996), 6095-6100.

[7] K. Dunn, *Two-fluid cosmological models in Gödel-type spacetimes*, Gen. Rel. Grav., **21** (1989), 137-147 .

[8] J. J. Ferrando, J. A. Morales, M. Portilla, *Two-perfect fluid interpretation of an energy tensor*, Gen. Rel. Grav., **22** (1990), 1021-1032.

[9] V. Husain, *Exact solutions for null fluid collapse*, Phys. Rev., **D 53** (1996), 1759-1762.

[10] J. P. S.Lemos, *Collapsing shells of radiation in anti-de Sitter spacetimes and the hoop and cosmic censorship conjectures*, Phys. Rev., **D 59** (1999), 044020 [gr-qc/9812078].

[11] A. Wang, Y. Wu, *Generalized Vaidya solutions*, Gen. Rel. Grav., **31** (1999), 107-114.

[12] T. Harko, K. S. Cheng, *Collapsing strange quark matter in Vaidya geometry*, Phys. Lett., **A 266** (2000), 249-253.

[13] E. N. Glass, J. P. Krisch, *Radiation and string atmosphere for relativistic stars*, Phys. Rev., **D 57** (1998), 5945-5947.

[14] E. N. Glass, J. P. Krisch, *Two-fluid atmosphere for relativistic stars*, Class. Quantum Grav., **16** (1999) 1175-1184.

[15] E. N. Glass, J. P. Krisch, *Scale symmetries of spherical string fluids*, J. Math. Phys., **40** (1999), 4056-4063.

[16] S. Kar, *Stringy black holes and energy conditions*, Phys. Rev., **D 55** (1997), 4872-4879.

[17] F. Larsen, *A string model of black hole microstates*, Phys. Rev., **D 56** (1997), 1005-1008.

[18] P. S. Letelier, *Anisotropic fluids with two-perfect fluid components*, Phys. Rev., **D 22** (1980), 807-813.

[19] P. S. Letelier, *Solitary waves of matter in general relativity*, Phys. Rev., **D 26** (1982), 2623-2631 .

[20] S. S. Bayin, *Anisotropic fluid spheres in general relativity*, Phys. Rev., **D 26** (1982), 1262-1274 .

[21] T. Harko, M. K. Mak, *Anisotropic relativistic stellar models*, arXiv:gr-qc/0302104 v1 26 Feb 2003.

[22] P. S. Letelier, P. S. C. Alencar, *Anisotropic fluids with multifluid components*, Phys. Rev., **D 34** (1986), 343-351.

[23] W. Zimdahl, *Reacting fluids in the expanding universe: a new mechanism for entropy production*, Mon. Not. R. Astron. Soc., **288** (1997), 665-673.

[24] W. Zimdahl, D. Pavon, R. Maartens, *Reheating and causal thermodynamics*, Phys. Rev., **D 55** (1997), 4681-4688.

[25] J. P. Krisch, L. L. Smalley, *Two fluid acoustic modes and inhomogeneous cosmologies*, Class. Quantum Grav., **10** (1993), 2615-2623.

[26] M. Cissoko, *Wavefronts in a relativistic cosmic two-component fluid*, Gen. Rel. Grav., **30** (1998), 521-534

[27] M. Cissoko, *Wave fronts in a mixture of two relativistic perfect fluids flowing with two distinct four-velocities*, Phys. Rev., **D 63** (2001), 083516.

[28] A. Murrone, *Modèles bi-fluides à six et sept équations pour les écoulement diphasiques à faible nombre de Mach*, PhD Thesis, Université de Provence, Aix-Marseille I, 2004.

[29] A. Murrone, H. Guillard, *A five equation reduced model for compressible two-phase flow computations*, J. Comput. Phys., **202** (2005), 664-698.

[30] A. Murrone, H. Guillard, *A five equation reduced model for compressible two phase flow problem*, INRIA, Rapport de recherche **4778**, 2003.

[31] E. H. van Brummelen, B. Koren, *A pressure-invariant conservative Godunov-type method for barotropic two-fluid flows*, J. Comput. Phys., **185** (2003), 289-308.

[32] J. Wackers, B. Koren, *Five-equation model for compressible two-fluid flow*, Report MAS-E0414, (2004).

[33] A. H. Taub, *Relativistic Rankine-Hugoniot equations*, Phys. Rev., **74** (1948), 328-334.

[34] A. Lichnerowicz, *Relativistic fluid Dynamics*, Cremonese, Roma, 1971.

[35] G. Boillat, *La propagation des ondes*, Gauthier-Villas, Paris, 1965.

[36] G. A. Maugin, *Conditions de compatibilité pour une hypersurface singulière en mécanique relativiste des milieux continus*, Ann. Inst. Henri Poincar, **24** (1976), 213-241.

[37] A. M. Anile, *Relativistic fluids and magneto-fluids*, Cambridge University Press, Cambridge, 1989.

[38] Y. Choquet-Bruhat, *Fluides relativistes de conductibilité infinie*, Astronautica Acta, **VI** (1960), 354-365.

[39] G. B. Wallis, *One-dimensional Two-Phase Flow*, McGraw-Hill, New York, 1969.