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EXISTENCE AND UNIQUENESS OF FUZZY SOLUTION FOR LINEAR VOLTERRA FUZZY INTEGRAL EQUATIONS, PROVED BY ADOMIAN DECOMPOSITION METHOD

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Abstract In the present paper, the existence and uniqueness of fuzzy solution for a the linear Volterra fuzzy integral equation is proved via the Adomian decomposition method.

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1. INTRODUCTION

The concept of integration of fuzzy functions has been introduced by Dubois and Prade [1], Goetschel and Voxman [2], Kaleva [3] and others. However, if the fuzzy function is continuous, all the various procedures yield the same result. The fuzzy integral is applied in fuzzy integral equations, such that there is a growing interest in fuzzy integral equations particularly in the past decade. The fuzzy integral equations have been studied by [4, 5, 6] and other authors.

Several criteria of existence of solutions of the Volterra fuzzy integral equation are given under the compactness-type conditions, by applying the embedding theorem in [7] and the Darbo fixed-points theorem in [8], and the Lipschitz condition, by applying the successive iterations of Picard method in [5, 9, 10]. In this paper, the existence theorems are proved for linear Volttera fuzzy integral equation under the Lipschitz condition and arbitrary kernels by means of the successive iterations of the Adomian decomposition method (ADM) [11, 12, 13] involving fuzzy set-valued function of a real variable where values are normal, convex, upper semicontinuous and compactly supported fuzzy sets in $\mathbb{R}^n$.

2. PRELIMINARIES

By $P_K(\mathbb{R}^n)$, we denote the family of all nonempty compact convex subsets of $\mathbb{R}^n$. Let $T = [a, b] \subset \mathbb{R}$ be a compact interval and denote [3]

$$E^n = \{u : \mathbb{R}^n \to [0, 1] |\, u \; satisfies \; (i) - (iv) \, below\},$$

where

i) u is normal i.e. there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,

ii) u is fuzzy convex,

iii) u is upper semicontinuous,

iv) $[u]^0 = \overline{\{x \in \mathbb{R} | u(x) > 0\}}$ is compact.

For $0 < \alpha \leq 1$ denote $[u]^\alpha = \{x \in \mathbb{R}^n | u(x) \geq \alpha\}$. Then from (i)-(iv), it follows that the α-level set $[u]^\alpha \in P_K(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$.

If $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a function, then using Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the relation

$$\tilde{g}(u, v)(z) = \sup_{z=g(x,y)} \min\{u(x), v(y)\}.$$

It is well known that $[g(u, v)]^\alpha = g([u]^\alpha, [v]^\alpha)$ for each $u, v \in E^n$, $0 \leq \alpha \leq 1$ and continuous function g. Moreover, we have $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[ku]^\alpha = k[u]^\alpha$, where $k \in \mathbb{R}$.

Define $D : E^n \times E^n \to \mathbb{R}^+$ by the relation $D(u, v) = \sup d([u]^\alpha, [v]^\alpha)$, where d is the Hausdorff metric defined in $P_K(\mathbb{R}^n)$. Then D is a metric in E^n. Furthermore, we know that [14]

- (E^n, D) is a complete metric space,
- $D(\theta u, \theta v) = |\theta| D(u, v)$ for every $u, v \in E^n$ and $\theta \in \mathbb{R}$,
- $D(u + w, v + w) = D(u, v)$ for all $u, v, w \in E^n$.

It can be proved that $D(u+v, w+z) \leq D(u, w)+D(v, z)$ for u, v, w and $z \in E^n$. Recall that the real numbers can be embedded to E^n by the correspondence

$$\tilde{c}(t) = \begin{cases} 1 & for \;\; t = c, \\ 0 & elsewhere. \end{cases}$$

Definition 2.1. *[3] We say that a function $F : T \to E^n$ is strongly measurable if for all $\alpha \in [0, 1]$ the set-valued function $F_\alpha : T \to P_K(\mathbb{R}^n)$ defined by*

$$F_\alpha(t) = [F(t)]^\alpha$$

is (Lebesgue) measurable, where $P_K(\mathbb{R}^n)$ is endowed with the topology generated by the Hausdorff metric d.

A function $F : T \to E^n$ is called integrable bounded if there exists an integrable function h such that $||x|| < h(t)$ for all $x \in F_0(t)$.

Definition 2.2. *[3] Let $F : T \to E^n$. The integral of F over T, denoted by $\int_T F(t)\,dt$ or $\int_a^b F(t)\,dt$, is defined levelwise by*

$$[\int_T F(t)\,dt]^\alpha =$$
$$\int_T F_\alpha(t)\,dt = \{\int_T f(t)\,dt | f : T \to \mathbb{R}^n \, is\, a\, measurable\, section\, for\, F_\alpha\}$$

for all $0 < \alpha \le 1$.

Proposition 2.1. *[3] Let $F, G : T \to E^n$ be integrable and $\theta \in \mathbb{R}$. Then*

i) $\int(F + G) = \int F + \int G$,

ii) $\int \theta F = \theta \int F$,

iii) $D(F, G)$ *is integrable,*

iv) $D(\int F, \int G) \le \int D(F, G)$.

Definition 2.3. *[15] A function $F : T \to E^n$ is bounded if there exists a constant $M > 0$ such that $D(F(t), \tilde{0}) \le M$ for all $t \in T$.*

3. EXISTENCE AND UNIQUENESS OF FUZZY SOLUTION

We consider the linear Voltera fuzzy integral equation

$$x(t) = f(t) + \int_0^t k(t, s)x(s)\,ds, \quad t \ge 0, \tag{1}$$

where $\Omega = \{(t, s)|0 \le s \le t \le \gamma\}$.

Theorem 3.1. *Assume the following conditions are satisfied*

i) $f : [0, \gamma] \to E^n$ *is continuous and bounded,*

ii) $k : \Omega \to \mathbb{R}$ *is a continuous function,*

iii) if $x, y : [0, \gamma] \to E^n$ are continuous, then the Lipschitz condition

$$D(k(t, s)x(s), k(t, s)y(s)) \le L\, D(x(t), y(t)), \tag{2}$$

is satisfied, with $0 < L < \frac{1}{\gamma}$.

Then there exists a unique fuzzy solution $x(t)$ of (1) and the successive iterations (see Appendix)

$$\begin{aligned} &\varphi_0(t) = f(t), \\ &\varphi_{n+1}(t) = f(t) + \textstyle\sum_{i=1}^{n+1} \int_0^t k(t,s)\, x_{i-1}(s)\, ds, \qquad (n \geq 0), \end{aligned} \tag{3}$$

are uniformly convergent to $x(t)$ on $[0,\gamma]$, where

$$\begin{aligned} &x_0(t) = f(t), \\ &x_n(t) = \textstyle\int_0^t k(t,s) x_{n-1}(s)\, ds, \qquad (n \geq 1). \end{aligned} \tag{4}$$

First we prove the following Lemma.

Lemma 3.1. *If the conditions of Theorem 3.1 hold and x_n is given by (4) then*

I) $x_n(t)$ is bounded,

II) $x_n(t)$ is continuous.

Proof. I) Clearly $x_0(t) = f(t)$ is bounded by the assumption. Assume $x_{n-1}(t)$ is bounded. From (2) and (4) we have

$$\begin{aligned} D(x_n(t), \tilde{0}) &= D(\textstyle\int_0^t k(t,s)\, x_{n-1}(s)\, ds, \tilde{0}) \leq \int_0^t D(k(t,s)\, x_{n-1}(s), \tilde{0})\, ds \\ &\leq L \textstyle\int_0^t D(x_{n-1}(s), \tilde{0}) ds \leq \gamma L \sup_{t\in[0,\gamma]} D(x_{n-1}(t), \tilde{0}), \end{aligned}$$

hence by induction $x_n(t)$ is bounded.

II) To prove continuity, we suppose $0 \leq t \leq \hat{t} \leq \gamma$, hence

$$\begin{aligned} &D(x_n(t), x_n(\hat{t})) \\ &\quad = D(\textstyle\int_0^t k(t,s) x_{n-1}(s)\, ds, \int_0^{\hat{t}} k(\hat{t},s) x_{n-1}(s)\, ds) \\ &\quad = D(\textstyle\int_0^t k(t,s) x_{n-1}(s)\, ds, \int_0^t k(\hat{t},s) x_{n-1}(s)\, ds + \int_t^{\hat{t}} k(\hat{t},s) x_{n-1}(s)\, ds) \\ &\quad \leq D(\textstyle\int_0^t k(t,s) x_{n-1}(s)\, ds, \int_0^t k(\hat{t},s) x_{n-1}(s)\, ds) + D(\int_t^{\hat{t}} k(\hat{t},s) x_{n-1}(s)\, ds, \tilde{0}) \\ &\quad \leq \gamma \sup_{s\in[0,\gamma]} D(k(t,s) x_{n-1}(s), k(\hat{t},s) x_{n-1}(s)) + \\ &\qquad + (\hat{t} - t) \sup_{s\in[0,\gamma]} D(x_{n-1}(s), \tilde{0}). \end{aligned}$$

As a result we obtain

$$D(x_n(t), x_n(\hat{t})) \to 0 \quad as \quad t \to \hat{t}.$$

Thus $x_n(t)$ is continuous on $[0,\gamma]$.□

Proof of Theorem 3.1. We assert that all $\varphi_n(t)$ are bounded on $[0,\gamma]$. In fact, $\varphi_0(t) = f(t)$ is bounded by the assumption. Suppose $\varphi_{n-1}(t)$ is bounded. From (3) we have

$$\begin{aligned} D(\varphi_n(t),\tilde{0}) &= D(f(t) + \textstyle\sum_{i=1}^{n}\int_0^t k(t,s)x_{i-1}(s)\,ds,\tilde{0}) \\ &= D(f(t) + \textstyle\sum_{i=1}^{n-1}\int_0^t k(t,s)x_{i-1}(s)\,ds + \int_0^t k(t,s)x_{n-1}(s)\,ds,\tilde{0}) \\ &= D(\varphi_{n-1}(t) + \textstyle\int_0^t k(t,s)x_{n-1}(s)\,ds,\tilde{0}) \\ &\le D(\varphi_{n-1}(t),\tilde{0}) + D(\textstyle\int_0^t k(t,s)x_{n-1}(s)\,ds,\tilde{0}) \\ &\le D(\varphi_{n-1}(t),\tilde{0}) + D(x_n(t),\tilde{0}), \end{aligned}$$

from induction and Lemma 3.1 part (I) we have that $\varphi_n(t)$ is bounded. Consequently, $\{\varphi_n(t)\}$ is a sequence of bounded functions on $[0,\gamma]$.

In the following, we prove that $\varphi_n(t)$ are continuous on $[0,\gamma]$. By Lemma 3.1 part (II) for $0 \le t \le \hat{t} \le \beta$, we have

$$\begin{aligned} &D(\varphi_n(t),\varphi_n(\hat{t})) \\ &\le D(f(t),f(\hat{t})) + D(\textstyle\sum_{i=1}^{n}\int_0^t k(t,s)x_{i-1}(s)\,ds, \sum_{i=1}^{n}\int_0^{\hat{t}} k(\hat{t},s)x_{i-1}(s)\,ds) \\ &\le D(f(t),f(\hat{t})) + D(\textstyle\sum_{i=1}^{n}\int_0^t k(t,s)x_{i-1}(s)\,ds, \sum_{i=1}^{n}\int_0^{t} k(\hat{t},s)x_{i-1}(s)\,ds) + \\ &\quad + D(\textstyle\sum_{i=1}^{n}\int_t^{\hat{t}} k(\hat{t},s)x_{i-1}(s)\,ds,\tilde{0}) \\ &\le D(f(t),f(\hat{t})) + \textstyle\int_0^t D(\sum_{i=1}^{n} k(t,s)x_{i-1}(s), \sum_{i=1}^{n} k(\hat{t},s)x_{i-1}(s))\,ds + \\ &\quad + \textstyle\int_t^{\hat{t}} D(\sum_{i=1}^{n} k(\hat{t},s)x_{i-1}(s),\tilde{0})\,ds \\ &\le D(f(t),f(\hat{t})) + \gamma\sup_{s\in[0,\gamma]} D(\textstyle\sum_{i=1}^{n} k(t,s)x_{i-1}(s), \sum_{i=1}^{n} k(\hat{t},s)x_{i-1}(s)) + \\ &\quad + (\hat{t}-t)\sup_{s\in[0,\gamma]} D(\textstyle\sum_{i=1}^{n} k(\hat{t},s)x_{i-1}(s),\tilde{0}). \end{aligned}$$

Finally we obtain

$$D(\varphi_n(t),\varphi_n(\hat{t})) \to 0 \quad as \quad t \to \hat{t}.$$

Therefore the sequence $\{\varphi_n(t)\}$ is continuous on $[0,\gamma]$.

To prove uniform convergence of the sequence $\{\varphi_n(t)\}$, for $n \ge 1$ we have

$$\begin{aligned} &D(\varphi_{n+1}(t),\varphi_n(t)) \\ &\quad = D(f(t) + \textstyle\sum_{i=1}^{n+1}\int_0^t k(t,s)x_{i-1}(s)\,ds,\varphi_n(t)) \\ &\quad = D(\varphi_n(t) + \textstyle\int_0^t k(t,s)x_n(s)\,ds,\varphi_n(t)) \\ &\quad = D(\textstyle\int_0^t k(t,s)x_n(s)\,ds,\tilde{0}) \\ &\quad \le \textstyle\int_0^t D(k(t,s)x_n(s),\tilde{0})\,ds \\ &\quad \le \gamma L\sup_{t\in[0,\gamma]} D(x_n(t),\tilde{0}). \end{aligned}$$

Hence we obtain

$$\sup_{t\in[0,\gamma]} D(\varphi_{n+1}(t),\varphi_n(t)) \le \gamma L \sup_{t\in[0,\gamma]} D(x_n(t),\tilde{0}). \tag{5}$$

From another point of view, by (12) we can obtain for $n \ge 1$,

$$\begin{aligned} D(x_n(t),\tilde{0}) &= D(\textstyle\int_0^t k(t,s)x_{n-1}(s)\,ds,\tilde{0}) \\ &\le \textstyle\int_0^t D(k(t,s)x_{n-1}(s),\tilde{0})\,ds \\ &\le \gamma L \sup_{t\in[0,\gamma]} D(x_{n-1}(t),\tilde{0}) \\ &\quad\vdots \\ &\le (\gamma L)^n \sup_{t\in[0,\gamma]} D(x_0(t),\tilde{0}) = (\gamma L)^n \sup_{t\in[0,\gamma]} D(f(t),\tilde{0}), \end{aligned}$$

that is,

$$\sup_{t\in[0,\gamma]} D(x_n(t),\tilde{0}) \le Q(\gamma L)^n, \tag{6}$$

where $Q = \sup_{t\in[0,\gamma]} D(f(t),\tilde{0})$. For $n \ge 0$, from (5) and (6) we obtain

$$\sup_{t\in[0,\gamma]} D(\varphi_{n+1}(t),\varphi_n(t)) \le Q(\gamma L)^{n+1}.$$

The series $Q\gamma L\sum_{n=0}^{\infty}(\gamma L)^n$ is convergent, hence the series $\sum_{n=0}^{\infty} D(\varphi_{n+1}(t),\varphi_n(t))$ is controlled uniformly on $[0,\gamma]$ this implying the uniform convergence of the sequence $\{\varphi_n(t)\}$. If we denote $x(t) = \lim_{n\to\infty}\varphi_n(t)$, then $x(t)$ satisfies (1). It is obviously continuous and bounded on $[0,\gamma]$.

At last, we prove the uniqueness of solution. Let $x(t)$ and $y(t)$ be two continuous solutions of (1) on $[0,\gamma]$. Then

$$\begin{aligned} 0 \le D(x(t),y(t)) &= D(x(t)+\varphi_n(t),y(t)+\varphi_n(t)) \\ &\le D(x(t),\varphi_n(t)) + D(y(t),\varphi_n(t)), \end{aligned}$$

and since $\varphi_n(t)$ is convergent to solution of (1),

$$\begin{aligned} D(x(t),\varphi_n(t)) &\to 0, \\ D(y(t),\varphi_n(t)) &\to 0, \end{aligned}$$

when $n \to \infty$, then $D(x(t),y(t)) = 0$ that is $x(t) = y(t)$. This finishes the proof of Theorem 3.1.□

Theorem 3.2. *Let $0 < M < 1$. Suppose that the following conditions are satisfied:*

i) $f : [0,\gamma] \to E^n$ is continuous and bounded, *ii)* $k : \Omega \to \mathbb{R}$ *is a continuous function and* $\int_0^t |k(t,s)|\, ds \leq M$*,*

then there exists a unique fuzzy solution $x(t) : [0,\gamma] \to E^n$ *of (1) and the successive iterations*

$$\begin{aligned} &\varphi_0(t) = f(t), \\ &\varphi_{n+1}(t) = f(t) + \textstyle\sum_{i=1}^{n+1} \int_0^t k(t,s) x_{i-1}(s)\, ds, \qquad (n \geq 0), \end{aligned} \tag{7}$$

are uniformly convergent to $x(t)$ *on* $[0,\gamma]$*.*

Proof. Proving of uniqueness of solution, and that $\varphi_n(t)$ is bounded and continuous, is similar proving of Theorem 3.1 and is omitted.

We only prove the uniform convergence of the sequence $\{\varphi_n(t)\}$.

For $n \geq 1$ we have

$$\begin{aligned} &D(\varphi_{n+1}(t), \varphi_n(t)) \\ &\qquad = D(f(t) + \textstyle\sum_{i=1}^{n+1} \int_0^t k(t,s) x_{i-1}(s)\, ds, \varphi_n(t)) \\ &\qquad = D(\varphi_n(t) + \int_0^t k(t,s) x_n(s)\, ds, \varphi_n(t)) \\ &\qquad = D(\int_0^t k(t,s) x_n(s)\, ds, \tilde{0}) \\ &\qquad \leq \sup_{t\in[0,\gamma]} D(x_n(t), \tilde{0}) \int_0^t |k(t,s)|\, ds \\ &\qquad \leq M \sup_{t\in[0,\gamma]} D(x_n(t), \tilde{0}), \end{aligned}$$

hence we obtain

$$\sup_{t\in[0,\gamma]} D(\varphi_{n+1}(t), \varphi_n(t)) \leq M \sup_{t\in[0,\gamma]} D(x_n(t), \tilde{0}). \tag{8}$$

In the same way from (6) we have

$$\sup_{t\in[0,\gamma]} D(x_n(t), \tilde{0}) \leq QM^n,$$

which $Q = \sup_{t\in[0,\gamma]} D(f(t), \tilde{0})$, in result, from (8) we have

$$\sup_{t\in[0,\gamma]} D(\varphi_{n+1}(t), \varphi_n(t)) \leq QM^{n+1},$$

Since the series $QM \sum_{n=0}^{\infty} M^n$ is convergent, the series $\sum_{n=0}^{\infty} D(\varphi_{n+1}(t), \varphi_n(t))$ is uniformly controlled on $[0,\gamma]$ this implying the uniform convergence of the sequence $\{\varphi_n(t)\}$. If we denote $x(t) = \lim_{n\to\infty} \varphi_n(t)$, then $x(t)$ satisfies (1). It is obviously continuous and bounded on $[0,\gamma]$. This finishes the proof of Theorem 3.2.□

4. CONCLUSION

In this paper we proved, by using the ADM, the existence and uniqueness of fuzzy solution for the linear Voltera fuzzy integral equations with an arbitrary continuous kernel. Also, we represented uniform convergence to the exact unique fuzzy solution in the theorems.

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5. APPENDIX

The Adomian decomposition method

Consider the linear Voltera crisp integral equation as

$$x(t) = f(t) + \int_0^t k(t,s)x(s)\,ds, \tag{9}$$

where f and k are known functions and x is to be determined. The Adomian decomposition method consists of representing x as a series

$$x(t) = \sum_{n=0}^{\infty} x_n(t). \tag{10}$$

Because in eq. (9) there are no nonlinear terms, in the infinite series do not appear the so-called Adomian polynomials. Now by replacing (10) in (9), we will have

$$\sum_{n=0}^{\infty} x_n(t) = f(t) + \int_0^t k(t,s) \sum_{n=0}^{\infty} x_n(t)\,ds. \tag{11}$$

Following Adomian analysis, Adomian decomposition method uses the recursive relations

$$\begin{aligned} &x_0(t) = f(t), \\ &x_{n+1}(t) = \textstyle\int_0^t k(t,s)x_n(s)\,ds, \quad n \geq 0. \end{aligned} \tag{12}$$

We assume $\varphi_n(t) = \sum_{i=0}^{n} x_i(t)$, obviously we have

$$x(t) = \lim_{n\to\infty} \varphi_n(t),$$

hence we rewrite successive iterations (12) as follows

$$\begin{aligned} &\varphi_0(t) = f(t), \\ &\varphi_{n+1}(t) = f(t) + \textstyle\sum_{i=1}^{n+1} \int_0^t k(t,s)x_{i-1}(s)\,ds, \quad n \geq 0. \end{aligned} \tag{13}$$

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