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NEW PROPERTIES FOR A CLASS OF MULTIVARIATE OPERATORS

Loredana-Florentina Galea
Agora University, Oradea, Romania
loredana.galea@univagora.ro

Abstract In this paper we are concerned with a class of multivariate operators. Relying on the results of the weakly Picard operator's theory, our aim is to study the good and special weakly Picard properties for these operators.

Keywords: Picard operators, weakly Picard operators, good Picard operators, special Picard operators.
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1. INTRODUCTION

The class of multivariate operators was introduced in 2008 by C. Bacoţiu [2]. In this paper our aim is to study the good and special weakly Picard properties for these operators.

The notions of good and special weakly Picard operator are defined by I.A. Rus in 2003, [5], and good and special convergence of type M of the sequence of successive approximation in metric spaces was introduced by L. D'Apuzzo in 1976, [1].

Let (X, d) be a metric space and $A : X \to X$ an operator. In this paper we use the following notations:
$P(X) = \{Y \subset X \,|\, Y \neq \emptyset\}$;
$F_A = \{x \in X \,|\, A(x) = x\}$ - the fixed point set of A;
$I(A) = \{Y \in P(X) \,|\, A(Y) \subset Y\}$- the family of nonvoid invariant subsets of A;
$A^0 := 1_X$, $A^1 := A, ..., A^{n+1} := A \circ A^n$, $n \in \mathbb{N}$.

In what follows we present the basic features of the theory while in the next section we state the main results.

Definition 1.1. *(I.A. Rus - [6], [7], [5]) Let (X, d) be a metric space.*
1) An operator $A : X \to X$ is a weakly Picard operator (briefly WPO) if the sequence of successive approximations $(A^m(x_0))_{m\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.
2) If the operator $A : X \to X$ is WPO and $F_A = \{x^*\}$, then by definition the operator A is a Picard operator (briefly PO).
3) If the operator $A : X \to X$ is WPO, then the operator A^∞ defined by $A^\infty : X \to X$, $A^\infty(x) := \lim_{m\to\infty} A^m(x)$ can be considered.

The basic result in the WPO's theory is the following:

Theorem 1.1. *(Characterization theorem [6], [7], [5]) An operator $A : X \to X$ is WPO if and only if there exists a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, such that:*
(a) $X_\lambda \in I(A)$, $\forall\lambda \in \Lambda$;
(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ is PO, $\forall\lambda \in \Lambda$.

Definition 1.2. *Let (X, d) be a metric space and $A : X \to X$ a WPO.*
1) $A : X \to X$ is a good WPO, if the series $\sum_{m=1}^{\infty} d\left(A^{m-1}(x), A^m(x)\right)$ converges, for all $x \in X$ (see [7]). When the sequence $\left(d\left(A^{m-1}(x), A^m(x)\right)\right)_{m\in\mathbb{N}^*}$ is strictly decreasing for all $x \in X$, A is a good WPO of type M (see [1]).
2) $A : X \to X$ is a special WPO, if the series $\sum_{m=1}^{\infty} d(A^m(x), A^\infty(x))$ converges, for all $x \in X$(see [7]). When the sequence $(d(A^m(x), A^\infty(x)))_{m\in\mathbb{N}^*}$ is strictly decreasing for all $x \in X$, A is a special WPO of type M (see [1]).

Theorem 1.2. *(S. Mureşan, L.F. Galea - [3]) Let (X, d) be a metric space and $A : X \to X$ a WPO. If A is a special WPO then A is a good WPO.*

In [2], C. Bacoţiu introduced the class of multivariate approximation operators and investigated some properties for these operators.

We present the basic results regarding this class of multivariate approximation operators.

For a fixed integer $p \geqslant 1$, we consider the domain

$$D := \underbrace{[0,1] \times [0,1] \times ... \times [0,1]}_{p} = [0,1]^p$$

Let us denote:

$$\alpha^{<0>} := (0, 0, ..., 0) = 0_{\mathbb{R}^p},$$

$$\alpha_1^{<1>} := (1, 0, ..., 0),\ \alpha_2^{<1>} := (0, 1, ..., 0), ...,\ \alpha_p^{<1>} := (0, 0, ..., 1).$$

That is, with the notation $M_1 := \{i_1 : 1 \leqslant i_1 \leqslant p\} \subset \mathbb{N}$, $\alpha_{i_1}^{<i>}$, $(i_1 \in M_1)$ represents the vector from $\mathbb{R}^p$ which has 1 on the position i_1 and 0 in the rest.

Generally, for all $k \in \overline{1, p}$ and for all $1 \leqslant i_1 < i_2 < ... < i_k \leqslant p$ we define $\alpha_{i_1,i_2,...,i_k}^{<k>}$ as the vector from $\mathbb{R}^p$ which has 1 on the positions $i_1, i_2, ..., i_k$ and 0 in the rest. $< k >$ is the number of positions on which appears the number 1 and $i_1, i_2, ..., i_k$ indicate these positions. The rest of the components are equal to 0.

We denote

$$M_k := \{(i_1, i_2, ..., i_k) : 1 \leqslant i_1 < i_2 < ... < i_k \leqslant p\} \subset \mathbb{N}^k$$

and define the set: $\nu_D := \left\{\alpha^{<0>}\right\} \cup \left\{\alpha_{i_1,...,i_k}^{<k>} : k = \overline{1, p} \text{ and } (i_1, ..., i_k) \in M_k\right\}$.

Remark 1.1. *$Card\,(M_k) = \binom{p}{k}$, $\forall\ k \in \overline{0,p}$ and*

$$Card\,(\nu_D) = \sum_{k=0}^{p} \binom{p}{k} = 2^p := N.$$

For any $\left(m_1, m_2, ..., m_p\right) \in \mathbb{N}^p$, consider the p-net

$$\Delta_{m_k}^{k} := (0 = x_{k,m_k,0} < x_{k,m_k,1} < ... < x_{k,m_k,m_k} = 1)$$

and the following system of real functions with positive values:

$$0 \leqslant \psi_{k,m_k,i} \in C\,[0,1]\,,\ \forall\ i = \overline{0,m_k},\ \forall\ k = \overline{1,p}.$$

We assume that the following conditions are satisfied

1) $\sum_{i=0}^{m_k} \psi_{k,m_k,i}\,(x) = 1,\ \forall\ x \in [0,1]\,,\ \forall\ k = \overline{1,p}$;

2) $\sum_{i=0}^{m_k} x_{k,m_k,i}\psi_{k,m_k,i}\,(x) = x,\ \forall\ x \in [0,1]\,,\ \forall\ k = \overline{1,p}$;

3) $\psi_{k,m_k,0}\,(0) = \psi_{k,m_k,m_k}\,(1) = 1\ ,\ \forall\ k = \overline{1,p}.$

Definition 1.3. *(C. Bacoţiu - [2]) The operators $L_{m_1,...,m_p} : C\,(D) \to C\,(D)$ defined by $\left(L_{m_1,...,m_p}\,(f)\right)\left(x_1, ..., x_p\right) := \sum_{i_1=0}^{m_1} ... \sum_{i_k}^{m_k} ... \sum_{i_p}^{m_p} \psi_{1,m_1,i_1}\,(x_1)...\psi_{k,m_k,i_k}\,(x_k)\,...$*

$$...\psi_{p,m_p,i_p}\left(x_p\right) \cdot f\left(x_{1,m_1,i_1}, ..., x_{k,m_k,i_k}, ...x_{p,m_p,i_p}\right)$$

for any $f \in C\,(D)$ and for any $\left(x_1, ..., x_p\right) \in D$ are called multivariate operators.

For these operators we have the following properties regarding the sets $X_\lambda := \left\{ f \in C\,(D) : f\,(\alpha) = f\left(\alpha_{i_1,...,i_k}^{<k>}\right) := \lambda_{\omega\left(\alpha_{i_1,...,i_k}^{<k>}\right)}, \forall \alpha = \alpha_{i_1,...,i_k}^{<k>} \in \nu_D \right\}$ for any $\lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{R}^N$,

(P1) X_λ is a closed subset of $C\,(D)$;

(P2) $X_\lambda \in I\left(L_{m_1,...,m_p}\right)$, for all λ;

(P3) $C\,(D) = \bigcup_{\lambda \in \mathbb{R}} X_\lambda$ is a partition of $C\,(D)$;

(P4) In the following notations:

$$K := \{0, 1, ..., m_1\} \times \{0, 1, ..., m_2\} \times ... \times \left\{0, 1, ..., m_p\right\}$$

$$\partial K := \left\{(0, 0, ..., 0)\,, (m_1, 0, ...0)\,, ..., \left(m_1, m_2, ..., m_p\right)\right\} \subset \mathbb{R}^p$$

$$u_{m_1,...,m_p}\left(x_1, ..., x_p\right) := \sum_{(i_1,..,i_p) \in \partial K} \psi_{m_1,1,i_1}\,(x_1)\,...\psi_{m_p,p,i_p}\left(x_p\right)$$

$$\sigma_{m_1,...,m_p} := \inf\left\{u_{m_1,...,m_p}\left(x_1,...,x_p\right) : \left(x_1,...,x_p\right) \in D\right\}$$

$L_{m_1,...,m_p}\big|_{X_\lambda} : X_\lambda \to X_\lambda$ is a contraction, having:

$$(1)\ \left\|L_{m_1,...,m_p}(f) - L_{m_1,...,m_p}(g)\right\|_{C(D)} \leqslant \left(1 - \sigma_{m_1,...,m_p}\right)\|f-g\|_{C(D)}$$

for all $f, g \in X_\lambda$.

Theorem 1.3. *(C. Bacoţiu - [2]) The multivariate operators $L_{m_1,...,m_p}$ are WPO, for $\sigma_{m_1,...,m_p} \neq 0$ and for $\left(m_1,...,m_p\right) \in \mathbb{N}^p$, we have:*

$$L^{\infty}_{m_1,...,m_p}(f)\left(x_1,...,x_p\right) = C^0_0 + \sum_{i_1 \in M_1} C^1_{i_1} x_{i_1} + \sum_{(i_1,i_2)\in M_2} C^2_{i_1,i_2} x_{i_1} x_{i_2} +$$

$$+ ... + \sum_{(i_1,...,i_k)\in M_k} C^k_{i_1,...,i_k} x_{i_1}...x_{i_k} + ... + C^p_{1,2,...,p} x_1 x_2 ... x_p$$

for all $f \in C(D)$ and for all $\left(x_1,...,x_p\right) \in D$, where C^0_0 and $C^k_{i_1,...,i_k}$, $\forall\, k \in \overline{1,p}$, $\forall\ (i_1,...,i_k) \in M_k$ are real numbers which depend on f, and there have the following expressions:

$$C^0_0 = f\left(\alpha^{<0>}\right),$$

$$C^k_{i_1,...,i_k} := (-1)^k f\left(\alpha^{<0>}\right) + (-1)^{k-1} \sum_{S_1=1} f\left(\alpha^{<1>}_{i_{S_1}}\right) + (-1)^{k-2} \sum_{1\leqslant S_1<S_2\leqslant k} f\left(\alpha^{<2>}_{i_{S_1},i_{S_2}}\right) + ...$$

$$+ ... + (-1)^{k-l} \sum_{1\leqslant S_1<S_2<...S_l\leqslant k} f\left(\alpha^{<l>}_{i_{S_1},...,i_{S_l}}\right) + (-1)^0 f\left(\alpha^{<k>}_{i_1,...,i_k}\right).$$

2. MAIN RESULTS

In this section we investigate some properties of the iterates of multivariate operators in the sense of good and special convergence.

Using the inequality (1), we obtain the estimation:

$$\left|L^1_{m_1,...,m_p}(f)\left(x_1,...,x_p\right) - L^{\infty}_{m_1,...,m_p}(f)\left(x_1,...,x_p\right)\right| =$$

$$= \left|L^1_{m_1,...,m_p}(f)\left(x_1,...,x_p\right) - L^1_{m_1,...,m_p}\left(L^{\infty}_{m_1,...,m_p}(f)\right)\left(x_1,...,x_p\right)\right| \leqslant$$

$$\leqslant \left(1 - \sigma_{m_1,...,m_p}\right)\left|f\left(x_1,...,x_p\right) - L^{\infty}_{m_1,...,m_p}(f)\left(x_1,...,x_p\right)\right| =$$

$$= \left(1 - \sigma_{m_1,...,m_p}\right)\left|f\left(x_1,...,x_p\right) - \left(C^0_0 + \sum_{i_1\in M_1} C^1_{i_1} x_1 + \sum_{(i_1,i_2)\in M_2} C^2_{i_1,i_2} x_{i_1} x_{i_2}\right.\right.$$

$$\left.\left. + \sum_{(i_1,...,i_k)\in M_k} C^k_{i_1,...,i_k} x_{i_1} x_{i_2} ... x_{i_k} + ... + C^p_{1,2,...,p} x_1 x_2 ... x_p\right)\right| \leqslant$$

$$\leqslant \left(1-\sigma_{m_1,\ldots,m_p}\right)\cdot C$$

where $C=\max\left\{diam\left(\operatorname{Im} f\right),(p+1)\max\limits_{x\in[0,1]^p}|f(x)|\right\}$.

By induction, for $n\in\mathbb{N}^*$, we have:

$$\left|L^n_{m_1,m_2,\ldots,m_p}(f)\left(x_1,\ldots,x_p\right)-L^{\infty}_{m_1,m_2,\ldots,m_p}(f)\left(x_1,\ldots,x_p\right)\right|=$$

$$\left|L^1_{m_1,m_2,\ldots,m_p}\left(L^{n-1}_{m_1,m_2,\ldots,m_p}(f)\right)\left(x_1,\ldots,x_p\right)-L^1_{m_1,m_2,\ldots,m_p}\left(L^{\infty}_{m_1,m_2,\ldots,m_p}(f)\right)\left(x_1,\ldots,x_p\right)\right|\leqslant$$

$$\leqslant\left(1-\sigma_{m_1,m_2,\ldots,m_p}\right)^n\cdot C,\ \forall\, x\in D,\ C=\max\left\{diam\left(\operatorname{Im} f\right),(p+1)\max_{x\in[0,1]^p}|f(x)|\right\}$$

Hence $\sum\limits_{m=1}^{\infty}\left|L^n_{m_1,\ldots,m_p}(f)\left(x_1,\ldots,x_p\right)-L^{\infty}_{m_1,\ldots,m_p}(f)\left(x_1,\ldots,x_p\right)\right|\leqslant$

$$\leqslant C\cdot\left(1-\sigma_{m_1,m_2,\ldots,m_p}\right)\cdot\frac{1-\left(1-\sigma_{m_1,m_2,\ldots,m_p}\right)^n}{\sigma_{m_1,m_2,\ldots,m_p}},\ \forall\, x\in D \text{ with}$$

$$C=\max\left\{diam\left(\operatorname{Im} f\right),(p+1)\max_{x\in[0,1]^p}|f(x)|\right\}.$$

Thus, we obtained the following result.

Theorem 2.1. *The multivariate operators $L_{m_1,\ldots,m_p}$ are special weakly Picard operators of type M on $C(D)$.*

On the other hand, we have:

$$\left|L^n_{m_1,m_2,\ldots,m_p}(f)\left(x_1,\ldots,x_p\right)-L^{n-1}_{m_1,m_2,\ldots,m_p}(f)\left(x_1,\ldots,x_p\right)\right|=$$

$$\left|L^1_{m_1,m_2,\ldots,m_p}\left(L^{n-1}_{m_1,m_2,\ldots,m_p}(f)\right)\left(x_1,\ldots,x_p\right)-L^1_{m_1,m_2,\ldots,m_p}\left(L^{n-2}_{m_1,m_2,\ldots,m_p}(f)\right)\left(x_1,\ldots,x_p\right)\right|\leqslant$$

$\leqslant\left(1-\sigma_{m_1,m_2,\ldots,m_p}\right)^{n-1}\cdot C_1,\ \forall\, x\in D,\ C_1=diam\left(\operatorname{Im} f\right)$

So, $\sum\limits_{m=1}^{\infty}\left|L^n_{m_1,\ldots,m_p}(f)\left(x_1,\ldots,x_p\right)-L^{n-1}_{m_1,\ldots,m_p}(f)\left(x_1,\ldots,x_p\right)\right|\leqslant C_1\frac{1}{\sigma_{m_1,\ldots,m_p}},\ \forall\, x\in D,\ \forall\, f\in C(D)$.

Relying on the above results, we have the following theorem.

Theorem 2.2. *The operators $L_{m_1,m_2,\ldots,m_p}$ are good weakly Picard operators of type M on $C(D)$.*

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