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A SEVEN EQUATION MODEL FOR RELATIVISTIC TWO FLUID FLOWS-II

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Abstract An interface-capturing model for relativistic two-fluid flow is presented. The flow equations are the bulk-fluid equations, combined with particle number and energy-momentum tensor equations for one of the two fluids. The latter equations, i.e. the ones describing one component of the mixture, contain source terms, accounting for the energy and momentum exchanges between the species. The fluid model enables the derivation of an exact expression for these source terms.

Weak discontinuities propagating into this relativistic two-fluid system are examined and, moreover, expressions for their speeds of propagation are obtained.

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1. INTRODUCTION

Two-fluid flow problems appear in many topics in general relativity [1]-[27], [39]. In these problems, the medium consists of two (or more) fluids, which do not mix. In fact, a sharp interface separates the pure fluids. One way of treating two-fluid models is that of interface-capturing techniques. These methods do not use an explicit interface model. Instead, the fluid is modeled as a mixture of the pure fluids everywhere.

In this paper, a capturing method is presented, which is a relativistic extension of the method introduced by Wackers and Koren [32] for classical compressible two-fluid flow.

On the physical ground, the model describing the relativistic two-fluid flow is based on a relativistic two-phase flow model, in which the entire flow domain is filled with a mixture of two fluids. However, in this underlying two-phase model, the fluids are supposed not to be mixed on the molecular level. So, the fluid is a mixture in the macroscopic sense.

Then, each fluid still has its own particle number density and specific internal energy, but a single pressure and a single four-velocity are defined for the the whole mixture. Moreover, heat conduction is not allowed between the fluid components.

Concerning the unknown variables in our relativistic two-fluid model, seven unknowns appear in our description the full two-phase flow: the four-velocity, the particle number density and the pressure of the whole mixture, and two more quantities accounting for the relative concentrations of the components.

This paper is organized as follows. Section 2 is devoted to a description of the relativistic mixture model and to the derivation of the flow equations. In Section 3, the source terms appearing in the flow equations are analyzed and an exact expression for this source terms is derived using a generalized Gibbs' equation. Then, in Section 4, the complete system of governing differential equations is derived. In Section 5, the weak discontinuities propagating in the mixture are examined and the expressions for their speeds of propagation are obtained. Section 6 refers to a special case in which each fluid is supposed to satisfy the equation of state of perfect gases and the results represent the relativistic extension of the ones obtained by Wackers and Koren in the classical framework.

2. DIFFERENTIAL EQUATIONS

The standard equations for a single-relativistic-fluid flow are valid for the two-fluid mixture as a whole. The particle number, r, and the energy-momentum tensor, $T^{\mu\nu}$, satisfy the conservation equations:

$$\nabla_\nu (r u^\nu) = 0, \tag{1}$$

$$\nabla_\nu T^{\nu\mu} = 0 \, , \tag{2}$$

where u^ν is the four-velocity and the stress-energy tensor is given by

$$T^{\nu\mu} = r f u^\nu u^\mu - p g^{\nu\mu} \, ; \tag{3}$$

here f is the relativistic specific enthalpy

$$f = 1 + h = 1 + \epsilon + \frac{p}{r} = \frac{\rho + p}{r} \, ,$$

being h the "classical" specific enthalpy, ϵ the specific internal energy, p the pressure and ρ,

$$\rho = r(1 + \epsilon) \, , \tag{4}$$

the energy density of the whole mixture.

The spatial projection and the projection along u^ν of equation (2) give, respectively,

$$r f u^\nu \nabla_\nu u^\mu - \gamma^{\nu\mu} \partial_\nu p = 0 \, , \tag{5}$$

$$u^\nu \partial_\nu \rho + (\rho + p) \nabla_\nu u^\nu = 0 \, . \tag{6}$$

However, suitable expressions for the bulk quantities, r, ϵ, ρ and f, have to be found. The volume fraction X_k and the mass fraction Y_k of fluid k ($k = 1, 2$) are introduced according to the following relations

$$Y_k = \frac{X_k r_k}{r} \, , \tag{7}$$

and in such a way that

$$\begin{cases} X_1 + X_2 = 1 \ , \\ \\ Y_1 + Y_2 = 1 \ . \end{cases} \tag{8}$$

The volume fraction $X = X_1$ and the mass fraction $Y = Y_1$ of fluid 1 are chosen as field variables. The quantities X and Y allow to define any bulk quantity; the particle number density r, the specific internal energy ϵ, the energy density ρ, the relativistic specific enthalpy f and the energy-momentum tensor $T^{\mu\nu}$, in terms of X and Y, are:

$$\begin{cases} r = X_1 r_1 + X_2 r_2 \ , \\ \\ \epsilon = Y_1 \epsilon_1 + Y_2 \epsilon_2 \ , \\ \\ f = Y_1 f_1 + Y_2 f_2 \ , \\ \\ \rho = X_1 \rho_1 + X_2 \rho_2 \ , \\ \\ rf = X_1 r_1 f_1 + X_2 r_2 f_2 \ , \\ \\ T^{\mu\nu} = X_1 T_1^{\mu\nu} + X_2 T_2^{\mu\nu} \ , \end{cases} \tag{9}$$

where r_k, ϵ_k, ρ_k, f_k, $T_k^{\mu\nu}$ represents, respectively, the particle number density, the specific internal energy, the energy density, the relativistic specific enthalpy and the energy momentum tensor of each fluid ($k = 1, 2$) and $X_2 = 1 - X$, $Y_2 = 1 - Y$.

Thus, for regular solutions, the mathematical study of the model can be performed with the following set of seven independent field variables u^ν, r, p, X, and Y. Since only five equations have already been introduced, i.e. (1), (5) and (6), two more equations are needed in order to close the governing system.

The first one must, of course, be the conservation equation of the particle number density for one of the two fluids, since the fluids are not supposed to change into each other. The conservation equation corresponding to the particle number density $X_1 r_1$ of fluid 1 is

$$\nabla_\nu \left(X_1 r_1 u^\nu \right) = 0 \ . \tag{10}$$

This equation, together with equation (1), implies that also the particle number density of fluid 2 is conserved:

$$\nabla_\nu \left(X_2 r_2 u^\nu \right) = 0 \ . \tag{11}$$

Consequently, as last equation, only one option remains: an equation for the energy-momentum tensor of fluid 1 which, clearly, have to be a balance equation,

$$\nabla_\nu \left(X_1 T_1^{\nu\mu} \right) = F^\mu \ , \tag{12}$$

with

$$T_1^{\nu\mu} = \left(\rho_1 + p \right) u^\nu u^\mu - p g^{\nu\mu} \ , \tag{13}$$

and where F^μ represents the loss and source term in the separate balance.

3. DERIVATION OF THE SOURCE TERMS

This section is devoted to handling the source term F^μ in equation (12).

The projection along u^ν and the spatial projection of equation (12) are, respectively,

$$Xu^\nu\partial_\nu\rho_1 + \rho_1 u^\nu\partial_\nu X + X(\rho_1 + p)\,\theta = u_\mu F^\mu \ , \tag{14}$$

where

$$\theta = \nabla_\nu u^\nu \ ,$$

and

$$X\left[(\rho_1 + p)\,u^\nu\nabla_\nu u^\mu - \gamma^{\nu\mu}\partial_\nu p\right] - \gamma^{\mu\nu}\partial_\nu X = \gamma^\mu_\nu F^\nu \ . \tag{15}$$

Next, using equations (5) and (15), the following relation is obtained

$$\gamma^{\mu\nu}F_\nu = (\chi - X)\,\gamma^{\mu\nu}\partial_\nu p - \gamma^{\mu\nu}X \ , \tag{16}$$

where $\chi = Yf_1/f$ denotes the relativistic enthalpy concentration.

Moreover, from equation (14), being

$$\rho_1 = r_1(1 + \epsilon_1) \ ,$$

we obtain

$$Xr_1u^\nu\left(\partial_\nu\epsilon_1 + p\partial_\nu\frac{1}{r_1} - \frac{p}{Xr_1}\partial_\nu X\right) = u^\nu F_\nu \ . \tag{17}$$

Let us turn our attention to the thermodynamic features of the model. We will assume the following axiom: the entropy S_k of each fluid component is a function of the specific internal energy ϵ_k, the particle number density r_k of fluid k and the volume fraction X of fluid 1

$$S_k = S_k(\epsilon_k, r_k, X) \ . \tag{18}$$

The laws of extended thermodynamics allow to express the derivatives of entropy in terms other observable variables. Thus,

$$\begin{cases} \dfrac{\partial S_k}{\partial\epsilon_k} = \dfrac{1}{T_k} \ , \\[2ex] \dfrac{\partial S}{\partial r_k} = -\dfrac{p}{r_k^2 T_k} \ , \\[2ex] \dfrac{\partial S}{\partial X} = -\dfrac{p}{Xr_kT_k} \ , \end{cases} \tag{19}$$

where T_k is the temperature of fluid component k. It follows from equations (19) that

$$T_k dS_k = d\epsilon_k + p\,d\frac{1}{r_k} - \frac{p}{Xr_k}dX \tag{20}$$

and then

$$T_k u^\nu \partial_\nu S_k = u^\nu \left(\partial_\nu \epsilon_k + p \partial_\nu \frac{1}{r_k} - \frac{p}{X r_k} \partial_\nu X \right) . \quad (21)$$

We now also assume that the entropy S_k is conserved along the flow lines

$$u^\nu \partial_\nu S_k = 0 \qquad (k = 1, 2) . \quad (22)$$

Thus, from (21), we can deduce that

$$u^\nu \left(\partial_\nu \epsilon_k + p\, \partial_\nu \frac{1}{r_k} - \frac{p}{X r_k} \partial_\nu X \right) = 0 . \quad (23)$$

At this point, (17) allows to write the following relation involving F_μ

$$u^\nu F_\nu = 0 . \quad (24)$$

Therefore, using equations (16) and (24), the source term F_μ can now be computed as

$$F_\nu = -\gamma^{\mu\nu} \partial_\nu X + (\chi - X)\, \gamma^{\nu\mu} \partial_\mu p . \quad (25)$$

4. PRESSURE AND VOLUME FRACTION EQUATIONS

The derivation of equations for pressure and volume fraction is rather involved, as it requires the two energy equations (6) and (14). From equation (6) for total energy-momentum tensor, using (3), we deduce

$$r^2 u^\nu \partial_\nu \epsilon + \rho u^\nu \partial_\nu r + r^2 f \theta = 0 . \quad (26)$$

The total specific internal energy can be expressed in terms of the variables r, p, X, Y by an equation of state of the form

$$\epsilon = \epsilon\,(r, p, X, Y) \ . \quad (27)$$

Now, using this equation, the bulk energy equation (26) becomes

$$r \frac{\partial \epsilon}{\partial p} u^\nu \partial_\nu p + r \frac{\partial \epsilon}{\partial X} u^\nu \partial_\nu X + \left(p - r^2 \frac{\partial \epsilon}{\partial r} \right) \theta = 0 . \quad (28)$$

The energy balance equation for fluid 1, (17), with equations (24) and (10), becomes

$$r_1 \frac{\partial \epsilon_1}{\partial p} u^\nu \partial_\nu p + r_1 \frac{\partial \epsilon_1}{\partial X} u^\nu \partial_\nu X + \left(p - r r_1 \frac{\partial \epsilon_1}{\partial r} \right) \theta = 0 . \quad (29)$$

From equation (28) and (29) we can deduce the following evolution equations for the pressure and the volume fraction, respectively,

$$u^\nu \partial_\nu p + \omega \theta = 0 , \quad (30)$$

$$u^\nu \partial_\nu X + \xi\theta = 0 \,, \tag{31}$$

where ω and ξ are defined by

$$\omega = \frac{r_1 \frac{\partial \epsilon_1}{\partial X}\left(p - r^2 \frac{\partial \epsilon}{\partial r}\right) - r \frac{\partial \epsilon}{\partial X}\left(p - r_1 r \frac{\partial \epsilon_1}{\partial r}\right)}{r r_1 \left(\frac{\partial \epsilon}{\partial p}\frac{\partial \epsilon_1}{\partial X} - \frac{\partial \epsilon}{\partial X}\frac{\partial \epsilon_1}{\partial p}\right)} \,, \tag{32}$$

$$\xi = \frac{r \frac{\partial \epsilon}{\partial p}\left(p - r r_1 \frac{\partial \epsilon_1}{\partial r}\right) - r_1 \frac{\partial \epsilon_1}{\partial p}\left(p - r^2 \frac{\partial \epsilon}{\partial r}\right)}{r r_1 \left(\frac{\partial \epsilon}{\partial p}\frac{\partial \epsilon_1}{\partial X} - \frac{\partial \epsilon}{\partial X}\frac{\partial \epsilon_1}{\partial p}\right)} \,, \tag{33}$$

generalizing the classical results due to Wackers and Koren.

To end this section, we recall that the complete system of governing differential equations may be written in term of variables u^ν, r, p, X and Y as

$$\begin{cases} u^\nu \partial_\nu r + r\theta = 0 \,, \\ r f u^\nu \nabla_\nu u^\mu - \gamma^{\mu\nu}\partial_\nu p = 0 \,, \\ u^\nu \partial_\nu p + \omega\theta = 0 \,, \\ u^\nu \partial_\nu X + \xi\theta = 0 \,, \\ u^\nu \partial_\nu Y = 0 \,. \end{cases} \tag{34}$$

5. WEAK DISCONTINUITIES

We assume that field variables u^ν, r, p, X and Y are C^0 and piecewise C^1; that the discontinuities of their first order derivatives can occur across an hypersurface Σ of local equation $\varphi(x^\mu) = 0$, φ being a C^2 function; that such discontinuities are well defined in each point of Σ as the difference of the limiting values of the derivatives of u^ν, r, p, X and Y obtained approaching each point of Σ by the two sides in which Σ divides the manifold V_4; and that the above derivatives are uniformly convergent to the limiting values, which are tensor functions defined on Σ, obtained on each of the two sides. Under such hypothesis [33]-[37], introducing the infinitesimal discontinuity operator, δ, we investigate the conditions under which the tensor distributions, δu^ν, δr, δp, δX and δY, supported with regularity by Σ, are not simultaneously zero. At the same time, we obtain the differential equation (i.e. the characteristic equation) which must be satisfied by the function φ. To this end, it is sufficient to apply the first order compatibility conditions to the above differential equations. Then, from system (34), we have the following linear homogeneous system in the distribution $N_\mu \delta u^\mu$, δr, δp, δX and δY:

$$\begin{cases} L\delta r + rN_\mu \delta u^\mu = 0 \,, \\ rfL\delta u^\nu - \gamma^{\nu\mu} N_\mu \delta p = 0 \,, \\ L\delta p + \omega N_\mu \delta u^\mu = 0 \,, \\ L\delta X + \xi N_\mu \delta u^\mu = 0 \,, \\ L\delta Y = 0 \,, \end{cases} \tag{35}$$

where N_μ is the normalized vector, ($N^\mu N_\mu = -1$), defined as

$$N_\mu = \frac{L_\mu}{\sqrt{-L^\nu L_\nu}} \,, \qquad L_\mu = \partial_\mu \varphi \,, \tag{36}$$

and $L = u^\mu N_\mu$.

Moreover, from the unitary character of u^ν we have

$$u_\nu \delta u^\nu = 0. \tag{37}$$

Let us turn now our attention to the normal speeds of propagation, λ_Σ, of the various waves with respect to an observer moving with the mixture velocity u^ν, given by

$$\lambda_\Sigma^2 = \frac{L^2}{l^2} \,, \qquad l^2 = 1 + L^2 \,. \tag{38}$$

The local causality condition, i.e. the requirement that the characteristic hypersurface Σ is time-like or null, is equivalent to the condition

$$0 \leq \lambda_\Sigma^2 \leq 1 \,. \tag{39}$$

As first, from the above system (35), the solution $L = 0$ can be obtained, representing a wave moving with the mixture. The corresponding discontinuities satisfy the equations

$$\begin{cases} N_\mu \delta u^\mu = 0 \,, \\ \delta p = 0 \,. \end{cases} \tag{40}$$

It can be easily seen that five degrees of freedom are still left in the coefficients characterizing the discontinuities, so we have 5 independent eigenvectors corresponding to the eigenvalue $L = 0$ in the space of the field variables.

In what follows, we suppose $L \neq 0$. The equation $(35)_2$, multiplied by N_μ, gives

$$rfLN_\nu \delta u^\nu - l^2 \delta p = 0 \,. \tag{41}$$

Consequently, equations $(35)_3$ and (41) represent a linear homogeneous system in the two scalar distributions $N_\nu \delta u^\nu$ and δp, which may admit non zero solutions only if the determinant of the coefficients vanishes. Therefore, the following equation must hold

$$\mathcal{H} \equiv rfL^2 - \omega l^2 = 0\,. \tag{42}$$

This equation yields the hydrodynamical waves propagating in such a two-fluid system. Their speeds of propagation are given by

$$\lambda_\Sigma^2 = \frac{\omega}{rf}\,, \tag{43}$$

where ω is given by (32), and the condition $0 < \frac{\omega}{rf} \leq 1$ ensures their spatial orientation. The associated discontinuities can be written in terms of

$$\psi = N_\mu \delta u^\mu \tag{44}$$

as follows

$$\begin{cases} \delta u^\nu = -\dfrac{1}{l} n^\nu \psi\,, \\[2ex] \delta r = -\dfrac{r}{L}\psi\,, \\[2ex] \delta p = -\dfrac{\omega}{L}\psi\,, \\[2ex] \delta X = -\dfrac{\xi}{L}\psi\,, \\[2ex] \delta Y = 0\,, \end{cases} \tag{45}$$

where n^ν is the unitary space-like four-vector defined by

$$n_\mu = \frac{1}{l}\left(N_\mu - Lu_\mu\right)\,. \tag{46}$$

If the condition given above, in order to have surfaces with space-like orientation, is verified, then the governing equations represents a (not strictly) hyperbolic system. In fact, all velocities (eigenvalues) are real, and there is a complete set of eigenvectors in the space of field variables: seven independent eigenvectors (five from $L = 0$ and two from $\mathcal{H} = 0$) for the seven independent field variables u^ν, r, p, X and Y.

6. APPLICATION

Now, we examine the application of the preceding solution procedure to a relativistic mixture of two fluids in which each fluid k ($k = 1, 2$) satisfy the equation of state of perfect gases

$$\epsilon_k = \frac{1}{\gamma_k - 1}\frac{X_k}{Y_k}\frac{p}{r}\,, \qquad k = 1, 2, \tag{47}$$

where γ_k is the ratio of the specific heat capacities at constant pressure and volume, respectively, of fluid k. Using (47), $(9)_2$ writes as

$$\epsilon = \left(\frac{X}{\gamma_1 - 1} + \frac{1 - X}{\gamma_2 - 1} \right) \frac{p}{r} \, . \tag{48}$$

Using again (47), the expressions of ω and ξ, (32) and (33) modify, respectively, as

$$\omega = \left[X\gamma_1 + (1 - X)\,\gamma_2 \right] p \, , \tag{49}$$

$$\xi = X\,(1 - X)\,(\gamma_1 - \gamma_2) \; . \tag{50}$$

Replacing the expression (49) of ω into equation (43) yields

$$\lambda_\Sigma^2 = \frac{p}{rf}\,(X_1\gamma_1 + X_2\gamma_2) \, . \tag{51}$$

Recalling that the hydrodynamical waves λ_k in each fluid k is given by

$$\lambda_k^2 = \gamma_k \frac{p}{r_k f_k} \, , \qquad k = 1, 2, \tag{52}$$

we can rewrite equation (51) under the form

$$rf\lambda_\Sigma^2 = X_1 r_1 f_1 \lambda_1^2 + X_2 r_2 f_2 \lambda_2^2 \, . \tag{53}$$

Equation (51) represent the relativistic generalization of the expressions of the normal velocity in a two-fluid system found by Wackers and Koren [32], and allows to express the acoustic modes speeds in such a two-fluid system as combination of the speeds of the individual modes.

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