

STRICTLY INCREASING MARKOV CHAINS AS WEAR PROCESSES

Mario Lefebvre

Department of Mathematics and Industrial Engineering, École Polytechnique, Montréal, Canada
mlefebvre@polymtl.ca

Abstract To model the lifetime of a device, increasing Markov chains are used. The transition probabilities of the chain are as follows: $p_{i,j} = p$ if $j = i + \delta$, and $p_{i,j} = 1 - p$ if $j = i + 2\delta$. The mean time to failure of the device, namely the mean number of transitions required for the process, starting from x_0 , to take on a value greater than or equal to $x_0 + k\delta$ is computed explicitly. A second version of this Markov chain, based on a standard Brownian motion that is discretized and conditioned to always move from its current state x to either $x + \delta$ or $x + 2\delta$ after ϵ time units, is also considered. Again the expected value of the time it takes the process to cross the boundary at $x_0 + k\delta$ is computed explicitly.

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1. INTRODUCTION

Let $X(t)$ denote the wear of a machine at time t . Although wear should obviously increase with time, some authors have used a one-dimensional diffusion process as a model for $X(t)$. For example, in a related problem, Tseng and Peng [4] (see also Tseng and Peng [3]) considered the following model for the lifetime of a device: they assumed that the device possesses a certain quality characteristic that is closely correlated with its lifetime, and they denoted by $D(t)$ the value of this quality characteristic at time t . Next, they assumed that $D(t)$ is a decreasing function of t and that it can be represented as follows:

$$D(t) = M(t) + \int_0^t s(u)dB(u), \quad (1)$$

in which $M(t)$ designates the mean value of the random variable $D(t)$, and $\{B(t), t \geq 0\}$ is a standard Brownian motion. The function $s(u)$ could be a constant (as in [3]). Thus, Tseng and Peng wanted to use a stochastic integral as a *noise* term. Finally, in their model, the device is considered to be worn out the first time $D(t)$ takes on a value smaller than or equal to the critical level c (a constant). Hence, the lifetime L_c of the device is defined by

$$L_c = \inf\{t > 0 : D(t) \leq c\}.$$

Now, if the function $s(\cdot)$ is very small, the model proposed by Tseng and Peng is probably a good approximation of reality. However, even if we suppose that $M(t)$ is a decreasing function, we cannot claim that $D(t)$ is decreasing as well. Indeed, a stochastic process that satisfies Eq. (1) can *both* increase and decrease on any interval.

Next, to obtain a function $X(t)$ that is strictly increasing with time, as should be, Rishel [2] proposed to use a degenerate two-dimensional diffusion process defined by the following system of stochastic differential equations:

$$\begin{aligned} dX(t) &= \rho[X(t), Y(t)]dt, \\ dY(t) &= f[X(t), Y(t)]dt + \sigma[X(t), Y(t)]dB(t). \end{aligned}$$

In this model, $\rho(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ are positive functions in the domain of interest, and $Y(t)$ is a variable (it could actually be a vector) that directly influences the wear. Notice that we could use this model for the remaining lifetime of a device by assuming instead that $\rho(\cdot, \cdot)$ is a negative function.

Recently, the author (see [1]) used the above model in the particular case when $\{Y(t), t \geq 0\}$ is a geometric Brownian motion and the function $\rho(\cdot, \cdot)$ is given by

$$\rho[X(t), Y(t)] = -\frac{\rho_0 Y(t)}{[X(t) - c]^\kappa},$$

where $\rho_0 > 0$ and $\kappa \geq 0$ are constants. He computed the expected value of the first passage time

$$T_c(x, y) = \inf\{t > 0 : X(t) = c \mid X(0) = x (> c), Y(0) = y\}$$

(that corresponds to the random variable L_c defined in [4]). This expected value yields the Mean Time To Failure (MTTF) for the device considered.

In discrete time, we don't have to define a two-dimensional stochastic process. Indeed, we can consider a strictly increasing one-dimensional Markov chain $\{X_n, n = 0, 1, \dots\}$. In this paper, we suppose that after the n th time unit, for $n = 1, 2, \dots$, the process $\{X_n, n = 0, 1, \dots\}$ takes on the value $X_{n-1} + \delta$ with probability $p \in (0, 1)$ or $X_{n-1} + 2\delta$ with probability $1 - p$. In Section 2, we compute the mean number of transitions required for X_n to become greater than or equal to $x_0 + k\delta$, with $k = 1, 2, \dots$, a value for which the device is considered to be worn out.

In Section 3, we consider a version of the Markov chain defined above that is based on a continuous-time stochastic process: we condition a discretized Brownian motion process to always move from its current state x to either $x + \delta$ or $x + 2\delta$ after ϵ time units. Again the expected value of the time it takes the process to cross the boundary at $x_0 + k\delta$ is computed explicitly.

2. A STRICTLY INCREASING MARKOV CHAIN

Let X_n , for $n \in \{0, 1, \dots\}$, represent the wear of a certain device after n time units. We assume that the initial wear is $X_0 = x_0 (\geq 0)$ and that $\{X_n, n = 0, 1, \dots\}$ is a Markov

chain with state space $\{x_0, x_0 + \delta, \dots, x_0 + (k + 1)\delta\}$, where $\delta > 0$ and $k \in \{1, 2, \dots\}$, and having transition probabilities

$$p_{i,j} = \begin{cases} p & \text{if } j = i + \delta, \\ 1 - p & \text{if } j = i + 2\delta \end{cases}$$

for $i = x_0, x_0 + \delta, \dots, x_0 + (k - 1)\delta$, where $0 < p < 1$. The Markov chain is thus strictly increasing: after each time unit, the process increases by δ or 2δ units. The critical level for the device is the value $x_0 + k\delta$.

Let $m(i)$, for $i = 0, 1, \dots, k$, denote the expected value of the random variable

$$T(i) = \inf\{n > 0 : X_n \geq x_0 + k\delta \mid X_0 = x_0 + i\delta\}.$$

Notice that $T(i)$ is the number of transitions required for the Markov chain to increase by at least $(k - i)\delta$ units, if it starts from $x_0 + i\delta$.

To obtain a difference equation satisfied by the function $m(i)$, we can condition on the first transition. We obtain that

$$m(i) = m(i + 1)p + m(i + 2)(1 - p) + 1. \tag{2}$$

Indeed, if $X_1 = x_0 + \delta$ (respectively $X_1 = x_0 + 2\delta$), then, by the Markov property, the process starts anew from $x_0 + \delta$ (respectively $x_0 + 2\delta$). Furthermore, we must take the first transition into account. Hence we add one time unit to the expected value from the new starting point.

To obtain the general solution of (2), we must first find two linearly independent solutions of the corresponding homogeneous equation, that is,

$$m(i) = m(i + 1)p + m(i + 2)(1 - p).$$

One such solution is trivially $m(i) \equiv c_0$, a constant. A second solution is given by

$$m(i) = (p - 1)^{k-i}.$$

It follows that the general solution of the homogeneous equation can be expressed as

$$m(i) = c_0 + c_1(p - 1)^{k-i},$$

in which c_0 and c_1 are arbitrary constants.

Next, we can check that

$$m(i) = c(k - i),$$

where c is a constant, is a solution of the non-homogeneous equation (2) if we take $c = (2 - p)^{-1}$. Hence, we can write that

$$m(i) = c_0 + c_1(p - 1)^{k-i} + \frac{k - i}{2 - p}.$$

Finally, to obtain the particular solution to our problem, we simply have to make use of the boundary conditions $m(k) = 0$ and $m(k-1) = 1$. Indeed, if the initial value of the Markov chain is $X_0 = x_0 + k\delta$ then the device is already worn out, whereas if $X_0 = x_0 + (k-1)\delta$ then X_1 will be equal to either $x_0 + k\delta$ or $x_0 + (k+1)\delta$.

We can now state the following proposition.

Propoziția 2.1. *The expected lifetime of the device, when $X_0 = x_0 + i\delta$, is given by*

$$m(i) = \frac{(1-p)}{(2-p)^2} \left[1 - (p-1)^{k-i} \right] + \frac{k-i}{2-p}$$

for $i = 0, 1, \dots, k$.

Thus, when $X_0 = x_0$ the expected lifetime of the device is

$$m(0) = \frac{(1-p)}{(2-p)^2} \left[1 - (p-1)^k \right] + \frac{k}{2-p}.$$

Notice that if p decreases to zero, then $m(0)$ tends to $k/2$ if k is an even integer, and to $(k+1)/2$ if k is an odd integer, as should be. Similarly, if p increases to 1, then $m(0)$ tends to k , which again is obviously correct.

3. A CONDITIONED BROWNIAN MOTION

If we want to use a Brownian motion (or Wiener process) to model the wear of a device, we can condition it to increase by a small quantity in a short interval. By doing so, we take into account the fact that there are always small errors when we measure wear, so that it can slightly decrease in a short interval. However, the probability that wear will decrease (or increase) greatly in a short interval is negligible.

Assume that $\{B(t), t \geq 0\}$ is a standard Brownian motion starting at $B(0) = x_0$. Let $X_{\delta,\epsilon}(0) = B(0) = x_0$ and

$$X_{\delta,\epsilon}(t) = B(t) \mid \{ \{B(t) = B(t-\epsilon) + \delta\} \cup \{B(t) = B(t-\epsilon) + 2\delta\} \}$$

for $t = \epsilon, 2\epsilon, \dots$. Next, consider the random variable

$$\tau_{\delta,\epsilon}(x_0) = \inf\{t > 0 : X_{\delta,\epsilon}(t) = x_0 + k\delta \mid X_{\delta,\epsilon}(0) = x_0\}.$$

We obtain the following difference equation for the expected value of the random variable $\tau_{\delta,\epsilon}(x_0)$, which we will denote simply by $\mu(k)$ to emphasize that the process must increase by k times δ from its current position:

$$\mu(k) = \epsilon + \mu(k-1)p_\epsilon + \mu(k-2)(1-p_\epsilon), \quad (3)$$

where

$$p_\epsilon = P[B(t) = B(t-\epsilon) + \delta \mid \{B(t) = B(t-\epsilon) + \delta\} \cup \{B(t) = B(t-\epsilon) + 2\delta\}].$$

Remark 3.1. Because, by definition, the Wiener process has stationary increments, only the distance that it must cover, in ϵ time units, is important.

The probability p_ϵ is given by

$$p_\epsilon = \frac{f_{B(\epsilon)}(\delta)}{f_{B(\epsilon)}(\delta) + f_{B(\epsilon)}(2\delta)},$$

in which $f_{B(\epsilon)}$ is the probability density function of a Gaussian distribution with mean 0 and variance ϵ . That is,

$$f_{B(\epsilon)}(x) = \frac{1}{\sqrt{2\pi\epsilon}} \exp\left\{-\frac{x^2}{2\epsilon}\right\}$$

for $x \in \mathbb{R}$. It follows that

$$p_\epsilon = \frac{\exp\{-\frac{\delta^2}{2\epsilon}\}}{\exp\{-\frac{\delta^2}{2\epsilon}\} + \exp\{-\frac{2\delta^2}{\epsilon}\}}.$$

If we write that $\delta^2 = \sigma_0^2\epsilon$, where $\sigma_0 > 0$, we have:

$$p_\epsilon = \frac{1}{1 + \exp\{-\frac{3\sigma_0^2}{2}\}}.$$

Hence, contrary to the case of the Markov chain in the preceding section, the probability p_ϵ cannot take on any value in the interval $(0, 1)$. Actually, we find that $p_\epsilon \in (1/2, 1)$.

Now, (3) is of the same form as (2). Proceeding as in Section 2, we obtain the following result.

Propoziția 3.1. Under the assumptions made in this section, the expected lifetime of the device, when $X_{\delta,\epsilon}(0) = x_0$, is

$$\mu(k) = \frac{(1-p)\epsilon}{(2-p)^2} \left[1 - (p-1)^k\right] + \frac{k\epsilon}{2-p},$$

where $p \in (1/2, 1)$.

Proof. The general solution of Eq. (3) can be written as

$$\mu(k) = k_1 + k_2(p-1)^k + \frac{k\epsilon}{2-p}.$$

The constants k_1 and k_2 are obtained from the boundary conditions $\mu(0) = 0$ and $\mu(1) = \epsilon$. ■

4. CONCLUSION

To model the wear of a device, in Section 2 we proposed a strictly increasing Markov chain. In Section 3, we used a standard Brownian motion that we discretized and conditioned to increase by either δ or 2δ units every ϵ time units. In both cases we obtained a second-order non-homogeneous difference equation with constant coefficients for the expected value of the random variable representing the time to failure of the device. We could try to compute the variance of the lifetime. Ideally, we would like to obtain the probability mass function of the lifetime.

Finally, in Section 2 we could consider the case when the process spends a random amount of time, say S , in a given state before making a transition. If the random variable S has an exponential distribution for each state, then the stochastic process $\{X(t), t \geq 0\}$, where $X(t)$ denotes the state of the process at time t , would be a continuous-time Markov chain. Otherwise, it would be a semi-Markov process.

Acknowledgments

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