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THE FACTORIZATION OF THE RIGHT PRODUCT OF TWO SUBCATEGORIES

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Abstract In the category of locally convex spaces the right product of two subcategories is a reflective subcategory. In the topological completely regular spaces a similar property is not always true. The factorization of this product according to a structure of factorization leads always to a reflective subcategory. Thus, some well known compactifications in the topology appear as this type of factorization.

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INTRODUCTION

In the sequel we use the following notations:

$\mathcal{E}_u$ (resp. $\mathcal{M}_u$) denotes the class of universal epi (resp. mono);

$\mathcal{E}_p$, (resp. $\mathcal{M}_p$) the class of precise epi (resp. mono) : $\mathcal{E}_p= \mathcal{M}_u^{\rceil}$, $\mathcal{M}_p= \mathcal{E}_u^{\lfloor}$;

$\mathcal{C}_2\mathcal{V}$ - the category of Hausdorff locally convex topological vector space;

$\mathcal{T}h$ - the category of Tikhonov spaces (the completely regular Hausdorff spaces); if $\mathcal{K}$ (resp. $\mathcal{R}$) is coreflective (resp. reflective) subcategory, then k: $\mathcal{C} \to\mathcal{R}$ (r: $\mathcal{C}\to\mathcal{R}$) is the coreflector (resp. reflector) functor;

$(\mathcal{E}pi, \mathcal{M}_f)$ - (the class of epimorphisms, the class of strict monomorphisms) = (the class of mappings with dense image, the class of topological inclusions with closed image);

$(\mathcal{E}_f, \mathcal{M}ono)$ - (the class of strict epimorphisms, the class of monomorphisms)

$(\mathcal{E}_u, \mathcal{M}_p)$-(the class of universal epimorphisms, the class of precise monomorphisms)=(the class of surjective mapping, the class of topological inclusions).

For concepts from general topology see [7], from topology of locally convex spaces see [8], and for those related to factorization structures see also [8].

The right product of a coreflective and reflective subcategory was introduced and examined in the paper [5]. Necessary and sufficient conditions for the product to be a reflective subcategory were identified. In the category $\mathcal{C}_2\mathcal{V}$ of Hausdorff locally convex topological vector spaces many cases when this product is a reflective subcategory were found.

The examination of the right product of two subcategories is requested by the following situations:

1. The right product appears in natural way when studying the semi-reflexive subcategories [2].

2. The relative torsions theories, that are so frequent in the category $\mathcal{C}_2\mathcal{V}$, can be performed as right product theories.

3. Many reflective subcategories can be explained as the right product of two subcategories of certain type [3].

In the category $\mathcal{C}_2\mathcal{V}$, as well as in the category of Tikhonov spaces, there are examples when this product is not a reflective subcategory (Theorem 1.1).

The properties of right product factorization are examined in this paper (Lemma 1.2). There are stipulated the conditions when this factorization defines a reflective subcategory (Theorem 1.3).

In Section 2 it is proved that τ-complete spaces [6], [11] could be constructed as such factorizations (Theorem 2.3).

In Section 3 it is shown how the subcategory of τ-complete spaces could be performed in two ways: either varying in the product the coreflective subcategory, or varying the factorization structure.

In Section 4 there are formulated some issues for the category of Tikhonov spaces.

1. THE RIGHT PRODUCT OF TWO SUBCATEGORIES

Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ - a reflective subcategory of the category $\mathcal{C}$. For any object X of the category $\mathcal{C}$ assume that $r^X : X \to rX$ is $\mathcal{R}$-replique of object X, and $k^X : kX \to X$ and $k^{krX} : krX \to rX$ are the $\mathcal{K}$-corepliques of respective objects. Then

$$r^X k^X = k^{rX} g \tag{1}$$

for some morphism g. Since $g = k(r^X)$ we can write the preceding equality

$$r^X k^X = k^{rX} k(r^X). \tag{2}$$

We assume that in the category $\mathcal{C}$ pushout squares exist and we construct the pushout square on the morphisms k^X and $k(r^X)$:

$$v^X k^X = g^X k(r^X). \tag{3}$$

Then there exists a morphism u^X so that

$$r^X = u^X v^X, \tag{4}$$

$$k^{rX} = u^X g^X. \tag{5}$$

$$
\begin{tikzcd}
kX \arrow[rr, dashed, "k(r^X)"] \arrow[dd, "k^X"'] & & krX \arrow[dl, dashed, "g^X"'] \arrow[dd, "k^{rX}"] \\
& vX \arrow[dr, dashed, "u^X"] & \\
X \arrow[ur, dashed, "v^X"] \arrow[rr, "r^X"] & & rX
\end{tikzcd}
$$

Figure 1.1

We denote by $\mathcal{K} *_d \mathcal{R}$ the full subcategory of all objects of category $\mathcal{C}$, isomorphic with the objects of vX, $X \in\mid \mathcal{C} \mid$ form.

Definition 1.1. *The subcategory $\mathcal{K} *_d \mathcal{R}$ is called the d-product or the right product of subcategories $\mathcal{K}$ and $\mathcal{R}$.*

Theorem 1.1. *Let $\mathcal{C}$ be a category with pull-back and pushout squares, $\mathcal{R}$ a monoreflective subcategory, $\mathcal{K}$ - a epireflective subcategory, and $\mathcal{V} = \mathcal{K} *_d \mathcal{R}$. Then, the following affirnations are equivalent:*

1. *$\mathcal{V}$ is a reflective subcategory of category $\mathcal{C}$.*
2. *For any object X of category $\mathcal{C}$ the morphism v^X is $\mathcal{V}$-replique of object X.*
3. *For any object X of category $\mathcal{C}$ the morphism v^X is an epi.*
4. *For any object X of category $\mathcal{C}$ the morphism u^X is $\mathcal{R}$-replique of object vX.*
5. *$X \in\mid \mathcal{V} \mid \Leftrightarrow r^X k^X$ is $\mathcal{K}$-coreplique of object rX.*

The proving is performed as in the case of category $\mathcal{C}_2\mathcal{V}$ ([5], Theorem 2.5).

We mention that there are known various cases when subcategory $\mathcal{V}$ is reflective (see [5], Theorems 3.2.-3.4. and [3] Theorem 2.6.).

Let us examine the right product of two subcategories in the category $\mathcal{T}h$ of Tikhonov spaces. Let $\mathcal{K}$ be a coreflective subcategory. Then it contains the subcategory $\mathcal{D}$ of the spaces with discrete topology. It follows that it is a monoreflective subcategory, and then it is $(\mathcal{E}_u \bigcap Mono)$ - coreflective, where $\mathcal{E}_u$ is class of universal epimorphisms (continued and surjective maps).

It is obvious that in the category $\mathcal{T}h$ the reflective subcategory $\mathcal{R}$ is monoreflective iff when $\mathcal{R}$ includes the subcategory of compact spaces: $\mathcal{C}omp \subset \mathcal{R}$.

Let $\mathcal{C}omp \subset \mathcal{R}$. We reffer to Figure 1.1. Let's presume that $\mathcal{V}$ is a reflective subcategory. Then u^X is $\mathcal{C}omp$-replique of object vX. So, u^X is a topological inclusion. And from the equality (5) it results that u^X is a surjective application. Therefore, u^X is an isomorphism. Thus, we proved:

Theorem 1.2. *Let $\mathcal{K}$ be a coreflective subcategory, $\mathcal{R}$ - a reflective subcategory in the category $\mathcal{T}h$ and $\mathcal{C}omp \subset \mathcal{R}$. Then, two cases are possible:*
1. $\mathcal{K} *_d \mathcal{R} = \mathcal{R}$.
2. $\mathcal{K} *_d \mathcal{R}$ is not a reflective subcategory of $\mathcal{T}h$ category.

But some examples of reflective subcategory of $\mathcal{T}h$ category show they can be obtained as a simple modification of the right product of two subcategories. In what follows we will describe this modification.

Definition 1.2. *The class of morphisms $\mathcal{A}$ of category $\mathcal{C}$ is called right-stable if, because*

$$g'f = f'g,$$

it is a pushout square, and from $f \in \mathcal{A}$ it follows that $f' \in \mathcal{A}$.

In the categories $\mathcal{T}h$ and $\mathcal{C}_2\mathcal{V}$ the pair $(\mathcal{E}_u, \mathcal{M}_p)$ = (the class of surjective morphisms, the class of topological inclusions) is a factorization structure. Therefore the class $\mathcal{E}_u$ is right-stable.

Lemma 1.1. *Let $\mathcal{C}$ be a category with pull-back and pushout squares in which the class $\mathcal{E}_u$ is right-stable, $\mathcal{K}$ - a monoreflective subcategory, and $\mathcal{R}$ - a reflective subcategory. Then for any abject X of the category $\mathcal{C}$, u^X is a monomorphism.*

Proof. Any monoreflective subcategory is $(\mathcal{E}_u \bigcap Mono)$-coreflective. We use the notations from the beginning of the section. In the pushout square (3) $k^X \in \mathcal{E}_u$ and according to the hypothesis that $g^X \in \mathcal{E}_u$. In the equality (5) we have $k^{rX} \in \mathcal{M}ono$ and $g^X \in \mathcal{E}_u$. According to the Lemma 1.3 [5], we deduce that $u^X \in \mathcal{M}ono$. ∎

Corollary 1.1. *In the categories $\mathcal{T}h$ and $\mathcal{C}_2\mathcal{V}$ for any two subcategories one coreflective and other reflective, u^X is always a monomorphism.*

Lemma 1.2. *Assume in the category $\mathcal{C}$ and in the subcategory $\mathcal{K}$ and $\mathcal{R}$, for any object X, u^X is a monomorphism. Then:*

1. For any object X of category $\mathcal{C}$ the morphism g^X is $\mathcal{K}$-coreplique of object vX : $g^X = k^{vX}$.

2. The correspondence $X \longmapsto vX$ defines a covariant functor $v : \mathcal{C} \to \mathcal{C}$.

3. $\mathcal{R}$ is a monoreflective subcategory of the category $\mathcal{K} *_d \mathcal{R}$.

4. For any object X of the subcategory $\mathcal{K} *_d \mathcal{R}$ the morphism v^X is sectionable.

Proof. 1. Consider $f : A \to vX$, where $A \in \mid \mathcal{K} \mid$. Then

$$u^X f = k^{rX} g$$

for some morphism g.

$$\begin{tikzcd} A \arrow[rr, dashed, "g"] \arrow[d, "f"'] & & krX \arrow[d, "k^{rX}"] \arrow[lld, "g^X"'] \\ vX \arrow[rr, "u^X"] & & rX \end{tikzcd}$$

Figure 1.2

We have

$$u^X f = k^{rX} g = u^X g^X g$$

i.e.

$$u^X f = u^X g^X g,$$

and since u^X is a mono, we deduce that

$$f = g^X g.$$

The uniqueness of the morphism g which satisfies the preceding equality results from the fact that k^{rX} is $\mathcal{K}$-coreplique of object rX.

So, we proved that $g^X = k^{vX}$.

2. Let us define the functor v on morphisms. Let $f : X \to Y \in \mathcal{C}$.

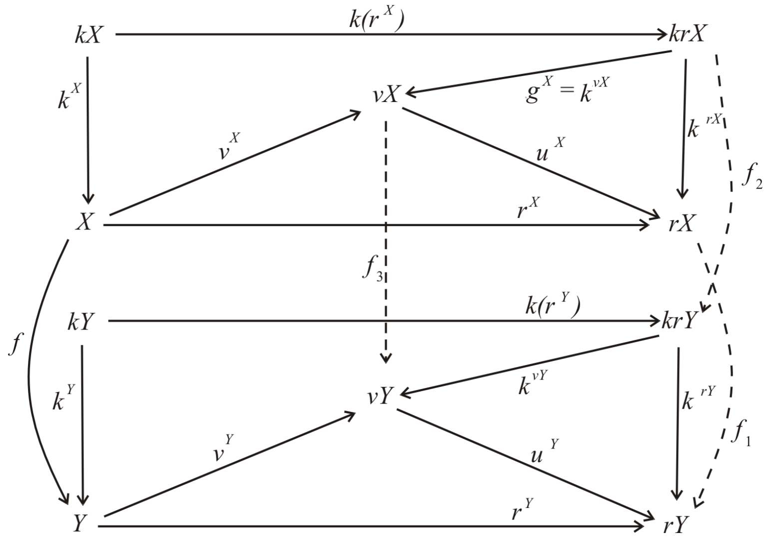

Figure 1.3

Then

$$r^Y f = f_1 r^X, \tag{6}$$

for some morphism f_1, where $f_1 = r(f)$;

$$f_1 k^{rX} = k^{rY} f_2, \tag{7}$$

for some morphism f_2, where $f_2 = k(f_1)$.

We have

$u^Y v^Y f k^X$ = (from (4), for object Y) = $r^Y f k^X$ =(from (6))= $f_1 r^X k^X$ =(from (2)) = $f_1 k^{rX} k(r^X)$ = (from (7)) = $k^{rY} f_2 k(r^X)$(= from (5) for object Y) = $u^Y k^{vY} f_2 k(r^X)$, i.e.

$$u^Y v^Y f k^X = u^Y k^{vY} f_2 k(r^X). \tag{8}$$

Since u^Y is a mono, from equality (8) we obtain

$$(v^Y f)k^X = (k^{vY} f_2)k(r^X). \tag{9}$$

From equality (9) and the fact that (3) is an pushout square, we deduce that

$$v^Y f = f_3 v^X, \tag{10}$$

$$k^{vY} f_2 = f_3 k^{vX}. \tag{11}$$

We define $v(f) = f_3$.

3. Let us get look again to Fig.1.1. Let $X \in\mid \mathcal{R} \mid$. Then r^X, and with him and $k(r^X)$ and v^X, are isomorphisms. Therefore $\mathcal{R} \subset \mathcal{K} *_d \mathcal{R}$.

Further on, let be $r^{vX} : vX \to rvX$ $\mathcal{R}$ replique of object vX. Then

$$u^X = t \cdot r^{vX}$$

for some morphism t. Since u^X is a mono of category $\mathcal{C}$, we deduce that r^{vX} is the same. The only thing we can add is: the subcategory $\mathcal{R}$ is $(\mathcal{E}pi \bigcap \mathcal{M}_u)$-reflective in the category $\mathcal{K} *_d \mathcal{R}$.

4. Let's complete the diagram from Fig. 1.1 with an analogous diagram built for the object vX.

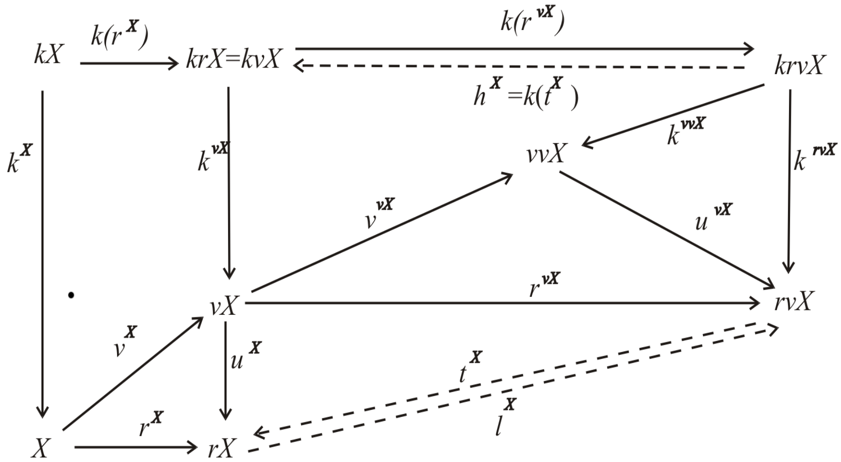

Figure 1.4

We have

$$u^X = t^X r^{vX} \tag{12}$$

for some morphism t^X. Then

$$t^X k^{rvX} = k^{rX} h^X \tag{13}$$

for some morphism $h^X = k(t^X)$.

We have

$k^{rX}h^X k(r^{vX})$= (from (13)) =$t^X k^{rvX} k(r^{vX})$=(from (2) for object vX) = $t^X r^{vX} k^{vX}$ =(from (12))= $u^X k^{vX}$ = (from (10) since $g^X = k^{vx}$) = k^{rX}, i.e.

$$k^{rX}h^X k(r^{vX}) = k^{rX}. \tag{14}$$

Therefore

$$h^X k(r^{vX}) = 1. \tag{15}$$

Then in the pushout square

$$v^{vX}k^{vX} = k^{vvX}k(r^{vX}) \tag{16}$$

the morphism $k(r^{vX})$ is sectional. Therefore the morphism v^{vX} is the same.

We can assert that

$$r^{vX}v^X = l^X r^X \tag{17}$$

for some morphism l^X. We have

$t^X l^X r^X$ = (from (17)) = $t^X r^{vX} v^X$ = (from (1)) = $u^X v^X$ = (from (4)) = r^X i.e.

$$t^X l^X r^X = r^X \tag{18}$$

or

$$t^X l^X = 1. \tag{19}$$

∎

We assume that the conditions of preceding lemma are satisfied, and $(\mathcal{P}, \mathcal{I})$ is a factorization structure in the category $\mathcal{C}$. Let $\mathcal{L}$ be the full subcategory of category $\mathcal{C}$ comprising $\mathcal{I}$-subobjects of objects of subcategories $\mathcal{K} *_d \mathcal{R}$. For any object X of category $\mathcal{C}$ let

$$v^X = i^X l^X \tag{20}$$

be $(\mathcal{P}, \mathcal{I})$-factorization of morphism v^X.

Theorem 1.3. *The correspondence $X \longmapsto (lX, l^X)$ defines the category $\mathcal{L}$ as a $\mathcal{P}$-reflective subcategory of the category $\mathcal{C}$.*

Proof. Let be $A \in\mid \mathcal{L} \mid$, and $f : X \longrightarrow A \in \mathcal{C}$. We show that the morphism f is extended through morphism l^X. According to the hypothesis there exists an object $B \in\mid \mathcal{K} *_d \mathcal{R} \mid$ and a morphism $i : A \longrightarrow B \in \mathcal{I}$. Then

$$v^B(if) = v(if)v^X, \tag{21}$$

or

$$(v^B i)f = (v(if)i^X)l^X. \tag{22}$$

Since v^B is sectional we deduce that $v^B i \in \mathcal{I}$, and $l^X \in \mathcal{P}$. Therefore $l^X \perp v^B i$, i.e.

$$f = gl^X, \tag{23}$$

$$v(if)i^X = v^B ig, \tag{24}$$

for some morphism g. The uniqueness of morphism g, that satisfies equality (23), results from the fact that l^X is an epi. ∎

$$\begin{tikzcd}
X \arrow[r, "l^X"] \arrow[d, "f"'] \arrow[rr, bend left, "v^X"] & lX \arrow[r, "i^X"] \arrow[dl, dashed, "g"'] & vX \arrow[d, dashed, "v(if)"] \\
A \arrow[r, "i"] & B \arrow[r, "v^B"] & vB
\end{tikzcd}$$

Figure 1.5

2. THE SUBCATEGORY OF τ-COMPLETE SPACES

We examine some coreflective subcategories of the category $\mathcal{T}h$ of Tikhonov spaces.

Definition 2.1. *Let τ be an cardinal.*

1. In a topological space the intersection of τ open sets is called $\mathcal{G}_\tau$ -set.

2. $\mathcal{P}_\tau^-$-change of the space (X, t) is called the space (X, t_τ^-), where the basis of topology t_τ^- is formed by $\mathcal{G}_\alpha$-set, $\alpha < \tau$.

3. $\mathcal{P}_\tau$-change of space (X, t) is called the space (X, t_τ), where the basis of topology t_τ is formed by $\mathcal{G}_\tau$-sets.

We note with $\mathcal{K}^-(\tau)$ (respectively $\mathcal{K}(\tau)$) the full subcategory of the category $\mathcal{T}h$ comprising all spaces (X, t) for which $t = t_\tau^-$ (respectively, $t = t_\tau$).

We observe that $\mathcal{K}(\tau) = \mathcal{K}^-(\tau^+)$, where τ^+ is the first cardinal which follows τ. If τ is limiting cardinal, then $\mathcal{K}^-(\tau) \neq \mathcal{K}^-(\lambda)$, for any cardinal λ. Therefore, it is enough to examine only the subcategories $\mathcal{K}^-(\tau)$.

It is easily checked that $\mathcal{K}^-(\tau)$ (similar by and $\mathcal{K}(\tau)$) are the coreflective subcategories of category $\mathcal{T}h$ with coreflective functors.

$$P_\tau^- : Th \longrightarrow \mathcal{K}^-(\tau),\ P_\tau^-(X, t) = (X, t_\tau^-),$$

$$P_\tau : Th \longrightarrow \mathcal{K}(\tau),\ P_\tau(X, t) = (X, t_\tau).$$

We mention the following properties of the subcategories $\mathcal{K}^-(\tau)$:

1. $\mathcal{K}^-(\tau) = \mathcal{T}h$ for $\tau \leq \omega$.
2. Let $\alpha < \beta$ be. Then $\mathcal{K}^-(\alpha) \supseteq \mathcal{K}^-(\beta)$.
3. Let $\mathcal{D}isc$ be the subcategory of discrete spaces. Then
$\cap\{\mathcal{K}^-(\tau)/\tau\} = \cap\{\mathcal{K}(\tau)/\tau\} = \mathcal{D}isc$.

Therefore, we can conclude that $\mathcal{D}isc = \mathcal{K}^-(\infty) = \mathcal{K}(\infty)$, considering that $\tau < \infty$ for every cardinal τ.

Theorem 2.1. *([4], Theorem1.2). Consider $\omega \leq \alpha < \beta$. Then:*

$$\mathcal{K}^-(\beta) \subset \mathcal{K}^-(\alpha) \text{ and } \mathcal{K}^-(\beta) \neq \mathcal{K}^-(\alpha).$$

Definition 2.2. *Let τ be an cardinal. The Tikhonov space X is called $Q(\tau)$-space (respectively $Q^-(\tau)$-space), if X is closed in $\mathcal{K}(\tau)$-coreplique (respectively $\mathcal{K}^-(\tau)$-coreplique) of space βX, where βX is $\mathcal{C}omp$-replique of spaces X.*

We note with $\mathcal{Q}(\tau)$ (respectively, $\mathcal{Q}^-(\tau)$) - the full subcategory of all $Q(\tau)$-spaces (respectively, $Q^-(\tau)$ -spaces).

The categories $\mathcal{Q}(\tau)$ have been studied by A. Cigoghidze [6], and the $Q^-(\tau)$ by H. Herrlich [11].

Let (X, t) be a Tikhonov space, $(Y, u) = \beta(X, t)$, and $(Y, u_\tau) - \mathcal{K}^-(\tau)$ -coreplique of space (Y, u). Let $\overline{X}$ be the closure of the set X in the space (Y, u_τ). On set $\overline{X}$ we induce the topology $u\prime$ out of space (Y, u). The topology space $(\overline{X}, u\prime)$ we note $v_t^- X$, or $v_t^-(X, t)$.

$$\begin{tikzcd}
v^-_\tau(X,t)=(\overline{X},u') \arrow[r, dashed] \arrow[dr] & (Y, u_\tau)=P^-_\tau(Y, u) \arrow[d, "p^Y_\tau"] \\
(X, t) \arrow[u, "v^X_\tau"] \arrow[r, "\beta^X"] & (Y, u)=\beta(X, t)
\end{tikzcd}$$

Figure 2.1

Evidently, $\mathcal{Q}(\tau) = \mathcal{Q}^-(\tau^+)$ for any limiting cardinals τ. But for limiting cardinals τ, the subcategories $\mathcal{Q}^-(\tau)$ are of other form that $\mathcal{Q}(\tau)$.

For a subcategory $\mathcal{A}$ of category $\mathcal{T}h$ we note with $Pr\mathcal{A}$ the subcategory that contains the products of objects of category $\mathcal{A}$, and with $\mathcal{M}_f(\mathcal{A})$ the subcategory that contains $\mathcal{M}_f$-subobjects of objects of subcategory $\mathcal{A}$.

Theorem 2.2. *([4], Theorem 2.6). 1. The subcategory $\mathcal{Q}^-(\tau)$ is a monoreflective subcategory (therefore also epireflective) of category $\mathcal{T}h$ with reflector functor*

$$v_t^- : \mathcal{T}h \longrightarrow \mathcal{Q}^-(\tau).$$

2. $\mathcal{Q}^-(\omega) = \mathcal{Q}(n) = \mathcal{C}omp, n \in \mathcal{N}$.
3. $\mathcal{Q}^-(\omega_I) = \mathcal{Q}(\omega) = \mathcal{Q}$ - the subcategory of Hewitt spaces.
4.$\mathcal{Q}^-(\tau) = \mathcal{M}_f Pr(R(\tau))$, where $R(\tau) = [-1; 1]^\tau \setminus \{-1; 1\}$.
5. $\mathcal{Q}^-(\tau) = \mathcal{M}_f Pr(E(\tau))$, where $E(\tau) = \Pi_{\alpha<\tau} R(\alpha)$.
6. $\alpha < \beta$ and $\omega < \beta$. Then $\mathcal{Q}^-(\alpha) \subset \mathcal{Q}^-(\beta)$ and $\mathcal{Q}^-(\alpha) \neq \mathcal{Q}^-(\beta)$.

Corollary 2.1. *Let τ be a limiting cardinal. Then*

$$\mathcal{Q}(\tau) = \mathcal{M}_f Pr(\cup \mathcal{Q}(\lambda) : \lambda > \tau).$$

Remark 2.1. *In [11] is defined the problem of existence of generators for subcategories $\mathcal{Q}^{-}(\tau)$, i.e. if there is a space A_τ, so that*

$$\mathcal{Q}^{-}(\tau) = \mathcal{M}_f Pr(A_\tau).$$

Ignoring the case in [6], the problem is solved for subcategories $\mathcal{Q}(\tau)$ and fully in the precedent theorem.

Theorem 2.3. *The reflective subcategory $Q^{-}(\tau)$ is $(\mathcal{E}pi, \mathcal{M}_f)$-factorization of the right product $\mathcal{K}^{-}(\tau) *_d \mathcal{C}omp$.*

Proof. For Tikhonov space (X, t) let $(Y, u) = \beta(X, t)$ the Stone-Čech compactification, and $p_\tau^X : (X, t_\tau) \longrightarrow (X, t)$ and $p_\tau^Y : (Y, u_\tau) \longrightarrow (Y, u)$ the $\mathcal{K}_\tau^{-}$-coreplique of respective objects.

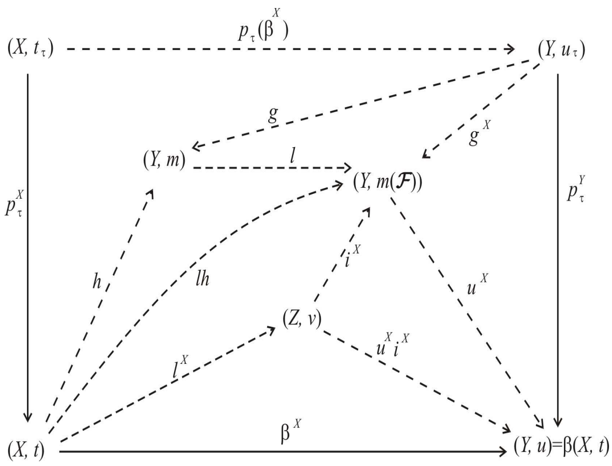

Figure 2.2

On the set Y we examine the inductive topology m, which is not mandatory being the Tikhonov topology, defined by applications β^X and p_τ^Y:

$$G \in m \Longleftrightarrow [(\beta^X)^{-1}(G) \in t \text{ and } (p_{\tau t}^Y)^{-1}(G) \in u_\tau].$$

The square

$$g p_\tau(\beta^X) = h p_\tau^X, \tag{25}$$

is the pushout square constructed on morphisms p_τ^X and $p_\tau(\beta(^X))$ in the category of topological spaces. In the sequel we construct the pushout square on these morphisms in the category $\mathcal{T}h$. Let $\mathcal{F}$ be the set of continued defined functions (Y, m) with values in the field of real numbers $\mathbb{R}$:

$$\mathcal{F} = Hom((Y, m), \mathbb{R}),$$

and $m(\mathcal{F})$ - the topology defined on the set Y of this set $\mathcal{F}$:

$$m(\mathcal{F}) = \{f^{-1}(G)|f \in \mathcal{F} \text{ and } G \text{ is open in } \mathbb{R}\}.$$

Let l be the canonic application. Then

$$(lh)p_\tau^X = g^X p_\tau(\beta^X) \tag{26}$$

is the pushout square constructed on morphisms p_τ^X and $p_\tau(\beta^X)$ in the category $\mathcal{T}h$.

Let Z be the closure of set X in the space $(Y, m(\mathcal{F}))$, and let the topology v be the one induced from space $(Y, m(\mathcal{F}))$. Since l^X is an epi in the category $\mathcal{T}h$ it follows that $u^X i^X$ is the Stone-Cech compactification of space (Z, v). Thus we can consider that the topology v is the one induced from application $u^X i^X = \beta^Z$ on set Z from space $\beta(X, t)$.

The closure of set X in the spaces (Y, m). A set A is closed in the space (Y, m) iff the set $h^{-1}(A)$ is closed in the space (X, t) and $g^{-1}(A)$ is closed in the space (Y, u_τ). If $X \subset A$, then the set A is closed iff the set $g^{-1}(A)$ is closed in the space (Y, u_τ).

The closure of set X in the spaces (Y, m) and (Y, u_τ) coincides: $cl_m X = cl_{u_\tau} X$. The last set we denote as $T : T = cl_{u_\tau} X$.

Let us prove that the closure of set X in the space (Y, u_τ) coincides with Z: $cl_{u_\tau} X = cl_{m(\mathcal{F})} X$.

Firstly we mention that $T \subset Z$ and will prove the reverse inclusion. Let be $y \in Y \backslash T$. Then exists a set $H \subset Y \setminus T$, that comprise point y so that:

1. H is closed in the topology u.
2. H is a G_τ-set in the topology u.
3. H is closed and open in the topology u_τ.
4. H remains open and closed in the topology m.
5. H remains open and closed in the topology $m(\mathcal{F})$.

The theorem is proved. ∎

3. THE CASE OF THE SUBCATEGORY $Q^-(\tau)$

Let be $\mathcal{R}$ and $\mathcal{L}$ two reflective subcategories of the category $\mathcal{C}$ and $\mathcal{L} \subset \mathcal{R}$. If $\mathcal{C}$ is local and colocal small with projective limits, then there exists a class $L(\mathcal{R})$ of factorization structure in the category $\mathcal{C}$ with the following property.

For any object X of category $\mathcal{C}$ let be $r^X : X \longrightarrow rX$ and $l^X : X \longrightarrow lX$ the respective replique. Then

$$l^X = v^X r^X \tag{27}$$

for some morphism v^X. If the subcategory $\mathcal{L}$ is monoreflective, then in the written equality all morphisms are bimorphisms.

We note

$$U = \{r^X | X \in |\mathcal{C}|\},$$
$$V = \{v^X | X \in |\mathcal{C}|\}.$$

Always $U \perp V$.

Assume
$\mathcal{P}''(\mathcal{R}) = \mathcal{P}'' = V^{\urcorner}$; $\mathcal{I}''(\mathcal{R}) = \mathcal{I}'' = V^{\urcorner\llcorner}$; $\mathcal{P}'(\mathcal{R}) = \mathcal{P}' = U^{\llcorner\urcorner}$; $\mathcal{I}'(\mathcal{R}) = U^{\llcorner}$.

Theorem 3.1. [1] *Let $\mathcal{C}$ be local and colocal small with projective limits and the subcategory $\mathcal{L}$ is monoreflective. Then:*

1. $(\mathcal{P}', \mathcal{I}')$ and $(\mathcal{P}'', \mathcal{I}'')$ are structures of factorization in the subcategory $\mathcal{C}$.

2. Let $(\mathcal{P}, \mathcal{I})$ be a structure of factorization in the category $\mathcal{C}$. The following affirmations are equivalent:

a) for any object X of category $\mathcal{C}$ the equality (1) is $(\mathcal{P}, \mathcal{I})$-factorization of morphism l^X;

b) $\mathcal{P}' \subset \mathcal{P} \subset \mathcal{P}''$.

Relying on this theorem we assert

Theorem 3.2. *Let τ be a cardinal, $\tau > \aleph_0$.*

1. The reflective subcategory $Q^-(\tau)$ could be obtained factorizing the right product (the morphism v^X) $\mathcal{K}^-(\tau) *_d \mathcal{C}omp$ following the structure of factorization $(\mathcal{E}pi, \mathcal{M}_f)$.

2. In the category $\mathcal{T}h$ there ar the structures of factorization $(\mathcal{P}, \mathcal{I})$, $\mathcal{P}'(Q^-(\tau)) \subset \mathcal{P} \subset \mathcal{P}''(Q^-(\tau))$, so that the reflective subcategory $Q^-(\tau)$ could be obtained doing the $(\mathcal{P}, \mathcal{I})$- factorization of right product $\mathcal{K}^-(\omega) *_d \mathcal{C}omp$.

4. PROBLEMS

In paper [4] some classes of coreflective subcategories in the category $\mathcal{T}h$ are examined.

Definition 4.1. *Let τ be an cardinal. The topological space (X, t) is called k_τ^--space, if any function definite on set X and continue on every compact $K \subset X, \mid K \mid < \tau$ is continue on space (X, t).*

Let $\mathcal{C}^-(\tau)$ a full subcategory of all k_τ^--spaces.

Theorem 4.1. *The subcategory $\mathcal{C}^-(\tau)$ is a coreflective subcategory of category $\mathcal{T}h$.*

Definition 4.2. *The weight $w(X, t)$ of the topological space (X, t) is called minimal of cardinals $|\mathcal{B}|$, where $\mathcal{B}$ is basis of spaces (X, t).*

Definition 4.3. *Let τ be an cardinal. The Tikhonov space is called b_τ^--space if every function definite on set X and continue on every compact $K \subset X$, $w(K) < \tau$ is continue on space (X, t).*

Let $\mathcal{B}^-(\tau)$ is a subcategory full of all b_τ^--spaces.

Theorem 4.2. *1. The subcategory $\mathcal{B}^-(\tau)$ is a coreflective subcategory of category $\mathcal{T}h$.*
2. Let be $\tau \leq \omega$. Then $\mathcal{B}^-(\tau) = \mathcal{D}isc$.
3. $\cap \mathcal{B}^-(\tau) = \mathcal{C}$ - subcategory of k-functionals space.

4. Let $\omega \leq \alpha < \beta$ and α a regular cardinal and $\beta \geq \alpha^+$. Then $\mathcal{B}^-(\alpha) \subset \mathcal{B}^-(\beta)$ and $\mathcal{B}^-(\alpha) \neq \mathcal{B}^-(\beta)$.

Problem 4.1. *1. Let us describe the subcategory $\mathcal{R}$ which is $(\mathcal{E}pi, \mathcal{M}_f)$-factorization of the right product $\mathcal{C}^-(\tau) *_d \mathcal{C}omp$.*

2. $\mathcal{R}$-replique of an arbitrary object X of category $\mathcal{T}h$ is also the closing of set X in the $\mathcal{C}^-(\tau)$-coreplique of space βX (with induce topology from βX)?

3. The same thing is valid for the right product $\mathcal{B}^-(\tau) *_d \mathcal{C}omp$.

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