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CLASSIFICATION OF A CLASS OF QUADRATIC DIFFERENTIAL SYSTEMS WITH DERIVATIONS

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Abstract The classification up to a center-affinity of a class of homogeneous quadratic differential systems defined on $\mathbb{R}^3$, which have a semisimple derivation with a 2-dimensional kernel is achieved. It is proved that there exist eighteen classes of affinely nonequivalent such systems.

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1. INTRODUCTION

The study of a homogeneous quadratic differential system (S) (briefly, HQDS) on a Banach space A can be achieved by using suitable algebraic tools. This is possible because a commutative algebra $A(\cdot)$ is naturally associated with any such a system (S). In general, algebra $A(\cdot)$ is not associative. There exists (see [11], [1]) a 1-to-1 correspondence between the classes of affinely equivalent HQDSs on A and the classes of isomorphic commutative algebras on A. A strong connection between the qualitative properties of HQDSs and the properties - invariant up to an isomorphism - of the corresponding commutative algebras emerges from this correspondence. In particular, any derivation/automorphism of (S) is a derivation/automorphism of its associated algebra $A(\cdot)$ (see [5], [9]). As it is well known (see, for example, [1]), if a derivation of $A(\cdot)$ having only real eigenvalues exists, then its semisimple and nilpotent parts are necessarily derivations, too. The commutative algebras on $\mathbb{R}^3$ having a nilpotent derivation were already classified (see [4], [6]). The problem of classification of 3-dimensional real commutative algebras having at least a semisimple derivation is an intricate task and it is still unsolved.

In this paper, in order to save the space, we shall classify - up to a center-affinity - only a particular class of HQDSs on $\mathbb{R}^3$ which have a semisimple derivation. It is proved that this class of HQDSs is naturally partitioned in eighteen mutually disjoint subclasses consisting of nonequivalent systems, i.e., every system in one subclass is not equivalent to any system in another subclass; moreover, two systems in the same subclass that correspond to different values of parameters are nonequivalent, too.

Our results are based on the remark: *the problem of classification up to an affine equivalence of HQDSs on $\mathbb{R}^n$ is equivalent with the problem of classification up to an isomorphism of all real n-dimensional commutative algebras*. That is why we shall classify, up to an isomorphism, the class of real 3-dimensional commutative algebras that correspond to the analyzed class of HQDSs. The lattices of the subalgebras of these algebras are used as a main tool for solving the isomorphism problems.

In a forthcoming paper we shall classify the set of all HQDSs on $\mathbb{R}^3$ which have a semisimple derivation.

2. REAL COMMUTATIVE ALGEBRAS ON $\mathbb{R}^3$ HAVING A SEMISIMPLE DERIVATION

Let us consider a nontrivial real 3-dimensional commutative algebra $A(\cdot)$. Then, any basis in A allows to identify A with $\mathbb{R}^3$.

Now, let us suppose that $A(\cdot)$ has a nonzero semisimple derivation. Then *Der A* necessarily contains a semisimple derivation D having its spectrum *SpecD* of the form $SpecD = \{1, \lambda, \mu\}$ $(\lambda,\ \mu \in \mathbb{R})$. Accordingly, there exists a basis $\mathcal{B} = (e_1, e_2, e_3)$ such that

$$D(e_1) = e_1, \quad D(e_2) = \lambda e_2, \quad D(e_3) = \mu e_3.$$

Obviously, the case $\lambda = \mu = 1$ is not possible. Since $D(e_1^2) = 2e_1^2$, then either $e_1^2 = 0$ (if $2 \notin SpecD$) or $e_1^2 \neq 0$ (if $2 \in SpecD$). More exactly, it follows

$$\begin{cases} e_1^2 = 0 & if \quad 2 \notin SpecD \\ e_1^2 = \kappa e_2 & if \quad \lambda = 2 \neq \mu \\ e_1^2 = \omega e_3 & if \quad \mu = 2 \neq \lambda \\ e_1^2 = \kappa e_2 + \omega e_3 & if \quad \lambda = \mu = 2. \end{cases}$$

Each of the following equations can be exploited in a similar way:

$$D(e_1e_2) = (1+\lambda)e_1e_2, \ \ D(e_1e_3) = (1+\mu)e_1e_3, \ \ D(e_2^2) = 2\lambda e_2^2,$$
$$D(e_2e_3) = (\lambda+\mu)e_2e_3, \ \ D(e_3^2) = 2\mu e_3^2.$$

In fact, at least one of the elements of the set $\{2,\ 1+\lambda,\ 1+\mu,\ 2\lambda,\ 2\mu,\ \lambda+\mu\}$ belongs to *SpecD*. A simple inspection of papers [4], [6], [10] provides the following cases of the spectrum of D that have to be considered:

$$\{1,0,0\},\ \{1,1,0\},\ \{1,1,2\},\ \{1,2,0\},\ \{1,2,2\},\ \{1,2,3\},\ \{1,2,4\},\{1,-1,0\}.$$

In what follows we deal with a class of algebras which have a semisimple derivation D with $SpecD = \{1,0,0\}$, only. This is the case when $dim_{\mathbb{R}}\ Ker\ D = 2$.

3. CASE *SPEC D* = {1, 0, 0}

There exists a basis $\mathcal{B} = \{e_1, e_2, e_3\}$ such that

$$D(e_1) = e_1, \quad D(e_2) = 0, \quad D(e_3) = 0.$$

Then, from the equations

$$D(e_1^2) = 2e_1^2,\ \ D(e_1e_2) = e_1e_2,\ \ D(e_1e_3) = e_1e_3,$$

$$D(e_2^2) = D(e_2e_3) = D(e_3^2) = 0$$

emerges the following multiplication table of algebra $A(\cdot)$

$$\begin{array}{ll} e_1^2 = 0 & e_2^2 = \alpha e_2 + \beta e_3 \\ e_1e_2 = \kappa e_1 & e_2e_3 = \gamma e_2 + \delta e_3 \\ e_1e_3 = \omega e_1 & e_3^2 = \varepsilon e_2 + \theta e_3. \end{array}$$

It results that $A = Im\ D \oplus Ker\ D$ where $\mathbb{R}e_1$ is an ideal and $Ker\ D$ is a subalgebra of $A(\cdot)$. The algebra $Ker\ D$ can have (see [11]) either a basis of nilpotent elements of index two or a single 1-subspace of such nilpotent elements, only. Among the algebras of the former kind, there exists one (see [11]) which has the following multiplication table:

$$e_2^2 = 0,\ \ e_2e_3 = e_3,\ \ e_3^2 = \gamma e_2 + e_3\ \ (\gamma \neq 0).$$

It suggests us to deal with algebras defined by the following multiplication table:

Table T

$$\begin{array}{ll} e_1^2 = 0 & e_2^2 = 0 \\ e_1e_2 = \alpha e_1 & e_2e_3 = e_3 \\ e_1e_3 = \beta e_1 & e_3^2 = \gamma e_2 + e_3 \end{array}$$

with $\alpha,\ \beta,\ \gamma \in \mathbb{R},\ \gamma \neq 0$.

We shall denote by $A(\alpha,\beta,\gamma)$ any algebra on $\mathbb{R}^3$ with the multiplication table **T** in basis $\mathcal{B}$.

Algebra A(α,β,γ)

Any algebra $A = A(\alpha,\beta,\gamma)$ $(\gamma \neq 0)$ is not simple because it contains the ideal $Im\ D$. Distinct triples of real numbers (α,β,γ) $(\gamma \neq 0)$ determine non-isomorphic algebras. Indeed, the following result is proved by a straightforward checking.

Theorem 3.1. *The algebras $A(\alpha,\beta,\gamma)$ $(\gamma \neq 0)$ and $A(p,q,r)$ $(r \neq 0)$ are isomorphic if and only if $\alpha = p,\ \beta = q,\ \gamma = r$.*

In order to find a partition for class $A(\alpha,\beta,\gamma)$ $(\gamma \neq 0)$ into subclasses of nonisomorphic subalgebras we look for the family of its 1-dimensional subalgebras. To this end we need to find its annihilator elements, nilpotent (of index two) elements and idempotent (of index two) elements.

This problem is solved in the next three propositions.

Proposition 3.1. *The set Ann (A) of all annihilator elements of $A = A(\alpha,\beta,\gamma)$ is:*

$$Ann\ (A) = \begin{cases} \{0\} & if \quad \alpha^2 + \beta^2 \neq 0 \\ Im\ D & if \quad \alpha = \beta = 0. \end{cases}$$

Proposition 3.2. *The set* $\mathcal{N}(A)$ *of all nilpotent elements* (*of index two*) *of* $A = A(\alpha, \beta, \gamma)$ *is:*

$$\mathcal{N}(A) = \begin{cases} \mathbb{R}e_1 \cup \mathbb{R}e_2 & if \quad \alpha \neq 0 \\ Span_{\mathbb{R}}\{e_1,\ e_2\} & if \quad \alpha = 0. \end{cases}$$

Proposition 3.3. *The set of idempotent elements* (*of index two*) *of any algebra* $A(\alpha, \beta, \gamma)$ $(\gamma \neq 0)$ *is:*

1° $\mathcal{I}(A) = \emptyset$ *if* $\gamma < -\frac{1}{8}$,
2° $\mathcal{I}(A) = \{-\frac{1}{2}e_2 + 2e_3\}$ *if* $\gamma = -\frac{1}{8}$ *and* $\alpha - 4\beta + 1 \neq 0$,
3° $\mathcal{I}(A) = \{xe_1 - \frac{1}{2}e_2 + 2e_3 \mid x \in \mathbb{R}\}$ *if* $\gamma = -\frac{1}{8}$ *and* $\alpha - 4\beta + 1 = 0$,
4° $\mathcal{I}(A) = \{y_1e_2 + z_1e_3,\ y_2e_2 + z_2e_3 \mid x \in \mathbb{R}\}$ *if* $\gamma > -\frac{1}{8},\ \gamma \neq 0$,
$2(\alpha y_1 + \beta z_1) - 1 \neq 0$ *and* $2(\alpha y_2 + \beta z_2) - 1 \neq 0$,
5° $\mathcal{I}(A) = \{y_1e_2 + z_1e_3\} \cup \{xe_1 + y_2e_2 + z_2e_3 \mid x \in \mathbb{R}\}$ *if* $\gamma > -\frac{1}{8},\ \gamma \neq 0$,
$2(\alpha y_1 + \beta z_1) - 1 \neq 0$ *and* $2(\alpha y_2 + \beta z_2) - 1 = 0$,
6° $\mathcal{I}(A) = \{xe_1 + y_1e_2 + z_1e_3,\ \mid x \in \mathbb{R}\} \cup \{y_2e_2 + z_2e_3\}$ *if* $\gamma > -\frac{1}{8},\ \gamma \neq 0$,
$2(\alpha y_1 + \beta z_1) - 1 = 0$ *and* $2(\alpha y_2 + \beta z_2) - 1 \neq 0$,
7° $\mathcal{I}(A) = \{xe_1 + y_1e_2 + z_1e_3 \mid x \in \mathbb{R}\} \cup \{xe_1 + y_2e_2 + z_2e_3 \mid x \in \mathbb{R}\}$ *if*
$\gamma > -\frac{1}{8},\ \gamma = 0,\ 2(\alpha y_1 + \beta z_1) - 1 = 0$ *and* $2(\alpha y_2 + \beta z_2) - 1 \neq 0$.

Corollary 3.1. *Every algebra* $A(\alpha, \beta, \gamma)$ $(\gamma < -\frac{1}{8})$ *is not isomorphic to any algebra* $A(p, q, \frac{1}{8})$ *or* $A(p, q, r)$ $(r > -\frac{1}{8}, r \neq 0)$.

Proof. It is enough to notice that any two such classes of algebras have distinct families of idempotent elements. ∎

Properties of algebra A(α, β, γ)

In what follows, we list the properties of these eighteen subclasses of algebras that help us to solve easily the isomorphism problem. Moreover, note that the lattices of subalgebras of each subclass of algebras were obtained.

Properties of algebra A1°=A(α, β, γ) $(\alpha \neq 0,\ \gamma < -\frac{1}{8},\ (\alpha - 1)^2 + (2\beta - 1)^2 \neq 0)$

- $AnnA = \{0\}$
- $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2$
- $\mathcal{I}(A) = \emptyset$
- *Subalgebras:* $\mathbb{R}e_1,\ \mathbb{R}e_2,\ Span_R\{e_1, e_2\},\ Span_R\{e_2, e_3\}$
- *Ideals* : $\mathbb{R}e_1$
- $A^2 = A$
- $Der\ A = \mathbb{R}D$
- $Aut\ (A) = \{T(x) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A2°=A$(1, \frac{1}{2}, \gamma)$ $(\gamma < -\frac{1}{8})$

- $AnnA = \{0\}$
- $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2$
- $\mathcal{J}(A) = \emptyset$
- *Subalgebras*: $\mathbb{R}e_1$, $\mathbb{R}e_2$, $Span_R\{e_2,\ ae_1 + be_3\}$ $(a^2 + b^2 \neq 0)$
- *Ideals* : $\mathbb{R}e_1$
- $A^2 = A$
- $Der\ A = \mathbb{R}D \oplus \mathbb{R}D_1$ where $[D_1] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $[D, D_1] = D_1$
- $Aut\ (A) = \{T(x, y) \mid x, y \in \mathbb{R},\ x \neq 0\}$ where $T(x, y) = \begin{bmatrix} x & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A3°=A$(0, \beta, \gamma)$ $(\beta \neq 0,\ \gamma < -\frac{1}{8})$

- $AnnA = \{0\}$
- $\mathcal{N}(A) = Span_{\mathbb{R}}\{e_1, e_2\}$
- $\mathcal{J}(A) = \emptyset$
- *Subalgebras:* $\mathbb{R}(pe_1 + qe_2)$ $(p^2 + q^2 \neq 0)$, $Span_R\{e_1, e_2\}$, $Span_R\{e_2, e_3\}$
- *Ideals* : $\mathbb{R}e_1$
- $A^2 = A$
- $Der\ A = \mathbb{R}D$
- $Aut\ (A) = \{T(x) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A4°=A$(\alpha, \beta, -\frac{1}{8})$ $(\alpha \neq 0, \alpha - 4\beta + 1 \neq 0)$

- $AnnA = \{0\}$
- $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2$
- $\mathcal{J}(A) = \{-\frac{1}{2}e_2 + 2e_3\}$
- *Subalgebras:* $\mathbb{R}e_1$, $\mathbb{R}e_2$, $\mathbb{R}(-\frac{1}{2}e_2 + 2e_3)$, $Span_R\{e_1, e_2\}$, $S\,pan_R\{e_2, e_3\}$, $Span_R\{e_1, -\frac{1}{2}e_2 + 2e_3\}$
- *Ideals* : $\mathbb{R}e_1$
- $A^2 = A$
- $Der\ A = \mathbb{R}D$
- $Aut\ (A) = \{T(x) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A5°=A$(0,\beta,-\frac{1}{8})$ $(\beta \neq 0,\ \beta \neq \frac{1}{4})$

- $AnnA = \{0\}$
- $\mathcal{N}(A) = Span_{\mathbb{R}}\{e_1,\ e_2\}$
- $\mathcal{J}(A) = \{-\frac{1}{2}e_2 + 2e_3\}$
- *Subalgebras:* $\mathbb{R}e_1$, $\mathbb{R}e_2$, $\mathbb{R}(pe_1+qe_2)$ $(p^2+q^2 \neq 0)$, $\mathbb{R}(-\frac{1}{2}e_2+2e_3)$, $Span_R\{e_1, e_2\}$, $Span_R\{e_2, e_3\}$, $Span_R\{e_1, -\frac{1}{2}e_2 + 2e_3\}$
- *Ideals* : $\mathbb{R}e_1$
- $A^2 = A$
- $Der\ A = \mathbb{R}D$
- $Aut\ (A) = \{T(x) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A6°=A$(4\beta - 1,\beta,-\frac{1}{8})$ $(\beta \notin \{\frac{1}{4},\ \frac{1}{2}\})$

- $AnnA = \{0\}$
- $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2$
- $\mathcal{J}(A) = \{xe_1 - \frac{1}{2}e_2 + 2e_3 \mid x \in \mathbb{R}\}$
- *Subalgebras*: $\mathbb{R}e_1$, $\mathbb{R}e_2$, $\mathbb{R}(xe_1 - \frac{1}{2}e_2 + 2e_3)$ $(x \in \mathbb{R})$, $Span_R\{e_1, e_2\}$, $Span_R\{e_2, e_3\}$, $Span_R\{e_1, -\frac{1}{2}e_2 + 2e_3\}$
- *Ideals* : $\mathbb{R}e_1$
- $A^2 = A$
- $Der\ A = \mathbb{R}D$
- $Aut\ (A) = \{T(x) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A7°=A$(1, \frac{1}{2}, -\frac{1}{8})$

- $AnnA = \{0\}$
- $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2$
- $\mathcal{J}(A) = \{xe_1 - \frac{1}{2}e_2 + 2e_3 \mid x \in \mathbb{R}\}$
- *Subalgebras:* $\mathbb{R}e_1$, $\mathbb{R}e_2$, $\mathbb{R}(xe_1 - \frac{1}{2}e_2 + 2e_3)$ $(x \in \mathbb{R})$, $Span_R\{e_1, e_2\}$, $Span_R\{e_2, e_3\}$, $Span_R\{e_1, -\frac{1}{2}e_2 + 2e_3\}$, $Span_R\{e_2, ae_1 + be_3\}$ $(a, b \in \mathbb{R})\}$, $Span_R\{xe_1 - \frac{1}{2}e_2 + 2e_3, \frac{cx}{2}e_1 + be_2 + ce_3\}$ $(b, c \in \mathbb{R},\ 4b + c \neq 0)$
- *Ideals* : $\mathbb{R}e_1$
- $A^2 = A$
- $Der\ A = \mathbb{R}D \oplus \mathbb{R}D_1$ where $[D_1] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $[D, D_1] = D_1$

• $Aut\ (A) = \{T(x,y) \mid x, y \in \mathbb{R},\ x \neq 0\}$ where $T(x,y) = \begin{bmatrix} x & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A8°=A$(0, \frac{1}{4}, -\frac{1}{8})$

• $AnnA = \{0\}$
• $\mathcal{N}(A) = Span_{\mathbb{R}}\{e_1,\ e_2\}$
• $\mathcal{I}(A) = \{xe_1 - \frac{1}{2}e_2 + 2e_3 \mid x \in \mathbb{R}\}$
• *Subalgebras:* $\mathbb{R}(pe_1 + qe_2)$ $(p^2 + q^2 \neq 0)$, $Span_R\{e_1, e_2\}$, $Span_R\{e_2, e_3\}$, $Span_R\{e_1, -\frac{1}{2}e_2 + 2e_3\}$, $Span_R\{pe_1 + qe_2, e_3\}(p, q \in \mathbb{R}^*)$, $Span_R\{-\frac{1}{2}e_2 + 2e_3, ae_1 + be_2 + ce_3\}$ $(a, b, c \in \mathbb{R},\ 4b + c \neq 0)$, $Span_R\{xe_1 - \frac{1}{2}e_2 + 2e_3, (1 - 2xq)e_1 + qe_2\}$ $(x, q \in \mathbb{R},\ xq \neq 0)$
• *Ideals* : $\mathbb{R}e_1$
• $A^2 = A$
• $Der\ A = \mathbb{R}D \oplus \mathbb{R}D_1$ where $[D_1] = \begin{bmatrix} 0 & 4 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $[D, D_1] = D_1$
• $Aut\ (A) = \{T(x,y) \mid x, y \in \mathbb{R},\ x \neq 0\}$ where $T(x,y) = \begin{bmatrix} x & 4y & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A9°=A(α, β, γ) $(\alpha \neq 0,\ \gamma \neq 0,\ \gamma > -\frac{1}{8},\ 2(\alpha y_1 + \beta z_1) - 1 \neq 0,\ 2(\alpha y_2 + \beta z_2) - 1 \neq 0)$

• $AnnA = \{0\}$
• $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2$
• $\mathcal{I}(A) = \{y_1e_2 + z_1e_3,\ y_2e_2 + z_2e_3\}$
• *Subalgebras*: $\mathbb{R}e_1$, $\mathbb{R}e_2$, $\mathbb{R}(y_1e_2 + z_1e_3)$, $\mathbb{R}(y_2e_2 + z_2e_3)$, $Span_{\mathbb{R}}\{e_1, e_2\}$, $Span_{\mathbb{R}}\{e_2, e_3\}$, $Span_{\mathbb{R}}\{y_1e_2 + z_1e_3,\ e_1\}$, $Span_{\mathbb{R}}\{y_1e_2 + z_1e_3,\ e_2\}$, $Span_{\mathbb{R}}\{y_2e_2 + z_2e_3,\ e_1\}$, $Span_{\mathbb{R}}\{y_2e_2 + z_2e_3,\ e_2\}$, $Span_{\mathbb{R}}\{y_1e_2 + z_1e_3,\ ae_1 + b[\gamma(\alpha - z_1 - \alpha z_1 + 2\beta z_1 + 1)e_2 + 2(\beta + \gamma z_1 + \alpha\gamma z_1)e_3]\}$ $(a, b \in \mathbb{R})$, $Span_{\mathbb{R}}\{y_2e_2 + z_2e_3,\ ae_1 + b[\gamma(\alpha - z_2 - \alpha z_2 + 2\beta z_2 + 1)e_2 + (\beta + \gamma z_2 + \alpha\gamma z_2)e_3]\}$ $(a, b \in \mathbb{R})$
• *Ideals* : $\mathbb{R}e_1$
• $A^2 = A$
• $Der\ A = \mathbb{R}D$
• $Aut\ (A) = \{T(x) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A10°=A$(0, \beta, \gamma)$ $(\gamma \neq 0,\ \gamma > -\frac{1}{8},\ 2\beta z_1 - 1 \neq 0,\ 2\beta z_2 - 1 \neq 0,\ \beta \neq 0)$

• $AnnA = \{0\}$

• $\mathcal{N}(A) = Span_{\mathbb{R}}\{e_1, e_2\}$

• $\mathcal{J}(A) = \{y_1e_2 + z_1e_3,\ y_2e_2 + z_2e_3\}$

• *Subalgebras*: $\mathbb{R}(pe_1 + qe_2)$ $(p^2 + q^2 \neq 0)$, $\mathbb{R}(y_1e_2 + z_1e_3)$, $\mathbb{R}(y_2e_2 + z_2e_3)$, $Span_{\mathbb{R}}\{e_1, e_2\}$, $Span_{\mathbb{R}}\{e_2, e_3\}$, $Span_{\mathbb{R}}\{y_1e_2 + z_1e_3,\ e_1\}$, $Span_{\mathbb{R}}\{y_2e_2 + z_2e_3,\ e_1\}$, $Span_{\mathbb{R}}\{y_1e_2 + z_1e_3,\ ae_1 + b[\gamma(1 - z_1 + 2\beta z_1)e_2 + 2(\beta + \gamma z_1)e_3]\}$ $(a, b \in \mathbb{R},\ \gamma \neq 1)$, $Span_{\mathbb{R}}\{y_2e_2 + z_2e_3,\ ae_1 + b[\gamma(1 - z_2 + 2\beta z_2)e_2 + 2(\beta + \gamma z_2)e_3]\}$ $(a, b \in \mathbb{R},\ \gamma \neq 1)$
There exists other subalgebras if $\alpha,\ \beta,\ \gamma$ are connected by some appropriate equations, namely:
$Span_{\mathbb{R}}\{pe_1 + qe_2, \frac{p}{q}(b+\beta c)e_1 + be_2 + ce_3\}$ *if* $\gamma = 2\beta^2 - \beta,\ \beta \neq \frac{1}{4}$ $(b, c, p, q \in \mathbb{R},\ q \neq 0)$

• *Ideals* : $\mathbb{R}e_1$

• $A^2 = A$

• *Der* $A = \mathbb{R}D$

• *Aut* $(A) = \{T(x) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A11°**=A**(α, β, γ) $(\alpha \neq 0,\ \gamma \neq 0,\ \gamma > -\frac{1}{8},$ $2(\alpha y_1 + \beta z_1) - 1 \neq 0,\ 2(\alpha y_2 + \beta z_2) - 1 = 0)$

• $AnnA = \{0\}$

• $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2$

• $\mathcal{J}(A) = \{y_1e_2 + z_1e_3\} \cup \{xe_1 + y_2e_2 + z_2e_3 \mid x \in \mathbb{R}\}$

• *Subalgebras:* $\mathbb{R}e_1$, $\mathbb{R}e_2$, $\mathbb{R}(y_1e_2+z_1e_3)$, $\mathbb{R}(xe_1+y_2e_2+z_2e_3)$ $(x \in \mathbb{R})$, $Span_{\mathbb{R}}\{e_1, e_2\}$, $Span_{\mathbb{R}}\{e_2, e_3\}$, $Span_{\mathbb{R}}\{y_1e_2 + z_1e_3,\ e_1\}$, $Span_{\mathbb{R}}\{y_2e_2 + z_2e_3,\ e_1\}$,

If $x \neq 0$ a biparametric family of 2-dimensional algebras of type $Span_{\mathbb{R}}\{xe_1 + y_2e_2 + z_2e_3,\ ae_1 + be_2 + ce_3\}$ is obtained; they correspond to the following conditions:

$$\begin{aligned} &4x(1-\alpha)b^2 + 4\gamma x(\beta z_2 - 1)c^2 + 2(z_2 - 2)ab + (4\gamma z_2 + z_2 - 1)ac + \\ &\qquad +4x(\gamma z_2 + \alpha\gamma z_2 - \beta + 1)bc = 0, \\ &-\gamma x c^3 + 2xb^2c + xbc^2 - 2(y_2 - \alpha y_2 + \beta z_2)abc + (-y_2 + 2\beta y_2 + \gamma z_2)ac^2 - \\ &\qquad -2\alpha z_2 ab^2 = 0. \end{aligned}$$

• *Ideals* : $\mathbb{R}e_1$

• $A^2 = A$

• *Der* $A = \mathbb{R}D$

• *Aut* $(A) = \{T(x) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A12° $=$ **A**$(0, \beta, \gamma)$ $(\gamma \neq 0,\ \gamma > -\frac{1}{8},\ 2\beta z_1 - 1 \neq 0,\ 2\beta z_2 - 1 =$ $0,\ i.e.\ \gamma = 2\beta^2 - \beta,\ \beta = -\gamma z_1,\ \beta \neq 0)$

• $AnnA = \{0\}$

• $\mathcal{N}(A) = Span_{\mathbb{R}}\{e_1, e_2\}$

• $\mathcal{J}(A) = \{y_1e_2 + z_1e_3\} \cup \{xe_1 + y_2e_2 + z_2e_3 \mid x \in \mathbb{R}\}$

• *Subalgebras:* $\mathbb{R}(pe_1 + qe_2)$, $\mathbb{R}(y_1e_2 + z_1e_3)$, $\mathbb{R}(xe_1 + y_2e_2 + z_2e_3)$ $(x \in \mathbb{R})$, $Span_{\mathbb{R}}\{e_1, e_2\}$, $Span_{\mathbb{R}}\{e_2, e_3\}$, $Span_{\mathbb{R}}\{y_1e_2 + z_1e_3, e_1\}$, $Span_{\mathbb{R}}\{y_2e_2 + z_2e_3, e_1\}$, $Span\{pe_1 + qe_2, \frac{p}{q}(b + \beta c)e_1 + be_2 + ce_3\}$ $(b, c, p, q \in \mathbb{R},\ pq \neq 0)$, $Span_{\mathbb{R}}\{y_1e_2 + z_1e_3, ae_1 + b[\gamma(2\beta z_1 - z_1 + 1)e_2 + 2(\beta + \gamma z_1)e_3]\}$ $(a, b \in \mathbb{R},\ \beta \neq -1)$, $Span_{\mathbb{R}}\{xe_1 + y_2e_2 + z_2e_3, ae_1 + b(\frac{2\beta-1}{2}e_2 + e_3)\}$ $(x, a, b \in \mathbb{R})$, $Span_{\mathbb{R}}\{xe_1 + y_2e_2 + z_2e_3, ae_1 + \frac{4\beta a - a - 4\beta^2 xb}{4\beta x}e_2 + be_3\}$ $(x, a, b \in \mathbb{R},\ x \neq 0)$

• *Ideals* : $\mathbb{R}e_1$

• $A^2 = A$

• $Der\ A = \mathbb{R}D \oplus \mathbb{R}D_1$ where $[D_1] = \begin{bmatrix} 0 & 1 & -\gamma z_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $[D, D_1] = D_1$

• $Aut\ (A) = \{T(x, y) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & y & -\gamma z_1 y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A13°=A(α, β, γ) $(\alpha \neq 0,\ \gamma \neq 0,\ \gamma > -\frac{1}{8},\ 2(\alpha y_1 + \beta z_1) - 1 = 0\ ,\ 2(\alpha y_1 + \beta z_2) - 1 \neq 0,\ (\alpha - 1)^2 + (2\beta - 1)^2 \neq 0,\ i.e.\ 2\gamma(1-\alpha)^2 + (2\beta - 1)(\alpha - 2\beta) = 0)$

• $AnnA = \{0\}$

• $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2$

• $\mathcal{J}(A) = \{xe_1 + y_1e_2 + z_1e_3 \mid x \in \mathbb{R}\} \cup \{y_2e_2 + z_2e_3\}$

• *Subalgebras:* $\mathbb{R}e_1$, $\mathbb{R}e_2$, $\mathbb{R}(y_1e_2 + z_1e_3)$, $\mathbb{R}(xe_1 + y_2e_2 + z_2e_3)$ $(x \in \mathbb{R})$, $Span_{\mathbb{R}}\{e_1, e_2\}$, $Span_{\mathbb{R}}\{e_2, e_3\}$, $Span_{\mathbb{R}}\{y_1e_2 + z_1e_3,\ e_1\}$, $Span_{\mathbb{R}}\{y_2e_2 + z_2e_3,\ e_1\}$,

If $x \neq 0$ a 2-parametric family of 2-dimensional algebras of type $Span_{\mathbb{R}}\{xe_1 + y_1e_2 + z_1e_3,\ ae_1 + be_2 + ce_3\}$ is obtained; they correspond to the following conditions:

$$\begin{aligned} &4x(1-\alpha)b^2 + 4\gamma x(\beta z_1 - 1)c^2 + 2(z_1 - 2)ab + (4\gamma z_1 + z_1 - 1)ac + \\ &\qquad +4x(\gamma z_1 + \alpha\gamma z_2 - \beta + 1)bc = 0, \\ &-\gamma xc^2 + 2xb^2c + xbc^2 - 2(y_1 - \alpha y_1 + \beta z_1)abc + (-y_1 + 2\beta y_1 + \gamma z_1)ac^2 - \\ &\qquad -2\alpha z_1 ab^2 = 0. \end{aligned}$$

• *Ideals* : $\mathbb{R}e_1$

• $A^2 = A$

• $Der\ A = \mathbb{R}D$

• $Aut\ (A) = \{T(x) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A14°=A$(1, \frac{1}{2}, \gamma)$ $(\gamma \neq 0,\ \gamma > -\frac{1}{8})$

• $AnnA = \{0\}$

• $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2$

• $\mathcal{J}(A) = \{xe_1 + y_1e_2 + z_1e_3 \mid x \in \mathbb{R}\} \cup \{y_2e_2 + z_2e_3\}$

• *Subalgebras:* $\mathbb{R}e_1$, $\mathbb{R}e_2$, $\mathbb{R}(xe_1+y_1e_2+z_1e_3)$ $(x \in \mathbb{R})$, $\mathbb{R}(y_2e_2+z_2e_3)$, $Span_{\mathbb{R}}\{e_1, e_2\}$, $Span_{\mathbb{R}}\{e_2, e_3\}$, $Span_{\mathbb{R}}\{e_1, y_1e_2 + z_1e_3\}$, $Span_{\mathbb{R}}\{e_1, y_2e_2 + z_2e_3\}$, $Span_{\mathbb{R}}\{e_2, ae_1 + be_3\}$ $(a, b \in \mathbb{R})$, $Span_{\mathbb{R}}\{xe_1 + y_1e_2 + z_1e_3, -4xe_1 + z_1(e_2 - 4e_3)\}$, $Span_{\mathbb{R}}\{y_2e_2 + z_2e_3, ae_1 + b(y_1e_2 + z_1e_3)$ $(a, b \in \mathbb{R}^*)$

• *Ideals* : $\mathbb{R}e_1$

• $A^2 = A$

• *Der* $A = \mathbb{R}D \oplus \mathbb{R}D_1$ where $[D_1] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $[D, D_1] = D_1$

• *Aut* $(A) = \{T(x, y) \mid x, y \in \mathbb{R},\ x \neq 0\}$ where $T(x, y) = \begin{bmatrix} x & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A15°=A$(0, \beta, \gamma)$ $(\gamma \neq 0,\ \gamma > -\frac{1}{8},\ 2\beta z_1 - 1 = 0,\ 2\beta z_2 - 1 \neq 0,\ i.e.\ \gamma = 2\beta^2 - \beta,\ \beta = -\gamma z_2)$

• $AnnA = \{0\}$

• $\mathcal{N}(A) = S\,pan_{\mathbb{R}}\{e_1, e_2\}$

• $\mathcal{J}(A) = \{xe_1 + y_1e_2 + z_1e_3 \mid x \in \mathbb{R}\} \cup \{y_2e_2 + z_2e_3\}$

• *Subalgebras:* $\mathbb{R}(pe_1 + qe_2)$ $(p, q \in \mathbb{R})$, $\mathbb{R}(xe_1 + y_1e_2 + z_1e_3)$ $(x \in \mathbb{R})$, $\mathbb{R}(y_2e_2 + z_2e_3)$, $Span_{\mathbb{R}}\{e_1, e_2\}$, $Span_{\mathbb{R}}\{e_2, e_3\}$, $Span_{\mathbb{R}}\{y_1e_2 + z_1e_3, e_1\}$, $Span_{\mathbb{R}}\{y_2e_2+z_2e_3, e_1\}$, $Span\{pe_1+qe_2, \frac{p}{q}(b+\beta c)e_1+be_2+ce_3\}$ $(b, c, p, q \in \mathbb{R},\ pq \neq 0)$, $Span_{\mathbb{R}}\{y_2e_2 + z_2e_3, ae_1 + b[\gamma(2\beta z_2 - z_2 + 1)e_2 + 2(\beta + \gamma z_2)e_3]\}$ $(a, b \in \mathbb{R},\ \beta \neq -1)$, $Span_{\mathbb{R}}\{xe_1 + y_2e_2 + z_2e_3, ae_1 + b(\frac{2\beta-1}{2}e_2 + e_3)\}$ $(x, a, b \in \mathbb{R})$, $Span_{\mathbb{R}}\{xe_1 + y_1e_2 + z_1e_3, ae_1 + \dfrac{4\beta a - a - 4\beta^2 xb}{4\beta x}e_2 + be_3\}$ $(x, a, b \in \mathbb{R},\ x \neq 0)$

• *Ideals* : $\mathbb{R}e_1$

• $A^2 = A$

• *Der* $A = \mathbb{R}D \oplus \mathbb{R}D_1$ where $[D_1] = \begin{bmatrix} 0 & 1 & -\gamma z_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $[D, D_1] = D_1$

• *Aut* $(A) = \{T(x, y) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & y & -\gamma z_2 y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A16°=A$(0, 0, \gamma)$ $(\gamma < -\frac{1}{8})$

• $AnnA = \mathbb{R}e_1$

• $\mathcal{N}(A) = S\,pan_{\mathbb{R}}\{e_1, e_2\}$

• $\mathcal{J}(A) = \emptyset$

• *Subalgebras:* $\mathbb{R}(pe_1 + qe_2)$ $(p^2 + q^2 \neq 0)$, $Span_{\mathbb{R}}\{e_1, e_2\}$, $Span_{\mathbb{R}}\{e_2, e_3\}$

• *Ideals* : $\mathbb{R}e_1$, $Span_{\mathbb{R}}\{e_1, e_2\}$

• $A^2 = A$

• $Der\ A = \mathbb{R}D$

• $Aut\ (A) = \{T(x) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A17°=A$(0, 0, -\frac{1}{8})$

• $AnnA = \mathbb{R}e_1$

• $\mathcal{N}(A) = Span_{\mathbb{R}}\{e_1, e_2\}$

• $\mathcal{I}(A) = \{-\frac{1}{2}e_2 + 2e_3\}$

• *Subalgebras:* $\mathbb{R}(pe_1 + qe_2)\ (p^2 + q^2 \neq 0)$, $\mathbb{R}(-\frac{1}{2}e_2 + 2e_3)$, $Span_{\mathbb{R}}\{e_1, e_2\}$, $Span_{\mathbb{R}}\{e_2, e_3\}$, $Span_{\mathbb{R}}\{e_1, -\frac{1}{2}e_2 + 2e_3\}$, $Span_{\mathbb{R}}\{e_2, -\frac{1}{2}e_2 + 2e_3\}$

• $Ideals : \mathbb{R}e_1$, $Span_{\mathbb{R}}\{e_1, e_2\}$

• $A^2 = Span_{\mathbb{R}}\{-\frac{1}{8}e_2 + e_3, e_3\}$

• $Der\ A = \mathbb{R}D$

• $Aut\ (A) = \{T(x) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Properties of algebra A18°=A$(0, 0, \gamma)\ (\gamma \neq 0,\ \gamma > -\frac{1}{8})$

• $AnnA = \mathbb{R}e_1$

• $\mathcal{N}(A) = Span_{\mathbb{R}}\{e_1, e_2\}$

• $\mathcal{I}(A) = \{y_1e_2 + z_1e_3\} \cup \{y_2e_2 + z_2e_3\}$

• *Subalgebras:* $\mathbb{R}(pe_1 + qe_2)\ (p^2 + q^2 \neq 0)$, $\mathbb{R}(y_1e_2 + z_1e_3)$, $\mathbb{R}(y_2e_2 + z_2e_3)$, $Span_{\mathbb{R}}\{e_1, e_2\}$, $Span_{\mathbb{R}}\{e_2, e_3\}$, $Span_{\mathbb{R}}\{e_1, y_1e_2 + z_1e_3\}$, $Span_{\mathbb{R}}\{e_1, y_2e_2 + z_2e_3\}$

• $Ideals : \mathbb{R}e_1$, $Span_{\mathbb{R}}\{e_1, e_2\}$

• $A^2 = Span_{\mathbb{R}}\{\gamma e_2 + e_3, e_3\}$

• $Der\ A = \mathbb{R}D$

• $Aut\ (A) = \{T(x) \mid x \in \mathbb{R}^*\}$ where $T(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

These lists of properties of classes $Ai°$ allow us to assert the following result.

Theorem 3.2. *Every algebra of class $Ai°$ is not isomorphic to any algebra of class $Aj°$ $(i \neq j,\ i, j \in \{1, 2, ..., 18\})$.*

Accordingly, the class of homogeneous quadratic differential systems

$$(S)\ \begin{cases} x' = 2\alpha xy + 2\beta xz \\ y' = \gamma z^2 \\ z' = 2yz + z^2 \end{cases}$$

is naturally divided into eighteen classes of affinely nonequivalent systems. The last two equations of (S) supply the prime integral $F(y,z)$, namely

$$F(y,z)=\begin{cases} ln\,\dfrac{2y^2+yz-2\gamma z^2}{y^4}+\dfrac{1}{\gamma\sqrt{|8\gamma+1|}}arctg\,\dfrac{2\gamma z-y}{2\gamma y\sqrt{|8\gamma+1|}} & if\quad \gamma<-\frac{1}{8}\\ ln\,\dfrac{(4y+z)^2}{4y^4}-\dfrac{8y}{z+4y} & if\quad \gamma=-\frac{1}{8}\\ \left(\dfrac{|2y^2+yz-2\gamma z^2|}{y^4}\right)^{\frac{1}{2\sqrt{8\gamma+1}}}\dfrac{2\gamma z-(1+\sqrt{8\gamma+1})y}{2\gamma z-(1-\sqrt{8\gamma+1})y} & if\quad \gamma>-\frac{1}{8}. \end{cases}$$

Consequently, any trajectory of (S) lies on a cylinder $F(y,z)=const.$

4. THE LATTICE OF SUBALGEBRAS OF $\mathbf{A(\alpha,\beta,\gamma)}$

It is well known that isomorphic algebras have necessarily isomorphic lattices of their subalgebras. Consequently, the lattices of all subalgebras of two algebras become a natural tool in proving whether these algebras are or are not isomorphic.

Moreover, the lattice of subalgebras of any algebra A delivers a partition of its vector space in nonempty mutually disjoint cells. In its turn, this partition provides a partition of the set of all solutions of the associated homogeneous quadratic differential system. In the previous section we have obtained the lattice of all subalgebras of any algebra $A(\alpha,\beta,\gamma)$, even if, this finding is a customary task. For particular classes of algebras this problem can be easily solved, namely for algebras A with $Der\ A=\mathbb{R}D$ (i.e. algebras with fewer symmetries). For example, the lattice of subalgebras for algebra $A1^\circ$ is

$$\{\{0\},\ \mathbb{R}e_1,\ \mathbb{R}e_2,\ Span_R\{e_1,e_2\},\ Span_R\{e_2,e_3\},\ A\}.$$

These subalgebras define a partition of $\mathbb{R}^3$ by means of their ground subspaces Ox, Oy, xOy and yOz. Accordingly, any solution of the corresponding HQDS (S) has its orbit in one of the cells bounded by these subalgebras or it is contained in one of these subalgebras. Thus, some orbits lay either on the axis Ox or on Oy, other orbits are contained in the plane xOy or yOz (but have no intersection with Ox and Oy) and the other ones are contained inside of one of the four quadrants defined in $\mathbb{R}^3$ by planes xOy and yOz.

The lattice of subalgebras for algebra $A2^\circ$ is

$$\{\{0\},\ \mathbb{R}e_1,\ \mathbb{R}e_2,\ Span_R\{e_2,ae_1+be_3\}\ (a^2+b^2\neq 0),\ A\}.$$

A comparison of the two lattices allows us to decide that $A1^\circ$ and $A2^\circ$ are not isomorphic. On the other hand, each solution of the corresponding HQDS (S) is contained in a subalgebra, i.e. it is necessarily a null torsion curve. Indeed, a straightforward computation confirms us that the triple scalar product $(\mathbf{r'},\mathbf{r''},\mathbf{r'''})=0$ where $\mathbf{r'}=(2xy+xz)e_1+(\gamma z^2)e_2+(2yz+z^2)e_3$. Moreover, each trajectory of a HQDS associated to one of the algebras $A16^\circ$, $A17^\circ$, $A18^\circ$ has null torsion, too.

5. CONCLUSIONS

The analyzed class of commutative algebras is divided into eighteen classes of non-isomorphic algebras. Accordingly, the corresponding class of homogeneous quadratic differential systems is partitioned into eighteen classes of affinely nonequivalent systems. Since A is a vector direct sum of an ideal and a subalgebra, each such a system is decoupled into subsystems that can be solved.

The most part of classes of algebras have $Der\ A = \mathbb{R}D$, so that the corresponding orbits of solutions have just a few symmetries. The algebras in the classes $A2^\circ$, $A7^\circ$, $A8^\circ$, $A12^\circ$, $A14^\circ$, $A15^\circ$ are the only algebras that have two-dimensional derivation algebras.

References

[1] I. Burdujan, *Quadratic differential systems*, Publ. House PIM, 2008.(Romanian)

[2] I. Burdujan, *A classification of a class of homogeneous quadratic dynamical systems on $\mathbb{R}^3$ with derivations*, Bull. I.P. Iaşi, Sect. Matematica, Mecanica teoretica, Fizica, **T.LIV** (2008), 37–47.

[3] I. Burdujan, *Homogeneous quadratic dynamical systems on $\mathbb{R}^3$ having derivations with complex eigenvalues*, Libertas Mathematica, **XXVIII** (2008), 69–92.

[4] I. Burdujan, *Classification of Quadratic Differential Systems on $\mathbb{R}^3$ having a nilpotent of Order 3 Derivation*, Libertas Mathematica, **XXIX** (2009), 47-64.

[5] I. Burdujan, *Automorphisms and derivations of homogeneous quadratic differential systems*, Romai J., **6**, (12010), 15-28.

[6] I. Burdujan, *Classification of Quadratic Differential Systems on $\mathbb{R}^3$ having a nilpotent of Order 2 Derivation* (in press).

[7] T. Date, *Classifications and Analysis of Two-dimensional Real Homogeneous Quadratic Differential Equation Systems*, J. Diff. Eqs., **32**(1979), 311–334.

[8] I. Kaplansky, *Algebras with many derivations*, "Aspects of Mathematics and its Applications" (ed. J.A. Barroso), Elsevier Science Publishers B.V., 1986, 431.

[9] K. M. Kinyon , A. A. Sagle, *Quadratic Dynamical Systems and Algebras*, J. of Diff. Eqs, **117** (1995), 67–127 .

[10] K. M. Kinyon , A. A. Sagle, *Automorphisms ans Derivations of Differential Equations and Algebras*, Rocky Mountain J. of Mathematics, **24**, 1,(1994), 135–153.

[11] L. Markus, *Quadratic Differential Equations and Non-associative Algebras*, in "Contributions to the Theory of Nonlinear Oscillations" Annals of Mathematics Studies, no. 45, Princeton University Press, Princeton, N. Y., 1960.

[12] A.A. Sagle, R. Walde, *Introduction to Lie groups and Lie algebras*, Academic Press, New York, 1973.

[13] R. D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, New York, London, 1966.

[14] N. I. Vulpe, K. S. Sibirskiĭ, *Geometrical Classification of Quadratic differential systems* (Russian), Differentialnye Uravnenje, **13**, 5(1977), 803–814.