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# ANALYSIS AND NUMERICAL APPROACH TO UNIDIMENSIONAL ELASTO-PLASTIC PROBLEM WITH MIXED HARDENING

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**Abstract** In this paper is determined by the finite element method combined with the Euler method and Newton method, the solution of the problem of elasto-plastic deformation of a one-dimensional bar using weak formulation of the problem with initial and boundary data.

**Keywords:** elasto-plastic model, constitutive equation, weak solution, finite element method, numerical simulation.
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## 1. INTRODUCTION

The constitutive framework which describes the set of the constitutive and evolution equation in the mechanical problem can be found in the references [1], [5], and [11]. The initial and boundary value problem, which describes the dynamic behaviour of one-dimensional elasto-plastic material with isotropic and mixed hardening has been solved by J.C. Simo and T. J. Hughes in [10]. In order to solve the one-dimensional problem, the finite element method is applied together with the Newton method and the Return Mapping Algorithm. From the continuum model which describes the constitutive equation in the elasto-plastic model, through an elastic type constitutive equation together with the evolution equation for plastic strain and internal variables, the discrete algorithmic equations are obtained in [12] by applying an Euler type difference scheme. The Return Mapping Algorithm proposed in [12] involve the computation of the elastic trial stress together with the test for plastic loading. To solve the one-dimensional, initial and boundary value problem, we start from the quasi-static rate boundary value problem formulated at a generic moment of time and associated with the equilibrium equation. In order to determine the weak solution of the rate boundary value problem, which represents the time derivative of the displacement field $u$, i.e. the velocity at time $t$ in the one-dimensional body, we apply the finite element method, those main results are resumed from [2], [8] and [11]. The method proposed here works for loading elsto-plastic process only. To update the current values of $\sigma, \varepsilon^p, \alpha, k$, of the unknowns the Euler method and Newton method are applied in order to integrate the differential equation system. The numerical algorithms combine the finite element method with the numerical methods related to the differential type equations, i.e. the Euler and Newton method, and the coupled problems are simultaneously solved.

## 2. THE UNIDIMENSIONAL MATHEMATICAL MODEL OF ELASTO-PLASTIC PROBLEM WITH MIXED HARDENING

Let $x \in \Omega$ be a material point of the body $\Omega \subset R$. $\Omega$ is an one dimensional body, which deforms in time, say on the interval $I = [0, T)$. We denote by $u : \Omega \times I \to R$ - the *displacement field*, by $\sigma : \Omega \times I \to R$ - the Cauchy *stress field* and with $\varepsilon : \Omega \times I \to R$, with $\varepsilon(x, t) = \dfrac{\partial u}{\partial x}(x, t) \equiv \varepsilon(u)(x, t)$ -the strain. We introduce also the *hardening variables*: $\alpha : \Omega \times I \to R$ the *kinematic hardening variable* and $k : \Omega \times I \to R$ the *isotropic hardening variable*, with $k > 0$, which play different roles in describing the deformability of the surface of the plasticity in the stress space, during the irreversible deformation process. Following Cleja-Tigoiu and Cristescu [1985] (see also Chabauche [1989], Paraschiv-Munteanu and Cleja-Tigoiu [2004], we introduce the basic assumptions within the constitutive framework of the elasto-plastic model with mixed hardening. [see 2004]:

1. the rate of the strain tensor $\dot{\varepsilon}$ can be decomposed into the rate of elastic part and that of plastic part, denoted by $\dot{\varepsilon}^e$ and $\dot{\varepsilon}^p$, respectively

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p, \tag{1}$$

2. the elastic type constitutive equation is given by

$$\dot{\sigma} = E\dot{\varepsilon}^e, \tag{2}$$

3. the irreversible properties of the material are described in terms of the yield function $\mathcal{F}(\sigma, \alpha, k)$, which dependent on the stress and hardening variables; the rate of the plastic strain tensor is described by the associated flow rule through

$$\dot{\varepsilon}^p = \lambda \frac{\partial \mathcal{F}(\sigma, \alpha, k)}{\partial \sigma}, \tag{3}$$

where $\lambda$, the so-called *plastic factor* is a function of the state of the material, and it is defined through the Kuhn-Tucker and consistency conditions

$$\lambda \geq 0, \quad \mathcal{F} \leq 0, \quad \lambda \mathcal{F} = 0 \tag{4}$$

$$\lambda \dot{\mathcal{F}} = 0. \tag{5}$$

4. the variation of the kinematic hardening variable $\alpha$ is given by Armstrong and Frederik [1996] law

$$\dot{\alpha} = C\dot{\varepsilon}^p - \gamma \alpha \dot{k}, \tag{6}$$

C and $\gamma$ are material parameters, while the isotropic hardening variable $k$ is given by

$$\dot{k} = \sqrt{\frac{2}{3}\dot{\varepsilon}^p \cdot \dot{\varepsilon}^p} \tag{7}$$

5. we add the initial condition

$$\sigma(0) = 0,\ \varepsilon(0) = 0,\ \varepsilon^p(0) = 0,\ \alpha(0) = 0,\ k(0) = 0 \tag{8}$$

which correspond to a physical body undeformed and unstressed at time $t = 0$.

In the considered one-dimensional model the *yield function* is given by

$$\mathcal{F}(\sigma, \alpha, k) := |\sigma - \alpha| - F(k) \tag{9}$$

where F is a strictly increasing function, defined through

$$F(k) = R(k) + \kappa, \quad R(0) = 0, \quad R'(k) > 0, \quad k > 0 \tag{10}$$

$\kappa = \sigma_Y$ represents the initial yield condition in the uniaxial test. The isotropic changing in the dimension of the yield surface is represented by Chaboche [1989]

$$R(k) = Q(1 - e^{-bk}), \quad Q, \quad b > 0 \tag{11}$$

Q and b are material constants.

Taking into account the expression (10), the yield function is derived under the form

$$\mathcal{F}(\sigma, \alpha, k) = |\sigma - \alpha| - \left[Q(1 - e^{-bk}) + \sigma_Y\right] \tag{12}$$

The *plastic factor* is calculated from (5)

$$\lambda = \frac{\langle\beta\rangle}{h}\mathcal{H}(\mathcal{F}) \tag{13}$$

where the *complementary plastic factor* $\beta$ and the *hardening variable* h are defined through

$$\beta = \frac{E\varepsilon(\dot{u})(\sigma - \alpha)}{Q(1 - e^{-bk}) + \sigma_Y} \tag{14}$$

$$h = E + C - \sqrt{\frac{2}{3}}\gamma\alpha\frac{\sigma - \alpha}{Q(1 - e^{-bk}) + \sigma_Y} + \sqrt{\frac{2}{3}}Qbe^{-bk} \tag{15}$$

We introduce the supposition that $h > 0$ on $\mathcal{F}(\sigma, \alpha, k) = 0$. If $h$ becomes zero in the process, then the solution cannot be defined since $\dot{\varepsilon}^p \to \infty$. We can say that the material is damaged.

In relation (14)

$$\varepsilon(\dot{u}(x, t)) = \frac{\partial \dot{u}(x, t)}{\partial x} \equiv \frac{d\dot{u}(x, t)}{dx}, \tag{15}$$

with the last notation introduced for the sake of simplicity.

The Heaviside function, from the relationship (13) is

$$\mathcal{H}(\mathcal{F}) = \begin{cases} 0, & \text{if } \mathcal{F} < 0 \\ 1, & \text{if } \mathcal{F} \geq 0 \end{cases} \tag{17}$$

By eliminating the elastic part of the strain we arrive at the following result.

**Proposition 2.1.** *The one-dimensional elasto-plastic model with mixed hardening is described by the following differential system*

$$\begin{aligned} \dot{\varepsilon}^p &= \frac{\langle\beta\rangle}{h} \frac{\sigma - \alpha}{Q(1 - e^{-bk}) + \sigma_Y} \mathcal{H}(\mathcal{F}) \\ \dot{\alpha} &= \frac{\langle\beta\rangle}{h} \left( C \frac{\sigma - \alpha}{Q(1 - e^{-bk}) + \sigma_Y} - \sqrt{\frac{2}{3}} \gamma\alpha \right) \mathcal{H}(\mathcal{F}) \\ \dot{\sigma} &= E\varepsilon(\dot{u}) - E \frac{\langle\beta\rangle}{h} \frac{\sigma - \alpha}{Q(1 - e^{-bk}) + \sigma_Y} \mathcal{H}(\mathcal{F}) \\ \dot{k} &= \sqrt{\tfrac{2}{3}} \frac{\langle\beta\rangle}{h} \mathcal{H}(\mathcal{F}) \end{aligned} \tag{17}$$

*where $\beta$ as a function on $\dot{u}$ and $h$ are given by the relations (14) together with (16).*

The equilibrium equation in terms of one-dimensional Cauchy stress $\sigma(x,t)$ has the form:

$$div\, \sigma(x,t) + b_1(x,t) = 0 \quad \text{in} \quad \Omega \times I, \quad \text{with} \quad div\, \sigma := \frac{\partial \sigma}{\partial x} \tag{19}$$

where $b_1 \in R$ is the body force.

The boundary conditions are formulated on the boundary $\partial\Omega = \Gamma$ of the body which is divided into two parts $\Gamma_u$ and $\Gamma_\sigma$, such that $\Gamma_u \cup \Gamma_\sigma = \Gamma$ and $\Gamma_u \cap \Gamma_\sigma = \emptyset$. They are given by

$$\sigma_{11}(x,t)n = f_1(x,t) \quad \text{in} \quad \Gamma_\sigma \times I \quad ; \quad u(x,t) = g_1(x,t) \quad \text{in} \quad \Gamma_u \times I. \tag{19}$$

Here $n$ is the outward unit normal field on $\partial\Omega$ and $n = 1$ in the one-dimensional case. The functions $f_1(x,t)$ and $g_1(x,t)$ are given.

**Problem P**: Given the functions $b_1(x,t)$, $f_1(x,t)$ and $g_1(x,t)$, find the real valued functions $u, \sigma, \varepsilon, \varepsilon^p, \alpha, k$ that are defined on $\Omega \times [0,T)$ and satisfy (19) together with (20) and the differential type constitutive equations, listed in (18) together with (14) and (16).

To solve it we start from the variational formulation of the equilibrium problem for elasto-plastic one-dimensional bar.

## 3. THE VARIATIONAL FORMULATION

In what follows, we recall the main ideas from Johnson [1987] and Hughes [1987], which are appropriate to our description. Unlike Johnson and Hughes, which used a local momentum equation of the form $\frac{\partial}{\partial x}\sigma + \rho b = \rho \frac{\partial v}{\partial t}$ in $\Omega \times I$ where $v = \frac{\partial u}{\partial t}$, in this paper the equilibrium equation is given by relation (19).

We determine the weak solutions of the rate quasi-static boundary value problem associated at a generic moment $t$, which is derived by taking the time derivative of the equilibrium equation (19) together with the boundary condition (20)

$$\begin{aligned} & div\left[\frac{\partial \sigma(x,t)}{\partial t}\right] + \frac{\partial b_1(x,t)}{\partial t} = 0 \quad \text{in} \quad \Omega \times I \\ & \frac{\partial \sigma(x,t)}{\partial t} = \frac{\partial f_1(x,t)}{\partial t} \quad \text{in} \quad \Gamma_\sigma \times I \\ & \frac{\partial u(x,t)}{\partial t} = \frac{\partial g_1(x,t)}{\partial t} \quad \text{in} \quad \Gamma_u \times I \end{aligned} \tag{20}$$

For the problem P, at a generic stage of the process the current values, i.e. at the time t, the current plastic domain is the form:

$$\Omega_t^p = \{x \in \Omega | \mathcal{F}(\sigma(x,t), \alpha(x,t), k(x,t)) = 0\}. \tag{22}$$

The set of kinematically admissible velocity field is denoted by:

$$\mathcal{V}_{ad} = \left\{ w\, |w : \Omega \to R\,;\, \frac{\partial w}{\partial x} \in L^2(\Omega), \quad w/_{\Gamma_u} = \dot{g}_1 \right\} \subset H^1(\Omega) \tag{23}$$

where

$$L^2(\Omega) = \left\{ w \,\middle|\, w : \Omega \to R; \int_\Omega w^2 dx = \|w\|^2_{L^2(\Omega)} < \infty \right\} \tag{24}$$

is the space of the square integrable functions on $\Omega$, and

$$H^1(\Omega) = \left\{ w \in L^2(\Omega) \,\middle|\, \frac{\partial w}{\partial x} \in L^2(\Omega) \right\} \tag{25}$$

is the Sobolev space.

**Theorem 3.1.** *At every time t the rate of the displacement field, $\dot{u}$, satisfy the following relationships:*

$$\begin{aligned} & \int_\Omega E^{ep}(x,t) \frac{d\dot{u}(x,t)}{dx} \frac{dw(x,t)}{dx} dx = \int_\Omega \frac{db_1(x,t)}{dt} w(x,t) dx + \\ & + \left[ \frac{df_1(x,t)}{dt} w(x,t) \right] \Bigg|_{\Gamma_\sigma} + \left[ \frac{d\sigma(x,t)}{dt} \frac{dg_1(x,t)}{dt} \right] \Bigg|_{\Gamma_u} \end{aligned} \tag{25}$$

*which hold for every admissible vector field* $w \in \mathcal{V}_{ad}$.

*Proof.* By multiplying the first equation of equation (21) with an admissible displacement w, integrating on a domain Ω and applying Green's formula we get

$$\int_{\Omega} \frac{d\sigma(x,t)}{dt}\frac{dw(x,t)}{dx}dx = \left[\frac{d\sigma(x,t)}{dt}w(x,t)\right]\Big|_{\Gamma} + \int_{\Omega} \frac{db_1(x,t)}{dt}w(x,t)dx. \tag{27}$$

Further, we perform some transformation in the left hand side of the relation (27) by introducing the third relation from (18) and we have

$$\begin{aligned} &\int_{\Omega} E\frac{d\dot{u}(x,t)}{dx}\frac{dw(x,t)}{dx}dx - \int_{\Omega} E\lambda\frac{\sigma(x,t)-\alpha(x,t)}{Q(1-e^{-bk(x,t)})+\sigma_Y}\frac{dw(x,t)}{dx}\mathcal{H}\mathcal{F}dx = \\ &= \int_{\Omega}\frac{db_1(x,t)}{dt}w(x,t)dx + \left[\frac{df_1(x,t)}{dt}w(x,t)\right]\Big|_{\Gamma_\sigma} + \left[\frac{d\sigma(x,t)}{dt}\frac{dg_1(x,t)}{dt}\right]\Big|_{\Gamma_u} \end{aligned} \tag{28}$$

where the rate of strain $\varepsilon(\dot{u}(x,t))$ is replaced from (16).

The expression of $\beta$ and h are replaced by the relations (14) and (16), but the positive part of the expression of $\beta$ enters (28).

We remark that only under the assumption that

$$E\varepsilon(\dot{u}(x,t))(\sigma-\alpha) > 0 \tag{29}$$

along the process, i.e. when *no unloading is produced*, (28) becomes

$$\begin{aligned} &\int_{\Omega}\left[E\left(1-\frac{E}{h(x,t)}\left(\frac{(\sigma(x,t)-\alpha(x,t))}{Q(1-e^{-bk(x,t)})+\sigma_Y}\right)^2\mathcal{H}(\mathcal{F})\right)\right]\frac{d\dot{u}(x,t)}{dx}\frac{dw(x,t)}{dx}dx = \\ &\int_{\Omega}\frac{db_1(x,t)}{dt}w(x,t)dx + \left[\frac{df_1(x,t)}{dt}w(x,t)\right]\Big|_{\Gamma_\sigma} + \left[\frac{d\sigma(x,t)}{dt}\frac{dg_1(x,t)}{dt}\right]\Big|_{\Gamma_u}. \end{aligned} \tag{29}$$

In (30) following notation has been introduced, but only under the hypothesis formulated in (29),

$$E^{ep}(x,t) = \begin{cases} E & \text{if } \mathcal{H}(\mathcal{F}) = 0, \\ E\left(1-\frac{E}{h(x,t)}\left(\frac{\sigma(x,t)-\alpha(x,t)}{Q(1-e^{-bk(x,t)})+\sigma_Y}\right)^2\right) & \text{if } \mathcal{H}(\mathcal{F}) = 1, \end{cases} \tag{31}$$

where $E^{ep}$ is the so-called elasto-plastic modulus.

We introduced the supposition that $h > 0$ on $\mathcal{F}(\sigma,\alpha,k) = 0$. Consequently the equality (30) together with (31) becomes:

$$\begin{aligned} &\int_{\Omega} E^{ep}(x,t)\frac{d\dot{u}(x,t)}{dx}\frac{dw(x,t)}{dx}dx = \int_{\Omega}\frac{db_1(x,t)}{dt}w(x,t)dx + \\ &+ \left[\frac{df_1(x,t)}{dt}w(x,t)\right]\Big|_{\Gamma_\sigma} + \left[\frac{d\sigma(x,t)}{dt}\frac{dg_1(x,t)}{dt}\right]\Big|_{\Gamma_u}. \end{aligned} \tag{32}$$

The relation (32) is in fact the variational representation of the solution. ∎

**Remark.** In case when the displacement $u(x,t) = 0$ on the boundary $\Gamma_u$, i.e. $g_1(x,t) = 0$, the variational formulation given above becomes:

$$\int\limits_{\Omega} E^{ep}(x,t)\,\frac{d\dot{u}(x,t)}{dx}\frac{dw(x,t)}{dx}dx = \int\limits_{\Omega} \dot{b}_1(x,t)w(x,t)dx + \left[\dot{f}_1(x,t)w(x,t)\right]\Big|_{\Gamma_\sigma}, \tag{33}$$

written for all $w \in \mathcal{V}_{ad}$, which are vanishing on $\Gamma_u$.

As a consequence of the above theorem, the following statement holds.

**Theorem 3.2.** *Find a displacement field $u(\cdot,t)$, solution of the variational formulation*

$$a(\dot{u},w) = \langle L,w\rangle \quad \forall u \in V_{ad}, w \in V_{ad} \tag{34}$$

*where $a(\cdot,\cdot) : V_{ad} \times V_{ad} \to R$ is the bilinear and symmetric form defined by*

$$a(\dot{u},w) = \int\limits_{\Omega} E^{ep}(x,t)\varepsilon(\dot{u}(x,t))\varepsilon(w(x,t))\,dx. \tag{35}$$

*Here L is a linear functional*

$$\langle L,w\rangle = \int\limits_{\Omega} \dot{b}_1(x,t)w(x,t)dx + \left[\dot{f}_1(x,t)w(x,t)\right]\Big|_{\Gamma_\sigma}; \ \forall t \in [0,T] \tag{36}$$

*where*

$$E^{ep}(x,t) = \begin{cases} E & \text{if } \mathcal{H}(\mathcal{F}) = 0, \\ E\left(1 - \frac{E}{h(x,t)}\left(\frac{\sigma(x,t)-\alpha(x,t)}{Q(1-e^{-bk(x,t)})+\sigma_Y}\right)^2\right) & \text{if } \mathcal{H}(\mathcal{F}) = 1, \end{cases} \tag{37}$$

*but under the hypothesis written in (29).*

**Hypothesis.** We assume that the material properties are given in such a way to ensure that $a(\cdot,\cdot)$ be a bilinear, symmetric, continuous and coercive form.

Next,we replace the material particle $x \in \Omega$ with $\xi \in \Omega$, to avoid misunderstandings. The problem to be solved is presented below

**Problem P1.** Consider the following differential system

$$\frac{d}{dt}x(t) = f\left(x(t),\dot{u}(\xi,t)\right); \quad x(t_0) = x_0 \tag{38}$$

where the vector $x(t)$ has the components

$$x(t) = (x_1(t)\ \ x_2(t)\ \ x_3(t)\ \ x_4(t))^T \tag{39}$$

$$x_1 = \varepsilon^p,\ x_2 = \alpha,\ x_3 = \sigma,\ x_4 = k \tag{40}$$

while the vector valued function which defines the system (38) has the form:

$$f\,(x(t),\dot{u}\,(\xi,t)) = (f_1\,(x,\dot{u}) \quad f_2\,(x,\dot{u}) \quad f_3\,(x,\dot{u}) \quad f_4\,(x,\dot{u}))^T \tag{41}$$

$$\begin{cases} f_1\,(x,\dot{u}) = \lambda\,(x,\dot{u})\,\dfrac{x_3 - x_2}{Q(1-e^{-bx_4}) + \sigma_Y} \\[2ex] f_2\,(x,\dot{u}) = \lambda\,(x,\dot{u})\left(C\dfrac{x_3 - x_2}{Q(1-e^{-bx_4}) + \sigma_Y} - \sqrt{\dfrac{2}{3}}\gamma x_2\right) \\[2ex] f_3\,(x,\dot{u}) = E\,(\varepsilon\,(\dot{u}) - f_1\,(x,\dot{u})) \\[2ex] f_4\,(x,\dot{u}) = \sqrt{\dfrac{2}{3}}\lambda\,(x,\dot{u}) \end{cases} \tag{42}$$

$$\lambda\,(x(t),\dot{u}\,(\xi,t)) = \frac{\langle\beta\,(x(t),\dot{u}\,(\xi,t))\rangle}{h(x(t))}\mathcal{H}\,(\mathcal{F}\,(x(t))) \tag{43}$$

$$\beta\,(x(t),\dot{u}\,(\xi,t)) = \frac{E\varepsilon\,(\dot{u}\,(\xi,t))\,(x_3(t) - x_2(t))}{Q(1-e^{-bx_4(t)}) + \sigma_Y} \tag{44}$$

$$h(x(t)) = E + C - \sqrt{\frac{2}{3}}\gamma x_2(t)\frac{x_3(t) - x_2(t)}{Q(1-e^{-bx_4(t)}) + \sigma_Y} + \sqrt{\frac{2}{3}}Qbe^{-bx_4(t)} \tag{45}$$

$$\mathcal{H}(\mathcal{F}(x(t))) = \begin{cases} 0, & \text{if} \quad \mathcal{F}(x(t)) < 0 \\ 1, & \text{if} \quad \mathcal{F}(x(t)) \geq 0 \end{cases} \tag{46}$$

$$\mathcal{F}(x(t)) = |x_3(t) - x_2(t)| - \left[Q(1-e^{-bx_4(t)}) + \sigma_Y\right]. \tag{47}$$

Determine the displacement $u \in V_{ad}$ field, such that $\dot{u}(\cdot,t) \in V_{ad}$, the plastic deformation $\varepsilon^p$, the internal variables $\alpha, k$ and the stress $\sigma$ which satisfy at every time t the variational formulation

$$\begin{aligned} &\int_0^L E^{ep}\,(x(t))\,\varepsilon(\dot{u}(\xi,t))\varepsilon(w(\xi,t))d\xi = \\ &= \int_0^L \dot{b}_1(\xi,t)w(\xi,t)d\xi + \left[\dot{f}_1(\xi,t)w(\xi,t)\right]\Big|_{\xi=L} \end{aligned} \tag{48}$$

$$E^{ep}\,(x(t)) = \begin{cases} E & \text{if } \mathcal{H}(\mathcal{F}) = 0 \\ E\left(1 - \frac{E}{h(x(t))}\left(\frac{x_3(t)-x_2(t)}{Q(1-e^{-bx_4(t)})+\sigma_Y}\right)^2\right) & \text{if } \mathcal{H}(\mathcal{F}) = 1 \end{cases} \tag{49}$$

and having the time evolution for any fixed particle given by the above differential system.

## 4. DISCRETIZATION BY FINITE ELEMENT METHOD

Finite element method is generally used to solve a variational problem, or the discretization form of certain variational formulation of the problem. Here we use the finite element method to find the weak solutions of the problem, which satisfy the variational equation (48) coupled with the system of the differential equations (38).

The **problem P1** is solved using the finite element method, and hence we briefly presents this method, following the papers and books given by Ferreira [8], Fish [9], Johnson [10].

We consider one-dimensional body, which is identified with an interval of the real axis, i.e. $\Omega = [0, L]$. The time interval $[0, T]$ is discretized by $[0, T] = \bigcup_{n=1}^{N} [t_n, t_{n+1}]$. [0,L] is at its turn discretized in $n_e$ network elements, where a network element has the form $\Omega_e = \left[\xi_1^e, \xi_2^e, \xi_3^e\right]$, $e = \overline{1, n_e}$. Let $n_N$ the total number of nodes used in the discretization.

We apply the finite element method to the elasto-plastic problem formulated for the one-dimensional bar. Thus, we divide the bar into $n_e$ elements with $n_N$ nodes, each element having three nodes.

Since we consider the one-dimensional case, the number of degrees of freedom ngl is equal to one for each node in the network.

Concerning the boundary conditions: $\xi = 0$ $(\Gamma_u)$ is considered to be the fixed end of the bar thus the displacement is zero, while at a traction boundary condition is applied at $\xi = L$ $(\Gamma_\sigma)$.

Further we proceed to the effective implementation of finite element method proposed by Fish [9], Johnson [10]. The global approximation of the trial solution $u(\xi, t)$ is

$$u(\xi, t) = \sum_{e=1}^{ne} N^e(\xi) u^e(t) \equiv N(\xi) u(t). \tag{50}$$

The vectors $N(\xi)$ and $u(t)$ which enter the relation (50) have the form

$$N(\xi) = \left[N_1(\xi) \quad N_2(\xi) \quad N_3(\xi) \;...\; N_{n_N}(\xi)\right] \tag{51}$$

$$u(t) = \left[u_1(t) \quad u_2(t) \quad u_3(t) \;...\; u_{n_N}(t)\right]^T. \tag{52}$$

In the same way, the weight function $w(\xi, t)$ is calculated using the relationship

$$w(\xi, t) = \sum_{e=1}^{n_e} N^e(\xi) w^e(t) \equiv N(\xi) w(t) \tag{53}$$

where the vector $w(t)$ is given under the form

$$w(t) = \left[w_1(t) \quad w_2(t) \quad w_3(t) \;...\; w_{n_N}(t)\right]^T. \tag{54}$$

Also, the displacements $u^e(\xi, t)$ and the weight function $w^e(\xi, t)$ of the three nodes of each element can be calculated using the following relations:

$$u^e(\xi, t) = N^e(\xi)u^e(t) \equiv \left(N^e(\xi)L^e\right) u(t) \tag{55}$$

$$w^e(\xi, t) = N^e(\xi)w^e(t) \equiv \left(N^e(\xi)L^e\right) w(t) \tag{56}$$

where the vectors $u^e(t)$ and $w^e(t)$ are calculated using the relationships:

$$u^e(t) = L^e u(t) \tag{57}$$

$$w^e(t) = L^e w(t) \tag{58}$$

Here $L^e$, called the *selection matrix*, is composed by a number of lines equal to the number of degrees of freedom per element and a number of columns equal to the total number of degrees of freedom in the network and is build with the help of the relationships

$$L^e_{ij} = \delta_{Ine(i,e),j} = \begin{cases} 1, & Ine(i,e) = j \\ 0, & Ine(i,e) \neq j. \end{cases} \tag{59}$$

In the relation (59), *Ine* is the matrix of connection, that is a matrix that has a line for each item. The line *e* of the matrix *Ine* contains the nodes that make up the item.

In the relations (55) and (56), the vector of the interpolation function of an element *e*, if this element has three nodes, is of the form

$$N^e_i(\xi) = \begin{bmatrix} N^e_1(\xi) & N^e_2(\xi) & N^e_3(\xi) \end{bmatrix} \tag{60}$$

and the displacements vector for an element in the three nodes are

$$u^e(t) = \begin{bmatrix} u^e_1(t) & u^e_2(t) & u^e_3(t) \end{bmatrix} \tag{61}$$

$$w^e(t) = \begin{bmatrix} w^e_1(t) & w^e_2(t) & w^e_3(t) \end{bmatrix}. \tag{62}$$

For the unidimensional case when the network element contains three nodes, the shape functions $N^e_i(\xi)$ given by the relation (60) are built as is follows:

$$N^e_1(\xi) = \frac{\left(\xi - \xi^e_2\right)\left(\xi - \xi^e_3\right)}{\left(\xi^e_1 - \xi^e_2\right)\left(\xi^e_1 - \xi^e_3\right)} \tag{63}$$

$$N^e_2(\xi) = \frac{\left(\xi - \xi^e_1\right)\left(\xi - \xi^e_3\right)}{\left(\xi^e_2 - \xi^e_1\right)\left(\xi^e_2 - \xi^e_3\right)} \tag{64}$$

$$N^e_3(\xi) = \frac{\left(\xi - \xi^e_1\right)\left(\xi - \xi^e_2\right)}{\left(\xi^e_3 - \xi^e_1\right)\left(\xi^e_3 - \xi^e_2\right)}. \tag{65}$$

When we take the time derivate of the displacement written in (55), we have:

$$\frac{du^e(\xi,t)}{dt} \equiv \dot{u}^e(\xi,t) = (N^e(\xi)L^e)\,\dot{u}(t) \tag{66}$$

Thus the rate of the strain tensor is calculated using the following relationship:

$$\varepsilon(\dot{u}^e(\xi,t)) = \frac{d\dot{u}^e(\xi,t)}{d\xi} = \frac{dN^e(\xi)}{d\xi}L^e\dot{u}(t) \equiv (B^e(\xi)L^e)\,\dot{u}(t) \tag{67}$$

where $B^e(\xi) = \dfrac{dN^e(\xi)}{d\xi}$, and $B^e(\xi) = \begin{bmatrix} B_1^e(\xi) & B_2^e(\xi) & B_3^e(\xi) \end{bmatrix}$.

On the other hand, the represented of the strain measure is:

$$\varepsilon\left(w^e(\xi,t)\right) = (B^e(\xi)L^e)\,w(t). \tag{68}$$

In the weak form (48), the integral over $(0,L)$ is viewed as a sum of integrals over individual element domain, $\Omega^e$. Using the notations (67) and (68) in the variational formulation (48) we have:

$$\begin{array}{l} \sum\limits_{e=1}^{n_e}\int\limits_{\Omega^e}\left[B^e(\xi)L^e w(t)\right]^T (E^{ep})^e\,(x(t))\left[B^e(\xi)L^e\dot{u}(t)\right]d\xi = \\ \sum\limits_{e=1}^{n_e}\int\limits_{\Omega^e}\left[N^e(\xi)L^e w(t)\right]^T \dot{b}_1(\xi,t)d\xi + \sum\limits_{e=1}^{n_e}\left[\left[N^e(\xi)L^e w(t)\right]^T \dot{f}_1(\xi,t)\right]\Big|_{\Gamma_\sigma^e}. \end{array} \tag{69}$$

Moreover (69) can be written under the form

$$\begin{array}{l} w(t)^T\left[\sum\limits_{e=1}^{n_e} L^{eT}\left(\int\limits_{\Omega^e} B^e(\xi)^T\,(E^{ep})^e\,(x(t))\,(B^e(\xi))\,d\Omega^e\right)L^e\right]\dot{u}(t) = \\ = w(t)^T\left(\sum\limits_{e=1}^{n_e} L^{eT}\int\limits_{\Omega^e} N^e(\xi)^T\dot{b}_1(\xi,t)d\Omega^e + \sum\limits_{e=1}^{n_e} L^{eT}\left[N^e(\xi)^T\dot{f}_1(\xi,t)\right]_{\Gamma_\sigma^e}\right). \end{array} \tag{70}$$

We introduce the following notation relations

$$\begin{cases} K^e(x(t)) = \int\limits_{\Omega^e} B^e(\xi)^T\,(E^{ep}\,(x(t)))^e\,(B^e(\xi))\,d\Omega^e \\ R_b^e(t) = \int\limits_{\Omega^e} N^e(\xi)^T\dot{b}_1(\xi,t)d\Omega^e \\ R_f^e(t) = \left[N^e(\xi)^T\dot{f}_1(\xi,t)\right]_{\Gamma_\sigma^e} \end{cases} \tag{71}$$

which allow us to rewrite (70) in the form:

$$w(t)^T\left\{\left[\sum_{e=1}^{n_e} L^{eT}K^eL^e\right]\dot{u}(t) - \left[\sum_{e=1}^{n_e} L^{eT}{R_b}^e + \sum_{e=1}^{n_e} L^{eT}{R_f}^e\right]\right\} = 0. \tag{72}$$

The matrices introduced in (71) have the meaning, $K^e$ is the element stiffness matrix, ${R_b}^e$ is the matrix of the external forces and $R^e_f$ is the matrix of the internal forces. If

$$K(x(t)) = \sum_{e=1}^{n_e} L^{eT} K^e(x(t)) L^e \tag{73}$$

$$R(t) = R_b(t) + R_f(t) \to \begin{cases} R_b(t) = \sum\limits_{e=1}^{n_e} L^{eT} R^e_b(t) \\ R_f(t) = \sum\limits_{e=1}^{n_e} L^{eT} R^e_f(t) \end{cases} \tag{74}$$

then the relation (72) can be written in a shorter form

$$w(t)^T \left[K(x(t))\dot{u}(t) - R(t)\right] = 0 \; ; \quad \forall\, w(t) \tag{75}$$

As $w(t)$ is arbitrarily given, relationship (75) becomes

$$K(x(t))\dot{u}(t) = R(t). \tag{76}$$

Thus, in view of the above, in the following we apply the Gauss quadrature formula to the integral from relationship (70), i.e.:

$$K^e_{mn}(x(t)) = h_2 \sum_{i=0}^{n-1} A_i B^e_m(h_1 + h_2\tau_i)^T \left(E^{ep}(x(t))\right)^e B^e_n(h_1 + h_2\tau_i) \tag{77}$$

where the matrix $K^e$ is symmetric, and $m = \overline{1, n_N}$ and $n = \overline{1, n_N}$.

The integral $R^e_b(t)$ given by the relation from (71) can be similarly computed by using the Gauss quadrature formula, as follows:

$$\left(R^e_b\right)_m(t) = \left[h_2 \sum_{i=0}^{n-1} N^e_m(h_1 + h_2\tau_i)^T \dot{b}(h_1 + h_2\tau_i, t)\right], \; m = \overline{1, n_N} \tag{78}$$

where we used the notations $h_1 = \dfrac{b+a}{2}$, $h_2 = \dfrac{b-a}{2}$. We obtain the components

$$F_m(x(t), \dot{u}(t)) = \sum_{n=1}^{n_N} \left[K_{mn}(x(t))\dot{u}_n(t)\right] - R_m(t) \tag{79}$$

of the relation (76), that can be written in following form

$$F(x(t), \dot{u}(t)) := K(x(t))\dot{u}(t) - R(t) = 0. \tag{80}$$

To solve the system of equations (80) relative to the unknown $\dot{u}(t)$, we apply Newton's method. Thus we have:

$$\dot{u}(t_{n+1}) = \dot{u}(t_n) - \left[\frac{\partial F(x(t_n), \dot{u}(t_n))}{\partial \dot{u}(t_n)}\right]^{-1} F(x(t_n), \dot{u}(t_n)), n = 0, 1, \dots . \tag{81}$$

Simultaneously we apply the Euler method to the non-linear system of differential equations given by (38). The iterative formulae can be derived under the form:

$$x(t_{n+1}) = x(t_n) + \Delta t f(x(t_n), N(\xi)\dot{u}(t_n)), \ x(t_0) = 0, \ n = 0, 1, 2, ... \tag{82}$$

as a consequence of (50), where $\Delta t = t_{n+1} - t_n$ is the step of the method.

We present now the main steps of the algorithm applied to the formulated problem:

$$\begin{array}{l} \dot{u}(t_0) = 0; \quad x(t_0) = 0 \\ \left| \begin{array}{l} For \quad n = 0 \quad to \quad N \\ \left| \begin{array}{l} x(t_{n+1}) = x(t_n) + \Delta t f(x(t_n), N(\xi)\dot{u}(t_n)) \\ \dot{u}(t_{n+1}) = \dot{u}(t_n) - \left[ \dfrac{\partial F(x(t_n), \dot{u}(t_n))}{\partial \dot{u}(t_n)} \right]^{-1} F(x(t_n), \dot{u}(t_n)) \end{array} \right. \\ end \end{array} \right. \end{array} \tag{83}$$

To solve the problem P1, the algorithm presented in the relationship (83) is run in every point of the network.

## 5. NUMERICAL APPLICATION

Numerical application presented here, aims to highlight the numerical algorithm of solving the problem P1.

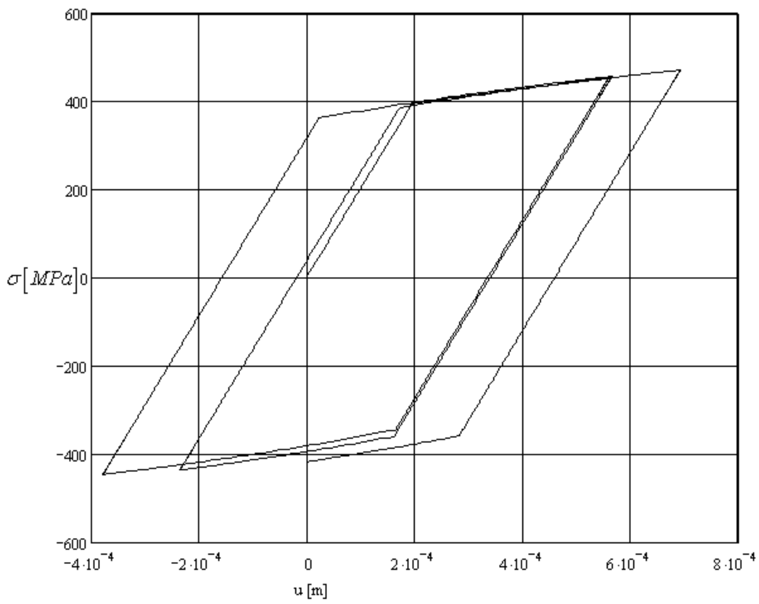


*Fig. 1.* The graph of the stress depending on the strain in the point 30

In this sense, the material chosen is steel DP 600, whose parameters are given below (Broggiato [2008]):

$$\left\{ \begin{array}{l} E = 182.000\,[MPa]\,;\sigma_Y = 349,4\,[MPa]\,;Q = 50,1\ \ [MPa] \\ C = 17.400\,[MPa]\,;b = 27,5\,[-]\,;\gamma = 125,9\,[-] \end{array} \right\}. \tag{84}$$

As an example we consider an uni-dimensional bar, whit $\Omega = [0,\ 30]$. The bar has length $L = 30$ $[cm]$. The bar is fixed in the node $\xi = 0$, and the traction force $f_1(\xi, t) = 450 \sin t$ is located in the node $\xi = 6$. The body force is considered by form $b_1 = 0,5\ t^2$. We specify also that for the meshing, the bar was divided into ten elements, each element having three nodes, total number of nodes in the network being 21. Since, the bar is fixed in the node $\xi = 0$, the displacement in this node is zero. In the network, the elements have been divided into equal intervals. In these conditions, after running the program, which implements the numerical algorithm of the problem P1, we obtain the following information presented in the figure 1.

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## References

[1] P. J. Armstrong, C. O. Frederick, *A mathematical representation of the multiaxial Bauschinger effect*, G.E.G.B. Report RD-B-N 731, 1996.

[2] J. Bathe, *Finite element procedure*, Vols. I, II, III, Prentice Hall, Englewood Cliffs, New Jersey, 1996.

[3] T. Belytschko, K. L. Wing, B. Moran, *Nonlinear Finite Elements for Continua and Structures*, British Library Cataloguing in Publication Data, Toronto, 2006.

[4] G. B.Broggiato , F. Campana, L. Cortese *The Chaboche nonlinear kinematic hardening model: calibration methodology and validation* , Meccanica, **43**, 2(2008), 115-124.

[5] J. L. Chaboche, *Constitutive equations for cyclic plasticity and cyclic visco-pasticity* , Int. J. Plasticity, 1989.

[6] S. Cleja-Tigoiu, N. Cristescu, *Teoria plasticitatii cu aplicatii la prelucrarea materialelor*, Ed. Univ. Bucuresti, 1985.

[7] S. Cleja-Tigoiu, E. Soos, *Elastoplastic model with relaxed configurations and internal state variables*, Appl. Mech. Rev., **43**, 7(1990), 131-151.

[8] A. J. M. Ferreira, *Matlab Codes for Finite Element Analysis Solid and Structures* , Springer, Berlin, 2009.

[9] J. Fish, T. Belytschko, *A First Course in Finite Elements* , John Wiley and Sons, Ltd., USA, 2007.

[10] C. Johnson, *Numerical solution of partial diferential equation by the finite element method* , Cambridge University Press, Cambridge, 2002.

[11] I. Paraschiv-Munteanu, S. Cleja-Tigoiu, E. Soos, *Plasticitate cu aplicaţii în geomecanică* , Ed. Universităţii din Bucureşti, 2004.

[12] J. C. Simo, T. J. R. Hughes, *Computational Inelasticity* , Springer, New-York,1998.