ROMAI J., **6**, 2(2010), 121–130

SOME NATURAL DIAGONAL STRUCTURES ON THE TANGENT BUNDLES AND ON THE TANGENT SPHERE BUNDLES

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Abstract We study the properties of some geometric structures defined on the tangent and tangent sphere bundles of a Riemannian manifold by using the natural lifts. These lifts are obtained from the Riemannian metric g of the base manifold. We get some almost Hermitian structures defined on the tangent bundle and find the conditions under which they are Kählerian. Then we study some specific properties of such structures (to have constant holomorphic sectional curvature, to be Einstein, etc). Similar problems are considered for the tangent sphere bundles T_rM endowed with the the induced almost contact structures (the quality of T_rM to be a Sasaki space form, or an η-Einstein manifold).

Keywords: tangent bundle, natural lift, Einstein structure, tangent sphere bundle, almost contact structure, Sasaki space form, η-Einstein manifold.

2000 MSC: 53C05, 53C15, 53C55.

1. INTRODUCTION

Some new interesting geometric structures on the tangent bundle of a Riemannian manifold (M, g) may be obtained by lifting the metric g from the base manifold.

In 1998, in a joint work with N. Papaghiuc ([16]), the second author introduced on the tangent bundle of a Riemannian manifold, a Riemannian metric defined by a regular Lagrangian depending on the energy density only.

The same author has studied some properties of a natural lift G, of diagonal type, of the Riemannian metric g (so that it is no longer obtained from a Lagrangian) and a natural almost complex structure J of diagonal type on TM (see [12]-[15]). The condition for (TM, G, J) to be a Kähler Einstein manifold leads to the conditions for (M, g) to have constant sectional curvature, and for (TM, G, J) to have constant sectional holomorphic curvature or to be a locally symmetric space. In [12] and [15], the second author excluded some important cases which appeared, in a certain sense, as singular cases. The case where the Riemannian metric G is defined by the Lagrangian L (excluded in [12]) was studied in [16]. Other singular cases excluded in [12] and [13] were studied by N. Papaghiuc in [18] and [19]. Other geometric structures obtained by considering natural lifts of the metric g from the base manifold to its tangent bundle have been studied in (see [1], [8], [17]).

In this paper we present a survey on the geometric structures defined by the natural diagonal lifts of the metric from the base manifold to the tangent bundle, as well as the structures defined by the restrictions of these lifts to the tangent sphere bundles of constant radius r, seen as hypersurfaces of the tangent bundles, consisting of the tangent vectors of norm equal to r only.

It is known that every almost Hermitian structure from the tangent bundle induces an almost contact structure on the tangent sphere bundle of constant radius r. Important results in this direction may be found in the recent surveys [1] and [9], but also in the papers [2]-[4], [11], [20]. The most part of the authors who worked in this field considered unitary tangent sphere bundles, endowed with the induced Sasaki metric (see [21]), but O. Kowalski and M. Sekizawa showed that the geometry of the tangent sphere bundles depends on the radius. They determined in [9] the g-natural metrics of the mentioned type on the tangent sphere bundles of constant radii, having constant scalar curvature, and in [2] M. T. K. Abassi and O. Kowalski obtained the conditions under which the g-natural metrics on the unit tangent sphere bundle are Einstein.

In the paper [5] we considered on the tangent bundle TM of a Riemannian manifold M a natural diagonal metric, obtained by the second author in [15], and denoted in the section 4 by $\widetilde{G}$. We proved that the tangent sphere bundle T_rM endowed with the Riemannian metric induced from $\widetilde{G}$ is never a space form, then we found the conditions under which (T_rM, G) is an Einstein manifold.

In [6] we obtained some almost contact metric structures (φ, ξ, η, G) on the tangent sphere bundles, induced by some almost Hermitian structures $(\widetilde{G}, J)$ of natural diagonal lift type on the tangent bundle of a Riemannian manifold (M, g). The above almost contact metric structures are not automatically contact metric structures. In order to get such structures we made some rescalings of the metric, of the fundamental vector field, and of the 1-form. Then we gave the characterization of the Sasakian structures of natural diagonal lift type on the tangent sphere bundles. In this case, the base manifold must be a space form. We studied in [7] the holomorphic ϕ-sectional curvature of the obtained Sasakian manifolds, and we proved that there are no natural diagonal tangent sphere bundles of constant holomorphic ϕ-sectional curvature. On the other hand, the study of the conditions under which the obtained Sasakian tangent sphere bundles $(T_rM, \varphi, \xi, \eta, G)$ are η-Eistein manifolds, namely the corresponding Ricci tensor field has the form

$$Ric = \rho G + \sigma\eta \otimes \eta,$$

where ρ and σ are smooth real functions, led in [6] to two cases, which will be presented in the final section of this paper.

The manifolds, tensor fields and other geometric objects we consider in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). The well known summation convention is used throughout this paper, the range of the indices h, i, j, k, l, r being always $\{1, \dots, n\}$.

2. PRELIMINARY RESULTS

Consider a smooth n-dimensional Riemannian manifold (M, g) and its tangent bundle $\tau : TM \to M$. The total space TM has a structure of a $2n$-dimensional smooth manifold, induced from the smooth manifold structure of M. This structure is obtained by using local charts on TM induced from usual local charts on M. If $(U, \varphi) = (U, x^1, \dots, x^n)$ is a local chart on M, then the corresponding induced local chart on TM is $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$, where the local coordinates $x^i, y^j,\ i, j = 1, \dots, n$, are defined as follows. The first n local coordinates of a tangent vector $y \in \tau^{-1}(U)$ are the local coordinates in the local chart (U, φ) of its base point, i.e. $x^i = x^i \circ \tau$, by an abuse of notation. The last n local coordinates $y^j,\ j = 1, \dots, n$, of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis in $T_{\tau(y)}M$ defined by the local chart (U, φ). Due to this special structure of differentiable manifold for TM, it is possible to introduce the concept of M-tensor field on it (see [10]).

Denote by $\dot{\nabla}$ the Levi Civita connection of the Riemannian metric g on M. Then we have the direct sum decomposition

$$TTM = VTM \oplus HTM \tag{1}$$

of the tangent bundle to TM into the vertical distribution $VTM = \mathrm{Ker}\ \tau_*$ and the horizontal distribution HTM defined by $\dot{\nabla}$ (see [22]). The vertical and horizontal lifts of a vector field X on M will be denoted by X^V and X^H respectively. The set of vector fields $\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\}$ on $\tau^{-1}(U)$ defines a local frame field for VTM, and for HTM we have the local frame field $\{\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}\}$, where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i}\frac{\partial}{\partial y^h}$, $\Gamma^h_{0i} = y^k\Gamma^h_{ki}$, and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set $\{\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\}_{i,j=\overline{1,n}}$, denoted also by $\{\partial_i, \delta_j\}_{i,j=\overline{1,n}}$, defines a local frame on TM, adapted to the direct sum decomposition (1).

Consider the energy density of the tangent vector y with respect to the Riemannian metric g

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y, y) = \frac{1}{2}g_{ik}(x)y^iy^k, \quad y \in \tau^{-1}(U). \tag{2}$$

Obviously, we have $t \in [0, \infty)$ for every $y \in TM$.

3. NATURAL DIAGONAL LIFTED STRUCTURES ON THE TANGENT BUNDLE

The second author constructed in [15] an (1,1)-tensor field J on the tangent bundle TM, obtained as natural 1-st order lift of the metric g from the base manifold to the tangent bundle TM:

$$J\delta_i = a_1(t)\partial_i + b_1(t)g_{0i}C, \quad J\partial_i = -a_2(t)\delta_i - b_2(t)g_{0i}\widetilde{C}, \tag{3}$$

where a_1, a_2, b_1, b_2 are smooth functions of the energy density, $C = y^h\partial_h$ is the Liouville vector field and $\widetilde{C} = y^h\delta_h$ is the geodesic spray.

The invariant expressions of the above (1,1)-tensor field are

$$\begin{aligned} JX^H_y &= a_1(t)X^V_y + b_1(t)g_{\tau(y)}(X,y)y^V_y, \\ JX^V_y &= -a_2(t)X^H_y - b_2(t)g_{\tau(y)}(X,y)y^H_y, \end{aligned} \quad \forall X \in \mathcal{T}^1_0(M),\ \forall y \in TM. \tag{4}$$

Studying the conditions under which $J^2 = -I$, it was easy to prove the following result.

Proposition 3.1. ([15]) *The* $(1,1)$*-tensor field* J *given by the relations* (3) *or* (4) *defines an almost complex structure on the tangent bundle if and only if* $a_2 = 1/a_1,\ b_2 = -b_1/[a_1(a_1+2tb_1)]$.

Then it was considered on TM a natural Riemannian metric of diagonal type G on TM, induced by g. The expression in local adapted frame are defined by the M-tensor fields

$$\begin{aligned} G^{(1)}_{ij} &= G(\partial_i, \partial_j) = c_1 g_{ij} + d_1 g_{0i} g_{0j}, \\ G^{(2)}_{ij} &= G(\delta_i, \delta_j) = c_2 g_{ij} + d_2 g_{0i} g_{0j}, \\ G(\partial_i, \delta_j) &= G(\delta_j, \partial_i) = 0. \end{aligned} \tag{5}$$

where the coefficients c_1, c_2, d_1, d_2 are smooth functions depending on the energy density $t \in [0.\infty)$ and the conditions for G to be positive definite are given by $c_1 > 0,\quad c_2 > 0,\quad c_1 + 2td_1 > 0,\quad c_2 + 2td_2 > 0$, for every $t \geq 0$.

The invariant expressions of the above metric are

$$\begin{aligned} G(X^H_y, Y^H_y) &= c_1(t)g_{\tau(y)}(X,Y) + d_1(t)g_{\tau(y)}(X,y)g_{\tau(y)}(Y,y), \\ G(X^V_y, Y^V_y) &= c_2(t)g_{\tau(y)}(X,Y) + d_2(t)g_{\tau(y)}(X,y)g_{\tau(y)}(Y,y), \\ G(X^V_y, Y^H_y) &= \widetilde{G}(X^H_y, X^V_y) = 0, \end{aligned} \tag{6}$$

$\forall X, Y \in \mathcal{T}^1_0(TM),\ \forall y \in TM.$

Theorem 3.1. ([14]) *Let* J *be a natural diagonal almost complex structure on* TM *defined by* g *and the functions* a_1, a_2, b_1, b_2*. Then the family of natural diagonal Riemannian metrics* G*, given by* (5) *or* (6) *is almost Hermitian with respect to* J*, i.e.* $G(JX, JY) = G(X, Y)$*, if and only if*

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = \lambda, \quad \frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \lambda + 2t\mu, \tag{7}$$

where $\lambda > 0,\ \mu > 0$ *are functions of* t.

In [14], the second author obtained some classes of natural almost Hermitian structures (G, J) on TM, these classes are obtained from the well known classification of the almost Hermitian structures in sixteen classes. The results concerning this classification are given by Theorems 4, 5, 6 and 7 in [14].

Some very important particular cases of the almost Hermitian structure obtained in Theorem 3.1 have been studied further. So, if we consider the case when $c_1(t) =$

$a_1(t) = u(t)$, $b_1(t) = d_1(t) = v(t)$, $b_2(t) = d_2(t) = w(t)$, $a_2(t) = c_2(t) = \frac{1}{u(t)}$, $\lambda(t) = 1$ and $\mu(t) = 0$, where $w(t) = -\frac{v}{u(u+2tv)}$, then all the conditions from Proposition 3.1 and Theorem 3.1 are satisfied, and we obtain the almost Hermitian structure defined and studied by the second author in [15].

Theorem 3.2. ([15]) *Under the above conditions, the almost Hermitian manifold (TM, G, J) is an almost Kähler manifold. Moreover, the almost complex structure J on TM is integrable (i.e. (TM, G, J) is a Kähler manifold) if (M, g) has constant sectional curvature c and the function v is given by*

$$v = \frac{c - uu'}{2tu' - u}.$$

In [15] the second author obtained a Kähler-Einstein structure even in the case where (M, g) has positive constant sectional curvature, but only on a tube around the zero section in TM ([15, Theorem 4.2]). He also obtained on TM a structure of Kähler manifold with constant holomorphic sectional curvature ([15, Theorem 5.1]).

The particular case of the above situation, when the function $u(t) = 1$, for all $t \in [0, \infty)$, has been studied by the same author in [12]. In this case he stated

Theorem 3.3. ([12]) *(TM, G, J) is an almost Kählerian manifold and the almost complex structure J is integrable if and only if (M, g) has constant sectional curvature c and $v = -c$. If $c < 0$ is obtained a Kähler structure on whole TM. In the case where the constant c is positive is obtained a Kähler structure in the tube around the zero section in TM defined by $t < \frac{1}{2c}$.*

Moreover, in the same paper it was proved

Theorem 3.4. *If (M, g) has negative constant sectional curvature then its tangent bundle has a structure of Kähler Einstein manifold. If (M, g) has positive constant sectional curvature then a tube around the zero section in TM has a structure of Kähler Einstein manifold.*

Another important case which appear as a singular case is when $v(t) = 0$ for all $t \in [0, \infty)$ which implies $w(t) = 0$. This case has been studied by N. Papaghiuc in [18], [19], and he obtained

Theorem 3.5. ([18]) *Assume that the function $u(t)$ satisfies the condition $u(0)u'(0) \neq 0$. Then the almost complex structure J on TM or on a tube around the zero section in TM is integrable if and only if the base manifold (M, g) has constant sectional curvature c and the function $u(t)$ satisfies the ordinary differential equation $uu' = c$, i.e. $u(t) = \sqrt{2ct + A}$, where A is an arbitrary real constant.*

Another result from [18] (see also [19]) is given by

Theorem 3.6. *If $n \neq 2$, then the Kählerian manifold (TM, G, J) cannot be an Einstein manifold and cannot have constant holomorphic sectional curvature. If $n = 2$, then the Kählerian manifold (TM, G, J) is Ricci flat.*

Remark that the singular case when $v(t) = u'(t)$ was studied in [16].

4. NATURAL DIAGONAL STRUCTURES ON THE TANGENT SPHERE BUNDLES OF CONSTANT RADIUS *R*

Let us recall some results from [5], concerning tangent sphere bundles endowed with a natural diagonal lifted metric.

We denote by $T_rM = \{y \in TM : g_{\tau(y)}(y, y) = r^2\}$, with $r \in (0, \infty)$, and the projection $\overline{\tau} : T_rM \to M$, $\overline{\tau} = \tau \circ i$, where i is the inclusion map.

The horizontal lift of any vector field on M is tangent to T_rM, but the vertical lift is not always tangent to T_rM. The tangential lift of a vector X to $(p, y) \in T_rM$, used in some recent papers like [3], [9], [11], [20], is defined by

$$X_y^T = X_y^V - \frac{1}{r^2} g_{\tau(y)}(X, y) y_y^V.$$

A generator system for the tangent bundle to T_rM is given by δ_i and $\partial_j^T = \partial_j - \frac{1}{r^2} g_{0j} y^k \partial_k$, $i, j, k = \overline{1, n}$. Remark that the vector fields $\{\partial_j^T\}_{j=\overline{1,n}}$ satisfy the relation $y^j \partial_j^T = 0$, so they are not independent. In any point of T_rM the vectors $\{\partial_i^T\}_{i=1,\dots n}$ span an $(n-1)$-dimensional subspace of TT_rM.

The vector field $N = y^i \partial_i$ is normal to T_rM in TM.

From now on we shall denote by $\widetilde{G}$ the metric G defined by the relations (5) or (6), and by G' the metric on T_rM induced by the metric of TM. Remark that the functions $c_1,\ c_2,\ d_1,\ d_2$ become constant, since in the case of the tangent sphere bundle of constant radius r, the energy density t becomes a constat equal to $\frac{r^2}{2}$. It follows

$$\begin{aligned} G'(X_y^H, Y_y^H) &= c_1 g_{\tau(y)}(X, Y) + d_1 g_{\tau(y)}(X, y) g_{\tau(y)}(Y, y), \\ G'(X_y^T, Y_y^T) &= c_2 [g_{\tau(y)}(X, Y) - \tfrac{1}{r^2} g_{\tau(y)}(X, y) g_{\tau(y)}(Y, y)], \\ G'(X_y^H, Y_y^T) &= G'(Y_y^T, X_y^H) = 0, \end{aligned} \tag{8}$$

$\forall X, Y \in \mathcal{T}_0^1(M)$, $\forall y \in T_rM$, where $c_1,\ d_1,\ c_2$ are constants. The conditions for G' to be positive are $c_1 > 0,\ c_2 > 0,\ c_1 + r^2 d_1 > 0$.

The Levi-Civita connection ∇, associated to the Riemannian metric G' on the tangent sphere bundle T_rM has the form

$$\begin{cases} \nabla_{\partial_i^T} \partial_j^T = A_{ij}^h \partial_h^T, \ \nabla_{\delta_i} \partial_j^T = \Gamma_{ij}^h \partial_h^T + B_{ji}^h \delta_h, \\ \nabla_{\partial_i^T} \delta_j = B_{ij}^h \delta_h, \ \nabla_{\delta_i} \delta_j = \Gamma_{ij}^h \delta_h + C_{ij}^h \partial_h^T, \end{cases}$$

the expressions of the involved M-tensor fields being given in [5].

In [5] we obtained the horizontal and tangential components of the curvature tensor field K, denoted by sequences of H and T, to indicate horizontal, or tangential argument on a certain position. For example, we may write

$$K(\delta_i,\delta_j)\partial_k^T = HHTH^h_{kij}\delta_h + HHTT^h_{kij}\partial_h^T.$$

Then we proved the following result

Theorem 4.1. ([5]) *The tangent sphere bundle T_rM, with the Riemannian metric induced by the natural metric $\widetilde{G}$ of diagonal lift type on the tangent bundle TM, has never constant sectional curvature.*

We computed the Ricci tensor field of the manifold (T_rM, G'), and we obtained the components $Ric(\delta_j,\delta_k) = RicHH_{jk}$ and $Ric(\partial_j^T,\partial_k^T) = RicTT_{jk}$:

$$\begin{aligned} RicHH_{jk} = Ric_{jk} &- \tfrac{d_1(2c_1+r^2d_1)}{2c_2(c_1+r^2d_1)}g_{jk} + \tfrac{d_1(2c_1+r^2d_1)[c_1n+r^2d_1(n-1)]}{2r^2c_1c_2(c_1+r^2d_1)}g_{0j}g_{0k} \\ &-\tfrac{c_2}{2c_1}R^h{}_{0kl}R^l{}_{jh0} + \tfrac{c_2d_1}{2c_1(c_1+r^2d_1)}R^h{}_{0k0}R_{h0j0} \end{aligned} \tag{9}$$

$$\begin{aligned} RicTT_{jk} = &\tfrac{r^4d_1^2-2c_1(c_1+r^2d_1)(n-2)}{2c_1r^2(c_1+d_1r^2)}(\tfrac{1}{r^2}g_{0j}g_{0k} - g_{jk}) + \tfrac{c_2^2}{4c_1^2}R_{hik0}R^{hi}{}_{j0} \\ &-\tfrac{c_2^2d_1}{2c_1^2(c_1+r^2d_1)}R^h{}_{0j0}R_{h0k0}. \end{aligned} \tag{10}$$

Analyzing the vanishing conditions of the difference $Ric(X,Y) - \rho G'(X,Y)$, for every $X,Y \in \mathcal{T}^1_0(T_rM)$ we stated

Theorem 4.2. ([5]) *The tangent sphere bundle T_rM of an n-dimensional Riemannian (M,g) of constant sectional curvature c is Einstein with respect to the metric G' induced by the natural diagonal lifted metric $\widetilde{G}$ defined on TM, i.e. it exists a real constant ρ such that $Ric(X,Y) = \rho G(X,Y)$, for every $X,Y \in \mathcal{T}^1_0(T_rM)$, if and only if*

$$c_1 = \frac{r^2d_1n}{n-2},\ c_2 = \frac{d_1n}{c(n-2)},\ \rho = \frac{c(n-1)^2(n-2)}{r^2d_1n^2}.$$

In the paper [6] we studied the almost contact metric structures (φ,ξ,η,G) on the tangent sphere bundles, induced by the natural diagonal almost Hermitian structures $(\widetilde{G},J)$ on TM, characterized in Theorem 3.1, and we proved

Theorem 4.3. *The almost contact metric structure (φ,ξ,η,G) on T_rM, with φ, ξ,η, and G given respectively by*

$$\begin{aligned} \varphi X^H = a_1X^T, \varphi X^T &= -a_2X^H + \tfrac{a_2}{r^2}g(X,y)y^H, \\ \xi = \tfrac{1}{2\lambda r^2\alpha}y^H,\ \eta(X^T) = 0,\ &\eta(X^H) = 2\alpha\lambda g(X,y),\ G = \alpha G', \end{aligned}$$

for every tangent vector fields $X,Y \in \mathcal{T}^1_0(M)$, and every tangent vector $y \in T_rM$, where $\alpha = \frac{c_1+r^2d_1}{4r^2\lambda^2}$ and the metric G' is given by (8), is a contact metric structure, *and it is Sasakian if and only if the base manifold has constant sectional curvature* $c = \frac{a_1^2}{r^2}$.

The contact metric manifold $(T_rM, \varphi, \xi, \eta, G)$ is $\eta-$Einstein if the corresponding Ricci tensor may be written as

$$Ric = \rho G + \sigma \eta \otimes \eta, \tag{11}$$

where ρ and σ are smooth real functions.

Using the expressions (9) and (10), we wrote the relation (11) with respect to the generator system $\{\delta_i, \partial_j^T\}_{i,j=\overline{1,n}}$, and taking into account Theorem 4.3 we proved

Theorem 4.4. ([6]) *The Sasakian manifold* $(T_rM, \varphi, \xi, \eta, G)$*, characterized in* Theorem 4.3, *is* $\eta-$*Einstein if and only if*

$$\text{Case I) } c_1 = cc_2r^2,\ \rho = cc_2(2n-3) - \frac{d_1}{2cc_2^2r^2},\ \sigma = \lambda^2\frac{d_1n - cc_2(n-2)}{2cc_2^2(cc_2+d_1)r^2};$$

$$\text{CaseII) } d_1 = \frac{cc_2r^2 - c_1(n-2)}{(n-1)r^2}, \rho = \frac{n(n-2)(c_1+cc_2r^2)}{2c_1c_2(n-1)r^2},$$

$$\sigma = \lambda^2\frac{2r^2cc_1c_2\{n[n(n-4)+6]-2\} - n^2(n-2)(c_1^2 + c^2c_2^2r^4)}{2c_1c_2(c_1+cc_2r^2)^2},$$

where $c = \frac{a_1^2}{r^2}$ *is the constant sectional curvature of the base manifold* M.

In [7] we studied the condition under which the Sasaki manifold $(T_rM, \varphi, \xi, \eta, G)$ characterized in Theorem 4.3 is a Sasaki space form, i.e. it has constant holomorphic ϕ-sectional curvature. The tensor field corresponding to the curvature of the Sasakian space form $(T_rM, \varphi, \xi, \eta, G)$ is given by:

$$K_0(X,Y)Z = \tfrac{k+3}{4}[G(Y,Z)X - G(X,Z)Y] + \tfrac{k-1}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X$$
$$+G(X,Z)\eta(Y)\xi - G(Y,Z)\eta(X)\xi + G(Z,\varphi Y)\varphi X - G(Z,\varphi X)\varphi Y + 2G(X,\varphi Y)\varphi Z],$$

where k is a constant.

With respect to the generator system $\{\delta_i, \partial_j^T\}_{i,j=1,\dots,n}$, we have six components of K_0. We exemplify by the expressions of two of them:

$$K_0(\partial_i^T, \partial_j^T)\partial_k^T = TTTT0_{kij}^h\partial_h^T,\ K_0(\partial_i^T, \partial_j^T)\delta_k = TTHH0_{kij}^h\delta_h,$$

where the M-tensor fields involved as coefficients have the expressions:

$$TTTT0_{kij}^h = \tfrac{k+3}{4}\alpha({G'}_{kj}^{(2)}\delta_i^h - {G'}_{ki}^{(2)}\delta_j^h),$$
$$TTHH0_{kij}^h = a_2^2\tfrac{k-1}{4}\alpha{G'}_{kl}^{(1)}\left\{\tfrac{1}{r^2}\left[g_{0i}(\delta_j^hy^l - \delta_j^ly^h) - g_{0j}(\delta_i^hy^l - \delta_i^ly^h)\right] + \delta_j^l\delta_i^h - \delta_i^l\delta_j^h\right\},$$

Studying the vanishing conditions of the difference between the curvature tensor field K of the Riemannian manifold (T_rM, G) and the curvature tensor field K_0 corresponding to the Sasaki manifold $(T_rM, \varphi, \xi, \eta, G)$, we obtained that some components of this difference vanish under certain very restrictive conditions, but $TTTT_{kij}^h-$

$TTTT0^h_{kij}$ takes the form

$$\frac{1}{r^2}\Big[\delta^h_i(g_{jk} - \frac{1}{r^2}g_{0j}g_{0k}) - \delta^h_j(g_{ik} - \frac{1}{r^2}g_{0i}g_{0k})\Big] \neq 0,$$

which is never equal to zero, so we proved the final theorem from [7].

Theorem 4.5. *The Sasakian manifold* $(T_rM, \varphi, \xi, \eta, G)$*, given by* Theorem 4.3, *is never a Sasaki space form.*

Acknowledgements. The first author was supported by the Program POSDRU/89/1.5/S/49944, "AL. I. Cuza" University of Iaşi, Romania.

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