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FORCING A CONTROLLED DIFFUSION PROCESS TO LEAVE THROUGH THE RIGHT END OF AN INTERVAL

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Abstract Let $\{X(t), t \geq 0\}$ be a one-dimensional controlled diffusion process evolving in the interval $[c, d]$. We consider the problem of finding the control that minimizes the mathematical expectation of a cost function with quadratic control costs on the way and a terminal cost function that is infinite if the process hits c before d. The optimal control is obtained explicitly and particular cases are presented.

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1. INTRODUCTION

Let $\{X(t), t \geq 0\}$ be a one-dimensional controlled diffusion process defined by the stochastic differential equation

$$dX(t) = m[X(t)]\,dt + b[X(t)]\,u(t)\,dt + \{v[X(t)]\}^{1/2}\,dB(t), \tag{1}$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion, $u(t)$ is the control variable and $b(\cdot) \neq 0$.

We define the random variable $T(x)$ by

$$T(x) = \inf\{t > 0 : X(t) = c \text{ or } d \mid X(0) = x\}.$$

Our aim is to determine the value of the control $u^*(t)$ that minimizes the expected value of the cost function

$$J(x) = \int_0^{T(x)} \tfrac{1}{2} q(x)\,u^2(t)\,dt + K[X(T), T],$$

where q is a positive function and K is the termination cost function.

Next, let $\{x(t), t \geq 0\}$ be the uncontrolled process obtained by setting $u(t) \equiv 0$ in (1), and let τ be the same as T, but for $\{x(t), t \geq 0\}$. If the condition

$$P[\tau(x) < \infty] = 1 \tag{2}$$

holds, and if the functions b, v and q are such that

$$\frac{b^2(x)}{q(x)\,v(x)} \equiv \alpha > 0, \tag{3}$$

then making use of a result proved by Whittle (see [2], p. 289), we can state that the optimal control u^* [$= u^*(0)$] is given by

$$u^* = \frac{v(x)}{b(x)} \frac{G'(x)}{G(x)}, \tag{4}$$

where

$$G(x) := E\left[\exp\{-\alpha\, K[x(\tau), \tau]\} \mid x(0) = x\right].$$

In Lefebvre [1], the author solved the problem of forcing the controlled process $\{X(t), t \geq 0\}$ to stay in the continuation region $C := (-\infty, d)$ until a fixed time t_0 by giving an infinite penalty if $T_1 < t_0$, where

$$T_1(x) = \inf\{t > 0 : X(t) = d \mid X(0) = x < d\}.$$

In the present paper, we consider the controlled process $\{X(t), t \geq 0\}$ in the interval $[c, d]$. We want the process to leave the continuation region through its right end. We will take

$$K[X(T), T] = K[X(T)],$$

where the function K is such that $K(c) = \infty$ and $K(d) \in \mathbb{R}$. That is, we give an infinite penalty if $X(t)$ reaches c (before d). The constant d can be chosen as large as we want. The larger it is, the longer it will take $X(t)$ to attain this value.

By giving an infinite penalty if the final value of $X(t)$ is equal to c, we force the process to avoid this boundary. We assume that there are no constraints on the control variable $u(t)$. In the next section, we will obtain an explicit formula for the optimal control u^*, and we will present some particular cases.

2. OPTIMAL CONTROL

Let

$$\pi_d(x) := P[x(\tau) = d \mid x(0) = x].$$

The function π_d satisfies the Kolmogorov backward equation

$$\frac{v(x)}{2} \pi_d''(x) + m(x)\, \pi_d'(x) = 0,$$

and is subject to the boundary conditions

$$\pi_d(c) = 0 \quad \text{and} \quad \pi_d(d) = 1.$$

We easily find that

$$\pi_d(x) = \frac{\int_c^x \exp\left\{\int_c^u -\frac{2\,m(s)}{v(s)}\,ds\right\}\,du}{\int_c^d \exp\left\{\int_c^u -\frac{2\,m(s)}{v(s)}\,ds\right\}\,du}. \tag{5}$$

We will prove the following proposition.

Proposition 2.1. *Assume that the conditions (2) and (3) are satisfied, and that the termination cost function is* $K[X(T), T] = K[X(T)]$, *with* $K(c) = \infty$ *and* $K(d) \in \mathbb{R}$. *Then, the optimal control is given by*

$$u^* = \frac{v(x)}{b(x)} \frac{\exp\left\{\int_c^x -\frac{2\,m(u)}{v(u)}\,du\right\}}{\int_c^x \exp\left\{\int_c^u -\frac{2\,m(s)}{v(s)}\,ds\right\}\,du} \quad \textit{for } c < x < d. \tag{6}$$

Proof. We can write that $P[x(\tau) = c \mid x(0) = x] = 1 - \pi_d(x)$. Hence, we deduce from Whittle's result that u^* is given by (4), with

$$\begin{aligned} G(x) &= E\left[\exp\{-\alpha\,K[x(\tau)]\} \mid x(0) = x\right] \\ &= e^{-\alpha K(c)}\,[1 - \pi_d(x)] + e^{-\alpha K(d)}\,\pi_d(x). \end{aligned}$$

Since $K(c) = \infty$, we obtain that

$$G(x) = e^{-\alpha K(d)}\,\pi_d(x),$$

so that

$$G'(x) = e^{-\alpha K(d)}\,\pi_d'(x).$$

Hence, the optimal solution (6) follows at once from (5). ∎

Remarks. i) Because the interval $[c, d]$ is bounded, the condition (2) is not restrictive. Furthermore, when b, q and v are all constant functions, then the condition (3) is automatically fulfilled.
ii) We see that the optimal control does not depend on the value of $K(d)$.
iii) In many applications, we would like to take $c = 0$. If the uncontrolled process $\{x(t), t \geq 0\}$ can attain the boundary at the origin, then we can indeed replace c by 0 in (6).

Particular cases.

I) First, if $m(x) \equiv 0$, then

$$\pi_d(x) = \frac{x - c}{d - c}$$

and

$$u^* = \frac{v(x)}{b(x)} \frac{1}{x - c} \quad \text{for } c < x < d.$$

Notice that this case includes the (controlled) standard Brownian motion, for which $m(x) \equiv 0$ and $v(x) \equiv 1$.

II) Next, assume that $q(x) \propto b^2(x)$. With $m(x) \equiv m_0 \neq 0$ and $v(x) \equiv v_0 > 0$, so that the uncontrolled process $\{x(t), t \geq 0\}$ is a Wiener process with drift coefficient m_0 and diffusion coefficient v_0, we find that

$$u^* = \frac{2\,m_0}{b(x)} \frac{1}{\exp\left\{\frac{2\,m_0}{v_0}\,(x-c)\right\} - 1} \quad \text{for } c < x < d.$$

III) Finally, if $\{x(t), t \geq 0\}$ is a geometric Brownian motion defined by $x(t) = e^{B(t)}$, then $m(x) = x/2$ and $v(x) = x^2$. The origin being a natural boundary for the geometric Brownian motion, we must take $c > 0$. The optimal control takes the form

$$u^* = \frac{x}{b(x)} \frac{1}{\ln(x/c)} \quad \text{for } 0 < c < x < d.$$

Here, $q(x)$ must be proportional to $b^2(x)/x^2$.

3. CONCLUSION

Based on the work presented in Lefebvre [1], we have solved the problem of optimally controlling a general diffusion process so that it leaves the continuation region (c, d) through the right-hand side of the interval. The objective could have been to leave through the left-hand side instead. Moreover, we could assume that $d = \infty$. In that case, we could try to maximize the time spent by the controlled process in the interval (c, ∞).

Finally, the same type of problem as the one solved here could be considered in two dimensions. For example, $(X_1(t), X_2(t))$ could be a controlled two-dimensional Brownian motion, and T be the first time that $X_1(t)$ hits the boundary $x_1 = c$. If the termination cost is a function of $X_2(T)$, then we would have to determine the distribution of this variable, which is a continuous rather than discrete random variable.

References

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