ROMAI J., **6**, 2(2010), 185–198

STUDIES AND APPLICATIONS OF ABSOLUTE STABILITY IN THE AUTOMATIC REGULATION CASE OF THE NONLINEAR DYNAMICAL SYSTEMS

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Abstract In this paper the methods of study of the automatic regulation of the absolute stability for some nonlinear dynamical systems are presented. Two methods for the absolute stability are specified: a) the Lurie method with the effective determination of the Lyapunov function; b) the frequencies method of the Romanian researcher V. M. Popov, that uses the transfer function in the critical cases. The applications envisaged here refer to metal cutting tools machine. Both analytical and numerical aspects are considered.

Keywords: : nonlinear systems, automatic control system, absolute stability, tools machine.

2000 MSC: 34D23.

1. INTRODUCTION

The automatic regulation for the stability of dynamical systems occupies a fundamental position in science and technique, and it is applied to the optimization of the technological process of cutting tools, of robots, of vehicles (or some machines components) movement regime, of energetic radioactive regimes, chemical, electromagnetic, thermal, hidroaerodynamic regimes, etc.

The complex technical achievements lead to complex mathematical models for closed circuits with input - output, following for the automatic regulation the integration of some mechanisms and devices with inverse reaction of response for the control and the fast and efficient elimination of the perturbations which can appears along these processes or dynamical regimes. Generally these dynamical regimes are nonlinear and some contributions and special achievements for automatic regulation, generating the automatic regulation of absolute stability (a.r.a.s.) for these classes of nonliniarities were necessary.

We highlight two special methods of a.r.a.s.:

1 Lyapunov's function method discovered by A.I. Lurie [10], [12], [17] and developed into a series of studies by M.A Aizerman, V.A. Iacubovici, F.R. Gantmaher, R.E. Kalman, D.R. Merkin [11] and others [1] [14];

2 Frequency method developed by researcher V. M. Popov [15] generalizing Nyquist's criterion, then developed in many studies [1], [2], [12].

We note the contributions of Romanian researchers recognized by the works and monographs on the stability and optimal control theory: C. Corduneanu, A. Halanay, V. Barbu, Vl. Rasvan, V. Ionescu, M.E. Popescu, S. Chiriacescu, A. Georgescu and also directly on a.r.a.s.: I. Dumitrache [4] D. Popescu 13], C. Belea [2], V. Rasvan [16], S. Chiriacescu [3] and other recent works [5] [6] [7] [8] [9].

The research has shown that both methods are equivalent, and studies can be performed qualitatively or numerically. In this paper we present the two methods and apply them to a concrete problem.

2. A.R.A.S. USING THE LYAPUNOV'S FUNCTION METHOD

In this part we'll present the Lurie's ideas and the effective method for finding the Lyapunov's function. [10] [11] [2] [16]

Generally, the systems of automatic regulation (s.r.a.) consist of the controlled processor system, sensor elements of measurement, acquisition board, and the mechanism feedback controller. The regulator comprises all the sensors and the acquisition board, but the controller is included in the feedback mechanism. Parameters characterizing the object control system (to control the working mode) are measured by sensors, and their records with the sensor response mechanism ζ is transmitted to the acquisition board.

This processes the command σ, which is mechanically transmitted to the controller which, on its turn, distributes the object state and interact simultaneously adjusting the response mechanism. We highlight below the dynamic system equations.

We denote by $x_1, x_2, \ldots, x_n$ the state parameters of the regime's subject which must be controlled, i.e. the coordinates and the sensorial speeds. We recall that the variation of these parameters of the open circuit (excluding the controller) system is described by linear differential equations with constant coefficients: $\dot{x}_k = \sum\limits_{j=1}^{n} a_{kj}x_j, k = 1, \ldots, n$. If the system is with closed loop, then the variables $x_1, x_2, \ldots, x_n$ will be under the influence of the regulation body, and we denote by ξ its state. In this case for the autonomous closed system we have the equations:

$$\dot{x}_k = \sum_{j=1}^{n} a_{kj}x_j + b_k\xi, k = 1, \ldots, n. \tag{1}$$

The relation between the output ζ and the input ξ is

$$\zeta = k\xi. \tag{2}$$

The acquisition board collects the signals and transmits them to the input sensors in order to obtain the embedded system

$$\sigma = \sum_{j=1}^{n} c_j x_j - r\xi, \tag{3}$$

where c_j, r are transfer numbers, $r > 0$ is the transfer coefficient of the inverse rigid connection (the regulator characteristics) [10], [11], [12].

The connection between the output (linear) function σ of the controller and the nonlinear input φ in the case of automatic regulation is expressed by the relation

$$\dot{\xi} = \varphi(\sigma). \tag{4}$$

The characteristic function of the controller $\varphi(\sigma)$, $\sigma \in (-\infty, +\infty)$ is continuous and satisfies the conditions [11], [6], [7]:

$$\begin{array}{ll} a) & \varphi(0) = 0, \\ b) & \sigma \cdot \varphi(\sigma) > 0, \quad \forall \sigma \neq 0, \\ c) & \int_0^{\pm\infty} \varphi(\sigma) d\sigma = \infty. \end{array} \tag{5}$$

The graph of the function φ lies in the quarters I, III. The functions $\varphi(\sigma)$ are named *admissible*. Moreover, we assume that the sector condition

$$0 < \frac{\varphi(\sigma)}{\sigma} < k \tag{6}$$

is satisfied, where k is the amplification coefficient.

Example 2.1. *1* $\varphi(\sigma) = sgn(\sigma) \cdot \ln(\sigma^2 + 1),\ k > 1;$

2 $\varphi(\sigma) = a(e^{\sigma} - 1),\ k \leq a.$

We assume that the $n \times n$ square matrix $A = \left\| a_{kj} \right\|$ is nonsingular. The equations (1), (3), (4) model the perturbed system with the zeros $x(0, 0, \ldots, 0),\ \xi = 0$.

By setting $B = \begin{pmatrix} b_1 \\ \cdots \\ b_n \end{pmatrix}$, $C = \begin{pmatrix} c_1 & \ldots & c_n \end{pmatrix}$, C' the transpose matrix of C, this system becomes

$$\dot{X} = AX + B\xi, \quad \dot{\xi} = \varphi(\sigma), \quad \sigma = C'X - r\xi, \quad X = \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}. \tag{7}$$

3. THE CANONICAL FORM AND THE LYAPUNOV FUNCTION OF SYSTEM (7)

We assume that A with $\det A = \Delta_0 \neq 0$ is Hurwitz, i.e. the characteristic polynomial

$$P(\lambda) = (-1)^n \det(A - \lambda E) \tag{8}$$

has simple roots with $Re(\lambda_k) < 0,\ k = 1, \ldots, n$.

The system (7) is brought to the canonical form if the matrix A is brought to the Jordan form $J = diagA = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$. There is a matrix $T = (t_{kj})$ such that

$$T^{-1}AT = J,\ (\Leftrightarrow\ AT = TJ),\ \det T \neq 0. \tag{9}$$

With the linear transform:

$$X = TY,\quad Y = \begin{pmatrix} y_1 \\ \cdots \\ y_n \end{pmatrix}, \tag{10}$$

from (7) we obtain: $T\dot{Y} = ATY + B\xi, \quad \dot{\xi} = \varphi(\sigma), \quad \sigma = C'TY - r\xi$, that implies:

$$\dot{Y} = JY + B_1\xi,\ \dot{\xi} = \varphi(\sigma),\ \sigma = C_1^{'}Y - r\xi,\ B_1 = T^{-1}B,\ C_1' = C'T. \tag{11}$$

Hence, with the linear transform:

$$Z = JY + B_1\xi,\ \sigma = C_1'Y - r\xi,\ Z = \begin{pmatrix} z_1 \\ \cdots \\ z_n \end{pmatrix}, \tag{12}$$

the problem (1),(4) becomes

$$\begin{cases} \dot{Z} = JZ + B_1\varphi(\sigma), \\ \dot{\sigma} = C_1'Z - r\varphi(\sigma). \end{cases} \tag{13}$$

The disturbed system (13) with the equilibrium solution ($z_k = 0, \sigma = 0$) is equivalent with system (7) with the equilibrium solution ($x_k = 0, \xi = 0$) and the transform (12) is nondegenerate if the determinant Δ is non-null:

$$\Delta = \begin{vmatrix} J & B_1 \\ C_1' & -r \end{vmatrix} \neq 0\ \Leftrightarrow\ r + C_1'J^{-1}B_1 \neq 0 \tag{14}$$

Returning to $J^{-1} = T^{-1}A^{-1}T,\ B_1 = T^{-1}B,\ C_1' = C'T$ transforms we obtain from (13) the final condition

$$r + C'A^{-1}B \neq 0. \tag{15}$$

Lurie's problem consists in establishing the conditions for the asymptotic stability of the the null solution ($x_k = 0,\ \xi = 0$) of system (7) (equivalent with (13)) with solution $z_k = 0,\ \sigma = 0$ for the initial perturbations and for any admissible functions $\varphi(\sigma)$ defined in (5), (6).

The conditions imposed on the matrix A and on the function $\varphi(\sigma)$ imply the absolute stability (a.s) of the system. [1] [13]

We remark that if $\varphi(\sigma)$ is linear, since the matrix A is Hurwitz, the systems (7), respectively (13) are asymptotic stable. The simplicity of system (13) entails immediate techniques for determining the Lyapunov function $V = V(z_1, \ldots, z_n, \sigma)$ attached to the system (13).

The function $V(z, \sigma)$ of class C^1 is a *Lyapunov function* for system (13) if $V(z = 0, \sigma = 0) = 0$, it is positive defined ($V(z, \sigma) > 0$), it is radially unlimited to ∞, with the absolute derivative $\dot{V} = \frac{dV}{dt}$ negative defined ($\frac{dV}{dt} < 0$) for ($z \neq 0, \sigma \neq 0$) in a vicinity of the equilibrium point. For the case of automatic regulation we search the function $V = V(z, \sigma)$ as the sum of a quadratic form z_k corresponding to the linear block A and an integral term corresponding to the non linear part

$$V(z, \sigma) = Z'PZ + \int_0^{\sigma} \varphi(\sigma) d\sigma = V_1(z, \sigma) + \int_0^{\sigma} \varphi(\sigma) d\sigma. \tag{16}$$

From theory [1] [4] $Z'PZ$ is the quadratic form defined strictly positive if the matrix P is symmetric ($P = P'$) and we have $A'P + PA = -Q$ where Q is symmetric and positive (with positive eigenvalues).

The integral term from (15) is strictly positive because of the conditions (5). Obviously, $V(z = 0, \sigma = 0) = 0$.

Next we impose the condition $\dot{V} < 0$ and obtain conditions for parameters c_k, r for a.r.a.s..

From (16), by using (13) and

$$Q = Q', P = P', B_1'PZ + Z'PB_1 = B_1'PZ + (PB_1)'Z = 2(PB_1)'Z,$$

for

$$\frac{dV(z, \sigma)}{dt} = Z'(J'P + PJ)Z - r\varphi^2(\sigma) + \varphi(\sigma)(B_1'PZ + Z'PB_1) + \varphi(\sigma)C_1,$$

we obtain

$$\frac{dV}{dt} = -Z'QZ - r\varphi^2(\sigma) + 2\varphi(\sigma)\left(PB_1 + \frac{1}{2}C_1\right)' Z, \tag{17}$$

$$\dot{V}(z = 0, \sigma = 0) = 0.$$

The connection between the matrices $P(p_{ij})$, $Q(q_{ij})$ can be established. From $\lambda_i + \lambda_j \neq 0, i, j = 1, \ldots, n, P = P', J = diagA$ and $Q = Q'$, we have $q_{ij} = -\left(\lambda_i p_{ij} + \lambda_j p_{ij}\right)$ that implies

$$p_{ij} = -\frac{q_{ij}}{\lambda_i + \lambda_j}. \tag{18}$$

Remark 3.1. *The matrix A is stable with $\lambda_i + \lambda_j \neq 0$ if Q is a quadratic form positive defined.*

Example 3.1. *If we choose $Q = E$ (E - the unit matrix) and P is obtained from (17), than the below observation is valid.*

We must have $(-\dot{V})$ positive defined (since $\dot{V} < 0$). Apply to the RHS of (17) the Silvester criterion demanding that all diagonal minors of (17) to be positive. Because Q is positive as quadratic form, than the first n inequalities are satisfied and the last inequality is

$$r > \left(PB_1 + \frac{1}{2}C_1\right)^{/} Q^{-1}\left(PB_1 + \frac{1}{2}C_1\right). \tag{19}$$

If the regulator parameters verify the conditions (15), (19) there are sufficient conditions for the asymptotic stability of the solution $(x = 0, \xi = 0)$ of the system (1), (3), (4) [10] [16].

Remark 3.2. *A choice technique of the quadratic form $V_1(z)$ for p_{ij} according to Lurie is:*

$$V_1(z) = \varepsilon \sum_{k=1}^{s} z_{2k-1}z_{2k} + \frac{\varepsilon}{2}\sum_{k=1}^{n-2s} z_{2s+k}^2 - \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{a_k z_k a_j z_j}{\lambda_k + \lambda_j}$$

where $a_1, a_2, \ldots, a_{2s}$ are complex conjugated, $a_{2s+1}, \ldots, a_n$ are real corresponding to roots λ_k determining the coefficients a_k .

Remark 3.3. *The two transforms for the diagonal system (1), (3), (4) to obtain (13) can be replaced directly by the transform [12]*

$$x_k = -\sum_{i=1}^{n} \frac{N_k(\lambda_i)}{D'(\lambda_i)} z_i, \tag{12$'$}$$

where from (7) $P(\lambda) = (-1)^n D(\lambda), N_k(\lambda) = \sum\limits_{i=1}^{n} b_i D_{ik}(\lambda)$, D_{ik} are the corresponding algebraic complements of (i,k) from $D(\lambda) = A - \lambda E$. In this case the simplified system analogous to (13) is:

$$\dot{z}_k = \lambda_k z_k + \varphi(\lambda), \quad \dot{\sigma} = \sum_{i=1}^{n} f_i z_i - r\varphi(\sigma), \quad k = 1, ..., n \tag{13$'$}$$

for which we will build easier $V(z, \varphi)$.

For the case when a root is null $(P(0) = 0)$ and all the others have $Re(\lambda_k) < 0, k = 1, ..., n-1$, the system (13), with $Z = \begin{pmatrix} \tilde{z} \\ z_1 \end{pmatrix}$, becomes:

$$\dot{\tilde{z}} = \tilde{J}\tilde{Z} + \tilde{B}_1\varphi, \quad \dot{z}_1 = b_0\varphi, \quad \dot{\varepsilon} = \tilde{C'}_1\tilde{Z} + C_0 z_1 - r\varphi, \tag{13$''$}$$

where $\tilde{J}$ is a $(n-1)\times(n-1)$ matrix, $\tilde{Z}$, $\tilde{B}_1$ are $(n-1,1)$ (column) matrices, while $\tilde{C}_1$ is a $(1,n-1)$ (row) matrix.

In this case the Lyapunov function is searched in the form

$$V(\tilde{z},z_1,\sigma) = az_1^2 + \left\{\tilde{z}'P\tilde{z} + \int_0^\sigma \varphi(\sigma)d\sigma\right\} \qquad (16').$$

For proofs and recent applications we recommend [2][12][11].

4. THE FREQUENCY METHOD FOR A.R.A.S.

This method obtained by V. M. Popov [15] is applied to the dynamical system with continuous nonlinearities. We present in this section the method and criteria given by Aizerman, Kalman, Jakubovici [16] [11].

Consider the dynamical, autonomous, non homogeneous system

$$\begin{aligned} \dot{x}_i &= \sum_{l=1}^{n} a_{il}x_l + b_i u, \quad i = 1,...,n, \\ \sigma &= \sum_{l=1}^{n} c_l x_l, \quad u = -\varphi(\sigma), \end{aligned} \tag{20}$$

where a_{il}, b_i, c_l are real constants, u is the arbitrary function of input, continuous, nonlinear with $\varphi(\sigma)$ and σ is the output function.

By using the Laplace transform, and by replacing the operator $\frac{d}{dt}$ with s we obtain from (20):

$$sx_i = \sum_{l=1}^{n} a_{il}x_l + b_i u, \quad \sigma = \sum_{l=1}^{n} c_l x_l, \quad i = 1,...,n. \tag{21}$$

Eliminating from (21) the characteristic parameters of the regulator we obtain

$$\sigma = W(s)u, \quad \sigma = W(s)(-\varphi), \tag{22}$$

where $W(s) = \frac{Q_m(s)}{Q_n(s)}$ is the transfer function and $Q(s)$ are polynomials of degrees m, respectively n, with $m < n$ [4] [5] [13].

The transfer function connects σ and φ; the function φ satisfies the conditions (5) and the sector condition (6) $0 < \frac{\varphi(\sigma)}{\sigma} < k \leq \infty$ - the plot $\varphi = \varphi(\sigma)$ in the plane (σ,φ) will be the sector $0 \leq \varphi(\sigma) \leq k\sigma$. The sector condition and the nonlinearity of φ determine the system (σ,φ) with closed loop through the impulse function φ.

We study the absolute stability of the null solution $(x = 0, u = 0)$ of system (20). Because the system is closed and nonlinear we can't apply directly the Nyquist criterion, [4] [5] [15]. If $\varphi \equiv k\sigma$ then the system is linear and this criterion can be applied.

Since the block $\sum a_{il}x_l$ is linear and $b_i u$ is nonlinear it results that the roots of the characteristic polynomial $P(\lambda) = (-1)(A - \lambda E) = 0, \quad P(\lambda_i) = 0$, the poles of $W(s)$ and k will influence the determination of the absolute stability criteria.

From $W(s = j\omega) = U(\omega) + jV(\omega), \quad j = \sqrt{-1}$ we have the hodograph for the axes (U, V) [2] [4] [5] [6] [12]:

$$U = U(\omega), \quad V = V(\omega), \quad 0 \leq \omega \leq \infty. \tag{23}$$

If all poles of $W(s)$ have $Re(s_i) < 0$ then the system is uncritical; if some of the poles of $W(s)$ are null or on the imaginary axis and the rest have $Re(s_i) < 0$ then the system is in the critical case.

We enunciate the criteria for absolute stability of automatic control a.r.a.s. by the frequency method.

Criterion 1. *(the uncritical case). Assume that the following conditions are satisfied for system (20):*

a) The function $\varphi(\sigma)$ satisfy (5), (6),

b) All poles of $W(s)$ have $Re(s_i) < 0$,

c) There is a $q \in R$ such that for any $\omega \geq 0$ the condition

$$\frac{1}{k} + Re\left[(1 + j\omega q)\, W(j\omega)\right] \geq 0 \tag{24}$$

holds.

Then the system (20) is automatic regulated and absolute stable for the null solution $(x = 0, u = 0)$.

From (24) we obtain

$$\frac{1}{k} + U(\omega) - q\omega V(\omega) \geq 0. \tag{24'}$$

From a geometric point of view, criterion (24) shows that in the plane $U_1 = U, \ V_1 = \omega V$ there exists the line

$$\frac{1}{k} + U_1 - qV_1 = 0 \tag{24''}$$

passing through $\left(-\frac{1}{k}, 0\right)$ such that the plot of the hodograph is under this line for $\omega \geq 0, \ k > 0$.

Criterion 2. *(the critical case when there is a simple null pole $s_0 = 0$). Assume that the following conditions are satisfied:*

a) The function φ verify (5), (6),

b) $W(s)$ has a simple null pole, and the others poles s_i have $Re(s_i) < 0$, *c)* $\rho = \lim_{s\to 0} sW(s) > 0$ *and there is a* $q \in R$ *such that for any* $\omega \geq 0$ *condition (24) holds. Then for the system (20) for the null solution we have a.r.a.s.*

Criterion 3. *(the critical case when* $s = 0$ *is a double pole). Assume that the following conditions are satisfied:*

a) The function $\varphi(\sigma)$ *verify (5), (6) and the sector condition for* $k = \infty$*,*

b) $W(s)$ *has a double pole in s=0 and the other poles have* $Re(s_i) < 0$*,*

c) $\rho = \lim_{s\to 0} s^2 W(s) > 0$, $\mu = \lim_{s\to 0} \frac{d}{ds}\left[s^2 W(s)\right] > 0$ *and* $\pi(\omega) = \omega Im W(j\omega) < 0$ *for* $\forall\omega \geq 0$.

Then for the system (20) we have a.r.a.s. for the null solution.

Remark 4.1. *The form of these criteria (1, 2, 3) has an analytical character but their verification for the construction of hodograph values of the coefficients must be done numerically. For special cases we recommend the monographs [2] [4] [12] [16].*

5. THE ABSOLUTE STABILITY IN THE AUTOMATIC REGULATION OF METAL CUTTING

The high precision of the metal cutting tools implies an automatic regulation of the processes.

In this section we present the original results for modeling and the for the study of the nonlinear dynamics of the cutting processes (CP) with tools into metal blocks, for composite materials, blocks or hardwood, practically defined in [3].

These (CP) are: CP of drilling, CP of milling, CP of grinding, screw machine, spindle bearing. Machine tool bar is provided with an inner elastic hard metal cutting, cutting inside to run the required geometric rotation and advancing to step slow. Because of the variation in hardness, density, coefficient of elasticity, material composition manufactured by the process disturbances will occur in work mode: transverse vibration due to shaft rotation or longitudinal vibrations to advance. The automatic controller is equipped with sensors, micrometers, tensiometers, rigid response mechanisms of signals output power amplifiers and accelerators. Their purpose is to adjust the characteristics to obtain asymptotic stability of the system work, resulting in high precision components. We apply the two methods described in the previous sections.

5.1. THE A.R.A.S. METHOD BY LYAPUNOV FUNCTION

Consider the dynamic system modeled mathematically, brought to a canonical autonomous form, that features automatic adjustment for absolute stability of dynamic cutting machining processes. [3] [14]

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + b_1\xi \\ \dot{x}_2 = a_{23}x_3 \\ \dot{x}_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \\ \dot{x}_4 = a_{44}x_4 + b_4\xi \end{cases} \quad \sigma = c_2x_2 + c_4x_4 - r\xi, \dot{\xi} = \varphi(\sigma) \tag{25}$$

where a_{ij}, b_i, r, ξ are constants $i, j = 1, 2, 3, 4$, and

$$\begin{array}{l} a_{11} = -m < 0, a_{31} = n > 0,\ a_{32} = -\varepsilon n < 0,\ a_{33} = -p < 0,\ a_{44} = -l < 0 \\ a_{23} = 1,\ c_2 = 1,\ c_4 = c < 0,\ b_1 = b > 0,\ b_4 = d - r > 0, r > 0. \end{array} \tag{26}$$

These represent mass inertia, elastic constants, strain or pressure coefficients, and σ, r, ξ are the characteristics of the controller. We assume that the input function φ is generally nonlinear and the conditions (5) (6) hold. We observe that the linear response function σ of the controller evaluate the elements x_2 - the speed of rotation of the cutting bar and x_4 - the speed of advancing its material, [3].

We check the absolute stability of the zero solution of the system $(x = 0, \xi = 0)$. We have $\det A \neq 0$. Moreover, $Re(\lambda_i) < 0, i = 1, 2, 3, 4$, as the following computations show:

$$\begin{aligned} P(\lambda) = D(\lambda) = &\begin{vmatrix} a_{11} - \lambda & 0 & 0 & 0 \\ 0 & -\lambda & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} - \lambda & 0 \\ 0 & 0 & 0 & a_{44} - \lambda \end{vmatrix} = \\ = &\ (a_{11} - \lambda)(a_{44} - \lambda)(\lambda^2 - \lambda a_{33} - a_{23}a_{32}) = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} &\lambda_1 = a_{11} = -m < 0, \lambda_4 = a_{44} = -l < 0, \\ &\lambda_{2,3} = \frac{1}{2}\left(-p \pm \sqrt{p^2 - 4\varepsilon n}\right) < 0, \lambda_i \in R \end{aligned} \tag{27}$$

In this case, following the diagonalization method (Section 3) with the formulas (9) - (13) or by directly choosing the alternative from Remark 3.3 we get the diagonal system in z_i and σ (12'), (13'):

$$\dot{z}_i = \lambda_i z_i + \varphi(\sigma);\quad \dot{\sigma} = \sum_{i=1}^{4} f_i z_i - r\varphi(\sigma),\quad i = 1, ..., 4, \tag{29}$$

$$\begin{aligned} f_1 = \frac{b_1a_{31}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)},\ f_2 = \frac{-b_1a_{31}}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)} \\ f_3 = \frac{b_1a_{31}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}, f_4 = b_4c_4 < 0. \end{aligned} \tag{29}$$

We observe that $f_1 + f_2 + f_3 = 0$ and whatever is the choice of order quantities, $\lambda_1, \lambda_2, \lambda_3$ are strictly negative, and always two of the functions $f_i, i = 1, 2, 3$ have the same sign and the third function takes opposite sign using relation (26).

In this case, if $f_4 < 0$ we can construct such Lyapunov function:

$$V(z,\sigma) = -\frac{1}{2} f_4 z_4^2 - \frac{1}{2}\sum_{i=1}^{3}\frac{a_i^2 z_i^2}{\lambda_i} - \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{a_i a_j}{\lambda_i + \lambda_j} z_i z_j + \int_0^{\sigma}\varphi(\sigma)d\sigma \tag{31}$$

where the real coefficients a_1, a_2, a_3 will be determined.

From $\lambda_i < 0,\ \lambda_i + \lambda_j < 0,\ V(0) = 0$, the terms after $i = 1, 2, 3$ determine a positive definite quadratic form and since the integral is positive, we have $V(z, \sigma) > 0$ in a neighborhood of the null solution.

We calculate $\dot{V}(z, \sigma)$ for the function V defined above:

$$\dot{V}(z,\sigma) = -f_4\lambda_4 z_4^2 - (a_1z_1 + a_2z_2 + a_3z_3)^2 - \varphi\sum_{i=1}^{3} z_i\left(\frac{a_i^2}{\lambda_i} + \sum_{j\neq i}\frac{2a_ia_j}{\lambda_i+\lambda_j} - f_i\right). \tag{32}$$

We remark that $\dot{V}(z = 0, \sigma = 0) = 0$ and in order to have the strict negativity, all the parentheses from the sum that multiplies φ must be null, that is

$$\frac{a_i^2}{\lambda_i} + \sum_{j\neq i}\frac{2a_ia_j}{\lambda_i+\lambda_j} - f_i = 0,\quad i = 1,2,3 \tag{33}$$

The system (33) comprises three equations $F_i(a_1, a_2, a_3) = 0, i = 1, 2, 3$ with three unknowns and the local existence of solutions is ensured by the condition on the system Jacobian: $J = \frac{D(F_1,F_2,F_3)}{D(a_1,a_2,a_3)} \neq 0$. If each equation from (33) is multiplied respectively by $\frac{1}{\lambda_i}$ and all equations are summed, then condition (33) proves to be equivalent to

$$S := \sum_{i=1}^{3}\frac{f_i}{\lambda_i} = \left(\sum_{i=1}^{3}\frac{a_i}{\lambda_i}\right)^2 > 0. \tag{34}$$

This condition indicates that the known sum (S) is strictly positive and in the parametric space (a_1, a_2, a_3) the plane (π_{12}) $\sum\limits_{i=1}^{3}\frac{a_i}{\lambda_i} = \pm\sqrt{S}$ where exists a solution, does not admit the null solution because $f_i \neq 0$.

The Jacobian J can be shown to be

$$J = \pm 8\frac{\sqrt{S}(a_1 + a_2 + a_3)[a_1(\lambda_2 + \lambda_3) + a_2(\lambda_1 + \lambda_3) + a_3(\lambda_1 + \lambda_2)]}{\lambda_1\lambda_2\lambda_3(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} \neq 0.$$

The condition above shows that the solution is contained in the plane (π_{12}), but not in the intersections of this plane with the planes $a_1 + a_2 + a_3 = 0$ or $a_1(\lambda_2 + \lambda_3) + a_2(\lambda_1 + \lambda_3) + a_3(\lambda_1 + \lambda_2) = 0$.

Analyzing the system (33) by the sign of f_i, we see that a_1, a_2, a_3 can not have all the same sign. If this would happen, then $f_i > 0$. We proved that there exist solutions of system (33) and, for these, $\dot{V} < 0$; it follows that the Lyapunov function provide the automatic regulation of the absolute stability. For this application sufficient conditions of type (15), (16) with numerical data, are also obtained.

5.2. THE FREQUENCY METHOD FOR A.R.A.S

In the following study the frequency method presented in Section 4 will be applied to problem (25), (26). Because the system (25) is equivalent with (29) the function $u = -\varphi(s)$ satisfies the sector conditions. By applying the Laplace transform, the transfer function $W(s)$ is found. So, from (29) is obtained, for $\sigma = W(s)(-\varphi)$

$$sz_i = \lambda_i z_i + \varphi; \quad s\sigma = \sum_{i=1}^{4} f_i z_i - r\varphi. \tag{35}$$

Eliminating z_i from (35) we obtain the transfer function from $\sigma = W(s)(-\varphi)$

$$W(s) = \frac{1}{s}\left(r - \sum_{i=1}^{4} \frac{\lambda_i f_i}{s - \lambda_i}\right). \tag{36}$$

Because the real roots $s_i = \lambda_i$ satisfy $Re(\lambda_i) < 0$, the transfer function has a simple pole in $s = 0$ and the rest of real roots with $Re(s_i) < 0$. In this case we may use the Criterion 2 of critical singularity from Section 4 for a.r.a.s.. Here, the conditions (15), (19) and II a), b) were verified in the preceding subsection and only condition c) must be verified.

$\rho = \lim\limits_{s\to 0} sW(s) = r + \sum\limits_{i=1}^{4} f_i = r + f_4 = r + b_4 c_4 > 0$ implies $r + c(d - r) > 0$ that means $r > \frac{cd}{1-c} > 0, \;\; d < 0$. From $W(s = j\omega) = U(\omega) + jV(\omega)$, we have:

$$U(\omega) = \sum_{i=1}^{4} \frac{f_i \lambda_i}{\lambda_i^2 + \omega^2}, \quad V(\omega) = -\frac{r}{\omega} + \frac{1}{\omega} \sum_{i=1}^{4} \frac{\lambda_i^2 f_i}{\lambda_i^2 + \omega^2}.$$

For given $k > 0$, from the condition $0 < \varphi(s) < k\sigma$ with $\varphi(\sigma)$ specified, some $q \in R$ verifying the condition (24') may be determined. The parameters λ_i, f_i are known from (28), (30), the nonlinear function φ is chosen with σ from (25) and for specified numerical k and, consequently, the delimitation of q are determined. The existence of these conditions can be checked hodographically for a.r.a.s. at this application.

6. CONCLUSION

In the paper, two methods for a.r.a.s. very useful in the fundamental and applicative research, are presented. Their use is exemplified in the application in Section 5. For other studies the published results of the researchers, the works [11] [12] [16] [17] are recommended.

7. ACKNOWLEDGEMENT

This work was supported by CNCSIS - IDEI Program of Romanian Research, Development and Integration National Plan II, Grant no. 1044/2007 and "Automatics, Process Control and Computers" Research Center from University "Politehnica" of Bucharest (U.P.B.-A.C.P.C.) projects.

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