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THE FIRST EXAMPLE OF MAXIMAL ITERATIVE ALGEBRA OF THE FUNCTIONS OF TOPOLOGICAL BOOLEAN ALGEBRA OF ORDER 16 WITH 3 OPEN ELEMENTS

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Abstract A.I. Mal'tsev [6] proposed the problem to obtain the description of iterative algebras of functions in propositional logics. From functional point of view a special interest represent the separation and description of maximal iterative algebras. L. Esakia and V. Meshi [1] and independly L.Maksimova [5] discovered the existence of pre-tabular modal logic EM4. It is approximated by the logics of a series of topological Boolean algebras $\Delta_i (i = 1, 2, \dots)$ of order 2^i with 3 open elements. In this work, for first time, a maximal subalgebra of iterative algebra of functions of topological Boolean algebra of order 16 with one open atom is built.

Keywords: iterative algebra, maximal algebra, topological boolean algebra.

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1. INTRODUCTION

The study of iterative algebras, initiated by E. Post [7] and A.I. Mal'tsev [6], is closely related to functional problems of the theory of multivalent logics and of logical calculi. Any iterative algebra is supported by a closed class of k-valent ($k = 2, 3, \dots$) logic functions or by a functions of a logics defined by logical calculi. The superposition, in its diverse variations, plays an essential role as signature of these algebras. But let us remember that any superpositions may be expressed easily by a finite number of special superposition indicated, for example, in the work [6, p.25]. We can conclude that iterative algebras have finite signature. Therefore the research in terms of such algebras is equivalent to the investigation of their bases, which, in most cases, are closed classes of k-valent logic functions.

The general problem of description of closed classes for k-valent logic, with $k \geq 3$, is equivalent to the problem of the description of corresponding iterative algebras, but it is complicated in principle by the existence of subalgebras with infinite numerable bases as well as by the existence of subalgebras without base [9, 11]. Consequently, the general problem of the description of iterative algebras remains open. In more details was examined the problem of (functional) completeness in some iterative al-

gebra. It requires to be clarified necessary and sufficient conditions that an arbitrary system of its elements to generate full algebra. A proper subalgebra of a given algebra is called *maximal* if supplementing it with any element of the difference of these algebras generate the whole given algebra [6]. The maximal subalgebras play a very special role in the completeness problem of systems in the iterative algebras. Remember that in any k-valent logic to any maximal subalgebra it corresponds respectively some pre-complete class of functions or formulas.

In this work for first time it is built one maximal subalgebra of iterative algebra of topological Boolean algebra of order 16 with 3 open elements including 2 trivial items and one atom.

2. PRELIMINARIES

By a topological Boolean algebra [8] we mean any universal algebra $< M;\ \&, \vee, \supset, \neg, \Box >$, such that the system, $< M;\ \&, \vee, \supset, \neg >$ is a Boolean algebra, and the symbol $\Box$ represents the operation of *taken inside*. In this work a significant role will be played by the series of following topological boolean algebras: Δ_1 with 2 elements, Δ_2 with 4 elements, ..., Δ_n with 2^n elements, ..., every of them having 2 trivial open elements and one open atom. This algebras can be represented by a series of diagrams, from which we present the following four (see the Figures 1 and 2):

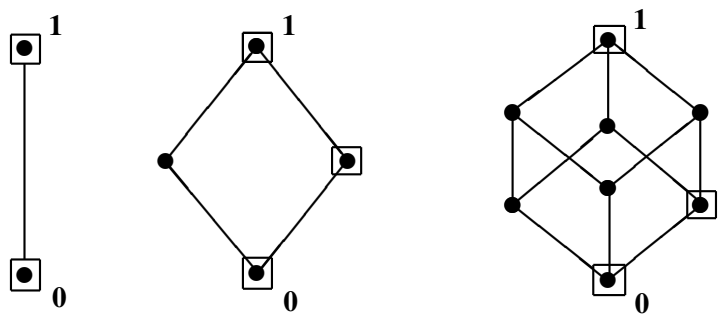

Fig. 1. The diagrams of Δ_1, Δ_2 and Δ_3 algebras

The elements 0 and 1 represent the smallest and respectively the greatest elements of algebra, while other elements of Δ_i $(i = 2, 3, 4)$ are denoted by small letters of Greek alphabet. The *open elements* of algebra are marked by $\Box$ (square). The *conjunction* of two elements of algebra is determined on the diagram by the *lowest edge up* and the *disjunction* - by the *biggest bound down* to those elements. The inside of any element α, i.e. $\Box\alpha$, is the *greatest open element* that is less than α. In [1, 5] it was remarked that the series of algebras $\Delta_1, \Delta_2, \ldots$ generates a variety of topological Boolean algebras. It is known [2] that the modal logics represent an important family of logics which, unlike k-valent logic, not every of them can be described by finite models. Between topological Boolean algebras and modal logics there is a closely

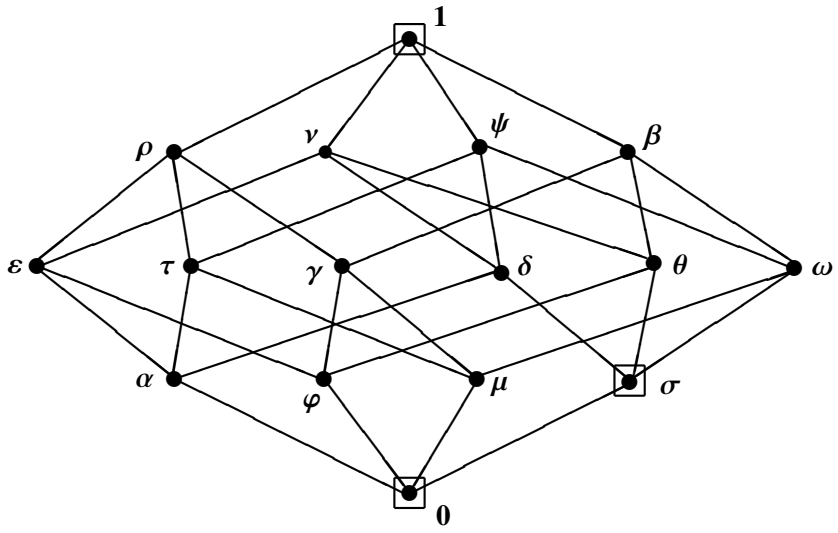

Fig. 2. The diagram of Δ_4 algebra

relation. Namely, for any modal logic L there exists a topological Boolean algebra whose logic coincides with L.

Modal formulas are traditionally built from variables $p, q, r, \ldots$ (sometimes with indexes) with logical operators & (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\neg$ (negation), $\Box$ (necessity) and parentheses. Operator $\neg\Box\neg$ (possibility) is denoted by diamond $\diamond$. Let us notice that the logical operators and the operations of topological Boolean algebra signature are denoted by the same symbol, which facilitates the interpretation of formulas on algebras. Formulas $(p \,\&\, \neg p)$ and $(p \supset p)$ are called *constants* and we note them by ciphers 0 and 1, respectively. The formulas usually we denote by capital letters of the Latin alphabet. The formula $(A \supset B)\,\&(B \supset A)$ is called the *equivalence* of formulas A and B and it is denoted by symbol $A \sim B$. An arbitrary function which is realized by a formula F after its interpretation on a topological Boolean algebra Δ will be called a *function* F of the algebra Δ.

When interpreting modal formulas on a topological Boolean algebra Δ, the set of all modal formulas identically equal to 1 on Δ constitutes a modal logic that is denoted by $L\Delta$. The best-known modal logics are the logic $S4$ and the logic $S5$ [2], the latter being simpler and it responding better to ancient thoughts about modalities.

A modal logic is called *tabular* if there is a finite topological Boolean algebra Δ such that $L = L\Delta$. A modal logic is called *pre-tabular*, if it is not tabular, but any of its proper extensions is already a tabular logic. The logic $S5$ is a well known example [5] of pre-tabular modal logic. In the works [1, 5] one new pre-tabular modal logic - the logic of $EM4$ was presented. This logic is obtained by adding to $S4$ the following three formulas as new axioms:

$$\Box \diamond p \supset \diamond\Box p, \qquad T(Z), \qquad T(Y),$$

where

$$Z = ((p \supset q) \vee (q \supset p)), \qquad Y = ((\neg\neg p \,\&((p \supset q) \supset q)) \supset q),$$

and the operator T means prescribing square $\Box$ before of each subformula. This modal logic corresponds fully to the variety of topological Boolean algebras generated by the series of algebras $\Delta_1, \Delta_2, \ldots$ [1, 5].

Following A.V. Kuznetsov [3, 4] , a formula F is called *expressible* in a logic L by a system of formulas Σ, if F can be obtained from the variables and formulas of Σ with a finite number of applications of weak rule of substitution which allows the transition from two formulae by substitution of one of them into the another instead of all entries of a variable, and the rule replacement by an equivalent in the logic L which allows the passage of one formula to any another formula equivalent to it in L. A formula is called *direct expressible* by Σ if it can be obtained from the variables and formulas of Σ with a finite number of applications of weak rule of substitution.

We will say that a modal formula $F(p_1, \ldots, p_n)$ preserves a predicate $R(x_1, \ldots, x_m)$ on topological Boolean algebra Δ if, for any elements $\alpha_{ij} \in \Delta$ $(i = 1, \ldots, m; j = 1, \ldots, n)$, from the fact that are true the sentences

$$R(\alpha_{11}, \ldots, \alpha_{m1}), \ldots, R(\alpha_{1n}, \ldots, \alpha_{mn})$$

it result that the statement

$$R(F[\alpha_{11}, \ldots, \alpha_{1n}], \ldots, F[\alpha_{m1}, \ldots, \alpha_{mn}])$$

are true. If the predicate R is defined on a finite algebra of a degree not too large, then R often can be given by its determining matrix

$$\beta_{ij} \; (i = 1, \ldots, m; j = 1, \ldots, l)$$

whose elements belong to the algebra Δ, so that following sentence is true

$$R(\beta_{1k}, \ldots, \beta_{mk}) \text{ for } k = 1, \ldots, l.$$

For example, each of formulae $\neg p$, $\Box p$ and $(p \,\&\, q)$ preserves on the algebra Δ_4 the following matrix that will be used below

$$\begin{pmatrix}
0 & \alpha & \varphi & \mu & \varepsilon & \tau & \gamma & \rho & \sigma & \delta & \theta & \omega & \nu & \psi & \beta & 1 \\
0 & \alpha & \mu & \varphi & \tau & \varepsilon & \gamma & \rho & \sigma & \delta & \omega & \theta & \psi & \nu & \beta & 1 \\
0 & \varphi & \alpha & \mu & \varepsilon & \gamma & \tau & \rho & \sigma & \theta & \delta & \omega & \nu & \beta & \psi & 1 \\
0 & \varphi & \mu & \alpha & \gamma & \varepsilon & \tau & \rho & \sigma & \theta & \omega & \delta & \beta & \nu & \psi & 1 \\
0 & \mu & \alpha & \varphi & \tau & \gamma & \varepsilon & \rho & \sigma & \omega & \delta & \theta & \psi & \beta & \nu & 1 \\
0 & \mu & \varphi & \alpha & \gamma & \tau & \varepsilon & \rho & \sigma & \omega & \theta & \delta & \beta & \psi & \nu & 1
\end{pmatrix}. \tag{1}$$

Taking into account the fact that disjunction and implication are expressible in classical logic by means of conjunction and negation, it follows that any formula preserves the matrix (1) on Δ_4 algebra.

3. THE MAIN RESULT

For any function $f(p_1, \ldots, p_n)$ of algebra Δ_4 let us consider it n-dimensional (i.e. with n entries) table. This picture (i.e. the part where are located function values and not the argument values) is divided into sectors so as be satisfied that: two arbitrary cells with two sets of coordinates $< \alpha_1, \ldots, \alpha_n >$ and $< \beta_1, \ldots, \beta_n >$ belong to the same sector then and only then when it take place the equality

$$\diamond\Box\alpha_i = \diamond\Box\beta_i \quad (i = 1, \ldots, n).$$

For example, on the tables below of some functions (i.e. derivative operations) of algebra Δ_4 the unique lines just divides the table in such sectors.

Table 1

p	0	α	φ	μ	ε	τ	γ	ρ	σ	δ	θ	ω	ν	ψ	β	1
$\Box p$	0	0	0	0	0	0	0	0	σ	σ	σ	σ	σ	σ	σ	1
$\diamond p$	0	ρ	ρ	ρ	ρ	ρ	ρ	ρ	1	1	1	1	1	1	1	1
$\diamond\Box p$	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
$\neg p \,\&\, \diamond p$	0	γ	τ	ε	μ	φ	α	0	ρ	γ	τ	ε	μ	φ	α	0
$p \supset \Box p$	1	β	ψ	ν	ω	θ	δ	σ	1	β	ψ	ν	ω	θ	δ	1
$\diamond p \supset p$	1	δ	θ	ω	ν	ψ	β	1	σ	δ	θ	ω	ν	ψ	β	1

A cell of sector is called *main*, if all its coordinates are equal to 0 or 1. Obviously, any sector has a single main cell. If the coordinates of the main cell of a sector constitute the collection $< \alpha_1, \ldots, \alpha_n >$, then we will denote this sector by symbol $Q < \alpha_1, \ldots, \alpha_n >$.

Theorem 3.1. *The iterative algebra of the functions of algebra Δ_4, which preserves the following line matrix with 12 elements*

$$\begin{pmatrix} 0 & \alpha & \mu & \varepsilon & \gamma & \rho & \sigma & \delta & \omega & \nu & \beta & 1 \end{pmatrix} \tag{2}$$

is maximal in the iterative algebra of all functions of Δ_4 algebra.

Indeed, we denote by Γ the iterative algebra of all functions of algebra Δ_4, which preserve the matrix (2). The fact that algebra Γ is proper subalgebra of iterative algebra of all functions of Δ_4 algebra results from fact that formula $p \vee q$ does not preserve matrix (2), because there is equality $\psi = \delta \vee \omega$ which shows that the disjunction of 2 elements δ and ω of the matrix (2) is equal to the element ψ that does not belong to the matrix (2).

Let us notice now that any element of the matrix (2) belongs to one of the following 2 isomorphic subalgebras of algebra Δ_4:

$$< 0, \alpha, \gamma, \rho, \sigma, \delta, \beta, 1; \Omega >, \qquad < 0, \mu, \varepsilon, \rho, \sigma, \omega, \nu, 1; \Omega >,$$

Table 2 $p \vee q$

$p\backslash q$	0	α	φ	μ	ε	τ	γ	ρ	σ	δ	θ	ω	ν	ψ	β	1
0	0	α	φ	μ	ε	τ	γ	ρ	σ	δ	θ	ω	ν	ψ	β	1
α	α	α	ε	τ	ε	τ	ρ	ρ	δ	δ	ν	ψ	ν	ψ	1	1
φ	φ	ε	φ	γ	ε	ρ	γ	ρ	θ	ν	θ	β	ν	1	β	1
μ	μ	τ	γ	μ	ρ	τ	γ	ρ	ω	ψ	β	ω	1	ψ	β	1
ε	ε	ε	ε	ρ	ε	ρ	ρ	ρ	ν	ν	ν	1	ν	1	1	1
τ	τ	τ	ρ	τ	ρ	τ	ρ	ρ	ψ	ψ	1	ψ	1	ψ	1	1
γ	γ	ρ	γ	γ	ρ	ρ	γ	ρ	β	1	β	β	1	1	β	1
ρ	ρ	ρ	ρ	ρ	ρ	ρ	ρ	ρ	1	1	1	1	1	1	1	1
σ	σ	δ	σ	ω	ν	ψ	β	1	σ	δ	θ	ω	ν	ψ	β	1
δ	δ	δ	ν	ψ	ν	ψ	1	1	δ	δ	ν	ψ	ν	ψ	1	1
θ	θ	ν	θ	β	ν	1	β	1	θ	ν	θ	β	ν	1	β	1
ω	ω	ψ	β	ω	1	ψ	β	1	ω	ψ	β	ω	1	ψ	β	1
ν	ν	ν	ν	1	ν	1	1	1	ν	ν	ν	1	ν	1	1	1
ψ	ψ	ψ	1	ψ	1	ψ	1	1	ψ	ψ	1	ψ	1	ψ	1	1
β	β	1	β	β	1	1	β	1	β	1	β	β	1	1	β	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

where $\Omega = \{\&, \vee, \supset, \neg, \Box\}$. Therefore, by applying any unary formula to the elements of matrix (2), only elements of this matrix can be obtained. Thus, any unary formula belongs to subalgebra Γ. It can be easily checked that the formulas $\Box p \,\&\, q$ and $\Box p \vee q$ belong to subalgebra Γ.

Let us denote by $S_0(p, q)$ the following binary formula

$$\Box(p \vee q) \vee \Box(p \supset q) \vee \Box(q \supset p) \vee \Box(\neg p \vee \neg q).$$

Notice that this formula belongs to algebra Γ because the formula S_0 can take only the values 0, 1 or σ which belong to the matrix (2). Let present the table of 2-dimensional function S_0.

Next we need following formula:

$$\begin{aligned} T_0(p, q) \quad = \quad & (\diamond\Box p \vee \diamond\Box q \vee S_0[\neg(\diamond p \supset p), \neg(\diamond q \supset q)])\& \\ & \&(\diamond\Box p \vee \Box\neg\Box q \vee S_0[\neg(\diamond p \supset p), q])\& \\ & \&(\Box\neg\Box p \vee \diamond\Box q \vee S_0[p, \neg(\diamond q \supset q)])\& \\ & \&(\Box\neg\Box p \vee \Box\neg\Box q \vee S_0(p, q)). \end{aligned}$$

For use this formula we present below her 2-dimensional table on algebra Δ_4.

On the basis of the tables of $p \vee q$ and of T_0 it is not difficult to verify that the following lemma holds.

Table 3 S_0

$p\backslash q$	0	α	φ	μ	ε	τ	γ	ρ	σ	δ	θ	ω	ν	ψ	β	1
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
α	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
μ	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ε	1	1	1	1	1	σ	σ	1	1	σ	σ	1	1	1	1	1
τ	1	1	1	1	σ	1	σ	1	1	σ	1	σ	1	1	1	1
γ	1	1	1	1	σ	σ	1	1	1	1	σ	σ	1	1	1	1
ρ	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
σ	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
δ	1	1	1	1	σ	σ	1	1	1	1	σ	σ	1	1	1	1
θ	1	1	1	1	σ	1	σ	1	1	σ	1	σ	1	1	1	1
ω	1	1	1	1	1	σ	σ	1	1	σ	σ	1	1	1	1	1
ν	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
β	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Lemma 3.1. *Formula*

$$D = (T_0(p,q) \supset (p \vee q))$$

belongs to subalgebra Γ.

The table of formula D on algebra Δ_4 can be described by the following scheme

$$D(p,q) = \begin{cases} \rho & \text{in 6 cells in which the value of } T_0 \text{ is } \sigma \text{ of the} \\ & \text{sector } Q < 0,0 >, \\ 1 & \text{in the other 18 cells where the value of } T_0 \text{ is } \sigma, \\ p \vee q & \text{in remaining cells.} \end{cases}$$

Next we need the following formula:

$$\begin{aligned} T_1(p,q) \quad = \quad & (\diamond\Box p \vee \diamond\neg\Box q \vee S_0(p,q))\& \\ & \&(\diamond\Box p \vee \Box\neg\Box q \vee S_0[p,(q \supset \Box q)])\& \\ & \&(\Box\neg\Box p \vee \diamond\Box q \vee S_0[(p \supset \Box p), q])\& \\ & \&(\Box\neg\Box p \vee \Box\neg\Box q \vee S_0[(p \supset \Box p),(q \supset \Box q)]). \end{aligned}$$

Lemma 3.2. *Formula* $K = T_1 \supset (p \,\&\, q)$ *belongs to subalgebra* Γ.

Table 4 T_0

$p\backslash q$	0	α	φ	μ	ε	τ	γ	ρ	σ	δ	θ	ω	ν	ψ	β	1
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
α	1	1	σ	σ	1	1	1	1	1	1	σ	σ	1	1	1	1
φ	1	σ	1	σ	1	1	1	1	1	σ	1	σ	1	1	1	1
μ	1	σ	σ	1	1	1	1	1	1	σ	σ	1	1	1	1	1
ε	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
τ	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
γ	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ρ	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
σ	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
δ	1	1	σ	σ	1	1	1	1	1	1	σ	σ	1	1	1	1
θ	1	σ	1	σ	1	1	1	1	1	σ	1	σ	1	1	1	1
ω	1	σ	σ	1	1	1	1	1	1	σ	σ	1	1	1	1	1
ν	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
β	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

By using the Lemma 3.1 and Lemma 3.2 it is not difficult to finish the proof of Theorem 3.1.

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