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ALTMAN ORDERING PRINCIPLES AND DEPENDENT CHOICES

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Abstract Some ordering principles related to Altman's [Nonlinear Analysis, 6 (1982), 157-165] are equivalent with the Dependent Choices Principle. This is also true for many intermediary statements, including the Szaz unboundedness criterion [Math. Moravica, 5 (2001), 1-6].

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1. INTRODUCTION

Let M be some nonempty set; and $(\leq)$, some *quasi-order* (i.e.: reflexive and transitive relation) over it. Denote $M(x, \leq) := \{t \in M; x \leq t\}$, $x \in M$; the subsets $M(x, \geq)$, $x \in M$, are introduced in a similar way, modulo $(\geq)$ (=the *dual* quasi-order). Further, take some application $F : M \times M \to R$ with the properties

(a01) F is $(\leq)$-pseudometric: $F(x, y) \geq 0$, when $x \leq y$

(a02) $\forall y \in M$, $F(\cdot, y)$ is decreasing on $M(y, \geq)$.

Let ψ_F stand for the function (from M to $R_+ \cup \{\infty\}$)

$$\psi_F(x) = \sup\{F(x, y); y \in M(x, \leq)\},\ x \in M.$$

From (a02), ψ_F is $(\leq)$-decreasing; for, if $x_1 \leq x_2$, then (denoting for simplicity $M_i = M(x_i, \leq)$, $i \in \{1, 2\}$),

$$\psi_F(x_1) = \sup\{F(x_1, y); y \in M_1\} \geq$$
$$\sup\{F(x_1, y); y \in M_2\} \geq \sup\{F(x_2, y); y \in M_2\} = \psi_F(x_2).$$

On the other hand, the alternative $\infty \in \psi_F(M)$ cannot be avoided, in general. But, under a regularity condition like

(a03) $F(x, \cdot)$ is bounded above over $M(x, \leq)$, $\forall x \in M$,

this happens [in the sense: $\psi_F(M) \subseteq R_+$].

Call $z \in M$, $(\leq, F)$-*semi-maximal* when: $z \leq w \in M \Longrightarrow F(z, w) = 0$; note that, in terms of the above introduced function, this amounts to: $\psi_F(z) = 0$. The following 1982 Altman's ordering principle [1] (in short: AOP) is our starting point:

Theorem 1.1. *Let (a01)–(a03) hold, as well as*

(a04) $(M, \leq)$ *is sequentially inductive:*
each ascending sequence is bounded above

(a05) $(M, \leq, F)$ *is semi regular:*
((x_n)=ascending) $\Longrightarrow \liminf_n F(x_n, x_{n+1}) = 0$.

Then, for each $u \in M$ *there exists a* $(\leq, F)$*-semi-maximal* $v \in M$ *with* $u \leq v$.

[In fact, the original statement was formulated in terms of the *dual* quasi-order $(\geq)$ and the function $\Phi(x, y) = -F(y, x)$, $x, y \in M$. Moreover (in terms of this translation), (a02) and (a05) were taken – respectively – in their stronger form

(a06) $\forall y \in M$, $F(\cdot, y)$ is decreasing $[x_1 \leq x_2 \Longrightarrow F(x_1, y) \geq F(x_2, y)]$

(a07) $(M, \leq, F)$ is regular: $[(x_n)$=ascending$] \Longrightarrow \lim_n F(x_n, x_{n+1}) = 0$.

But, the author's reasoning also works in this relaxed setting].

This result includes the 1976 Brezis-Browder ordering principle [2] (in short: BB). Moreover, as precise by the quoted authors, their statement includes Ekeland's variational principle [5] (in short: EVP); hence, so does AOP. Now, BB and EVP found some basic applications to control and optimization, generalized differential calculus, critical point theory and global analysis; see the quoted papers for a survey of these. So, it must be not surprising that AOP was subjected to different extensions. For example, a pseudometric enlargement of this ordering principle was obtained by Turinici [11]. Further contributions in the area were provided by Kang and Park [7]; see also Zhu and Li [15]. The obtained facts are interesting from a technical viewpoint. However, we must emphasize that, whenever a maximality principle of this type is to be applied, a substitution of it by the Brezis-Browder's is always possible. This raises the question of to what extent are these extensions of BB effective. As already shown in Turinici [12], the answer is negative for most of these. It is our aim in the following to provide a simpler proof of this fact, with respect to AOP; details will be given in Section 3. Note that, as a consequence, AOP is reducible to the Dependent Choices Principle (in short: DC) due to Tarski [10]; for a direct verification we refer to Section 2.

Now, in a close connection with Theorem 1.1, the following Szaz unboundedness criterion [9] (in short: S-U) is available:

Theorem 1.2. *Suppose that (in addition to (a01)+(a02))*

(a08) each ascending sequence (x_n) *with* $\sup_n F(x_0, x_n) < \infty$
is bounded above and $\liminf_n F(x_n, x_{n+1}) = 0$

(a09) $\forall x \in M$*:* $\psi_F(x) > 0$ *[i.e.:* $\exists y \in M(x, \leq)$ *with* $F(x, y) > 0$*]*

Then, $\psi_F(x) = \infty$*, for all* $x \in M$.

This statement is shown to imply a related one in Brezis and Browder [2]; which, as precise there, includes BB. Moreover [according to the author's observation] his result includes AOP as well. It is our aim in the following to show (in Section 4) that the reciprocal inclusion is also true; i.e., that S-U is nothing but a logical equivalent of AOP. Further aspects will be delineated elsewhere.

2. DC IMPLIES AOP

In the following, a direct proof of AOP is provided, via DC.

(A) Let M be a nonempty set; and $\mathcal{R} \subseteq M \times M$ stand for a relation on it. For each $x \in M$, denote $M(x, \mathcal{R}) = \{y \in M; x\mathcal{R}y\}$ (the *section* of $\mathcal{R}$ through x).

Proposition 2.1. (Dependent Choices Principle) *Suppose that*

(b01) $M(c, \mathcal{R})$ *is nonempty, for each* $c \in M$.

Then, for each $a \in M$ *there exists* $(x_n) \subseteq M$ *with* $x_0 = a$ *and* $x_n\mathcal{R}x_{n+1}$*, for all* n.

This principle, due to Tarski [10] is deductible from the Axiom of Choice (in short: AC), but not conversely; cf. Wolk [14]. Moreover, it suffices for constructing a large part of the "usual" mathematics; see Moore [8, Appendix 2, Table 4].

(B) We may now pass to the promised proof of AOP

Proof. Let the premises of the quoted statement be in use. If, by absurd, its conclusion is not true, there must be some $u \in M$ with (cf. Section 1)

(b02) $\psi_F(x) > 0$, for all x in $U := M(u, \leq)$.

Let γ be arbitrary fixed in $]0, 1[$. We introduce a relation $\mathcal{R}$ over U as:

(b03) $x\mathcal{R}y$ iff $x \leq y$, $F(x, y) > \gamma\psi_F(x)$.

From (b02), $U(x, \mathcal{R})$ is nonempty, for each $x \in U$. So, by (DC), there must be a sequence (x_n) in U with ($x_0 = u$ and) $x_n\mathcal{R}x_{n+1}$, $\forall n$; i.e.,

$$x_n \leq x_{n+1} \text{ and } F(x_n, x_{n+1}) > \gamma\psi_F(x_n), \text{ for all } n.$$

Let v be an upper bound (in U) of this (ascending) sequence (assured by (a04)). As ψ_F is decreasing, we have [by the above relations] $F(x_n, x_{n+1}) > \gamma\psi_F(v)$, $\forall n$. Passing to $\liminf$ as $n \to \infty$ yields $\psi_F(v) = 0$; in contradiction to (b02). Hence, this working assumption cannot be accepted; and the conclusion follows. ∎

Note that only the values of F on $\mathrm{gr}(\leq) := \{(x, y) \in M \times M; x \leq y\}$ were considered here. So, we may substitute (a01) with its extended version (b04) $F(x,y) \geq 0$, for all $x,y \in M$ (i.e.: $F(M \times M) \subseteq R_+$).

For, otherwise, the truncated function $F^*(x,y) = \max\{0, F(x,y)\}$, $x,y \in M$, (fulfilling (b04)) has all properties of F; and, from the conclusion of Theorem 1.1 written for F^* we derive the same fact (modulo F); because $F^* = F$ on $\mathrm{gr}(\leq)$. Suppose that this extension was effectively performed. In this case, we claim that the boundedness condition (a03) may be dropped. For, the associated function $G(x,y) = F(x,y)/(1 + F(x,y))$, $x,y \in M$, fulfills such a property; as well as all remaining ones (involving F). This, along with the observation that the notions of $(\leq, F)$-semi-maximal and $(\leq, G)$-semi-maximal are identical, proves our claim. Further technical aspects may be found in Turinici [11]; see also Kang and Park [7].

3. AOP $\Longrightarrow$ (S-U) $\Longrightarrow$ BB

Under these preliminary facts, we may now return to our initial setting.

(A) Concerning the former inclusion, its original proof uses Altman's ideas [1] for establishing Theorem 1.1. A slightly different reasoning is that given below.

Proof. Assume that conclusion of Theorem 1.2 would be false:

(c01) $\psi_F(x_0) < \infty$, for some $x_0 \in M$.

(Here, $\psi_F : M \to R \cup \{\infty\}$ is that of Section 1). Denote for simplicity $M_0 = M(x_0, \leq)$. Given the ascending sequence (y_n) in M_0, we have $F(y_0, y_n) \leq \psi_F(y_0) \leq \psi_F(x_0) < \infty$, $\forall n$; wherefrom (by (a08)), (y_n) is bounded above in M (hence in M_0) and $\liminf_n F(y_n, y_{n+1}) = 0$; so that, (a04)+(a05) hold over M_0. Moreover, (a03) is clearly true (over M_0) in view of (c01). Summing up, Theorem 1.1 applies to $(M_0, \leq)$ and F. It gives us, for the starting point $x_0 \in M_0$, some $z \in M_0$ with the property: $z \leq w \in M_0 \Longrightarrow F(z,w) = 0$; or, equivalently: $\psi_F(z) = 0$. But then, (a09) cannot hold; contradiction. Hence (c01) is false; and the conclusion follows. ∎

(B) Let again $(M, \leq)$ be a quasi-ordered structure; and $\varphi : M \to R$, some function. By the construction

(c02) $F(x,y) = \varphi(x) - \varphi(y), \quad x,y \in M$,

one gives, via (S-U), a basic result in Brezis and Browder [2, Theorem 1] (called: Brezis-Browder's unboundedness criterion; in short: BB-U).

Theorem 3.1. *Assume that φ is $(\leq)$-decreasing and*

(c03) each ascending sequence (x_n) with $\inf_n \varphi(x_n) > -\infty$ is bounded above

(c04) $\forall x \in M$, $\exists y \in M(x, \leq)$ with $\varphi(x) > \varphi(y)$.

Then, $\inf[\varphi(M(x, \leq))] = -\infty$, $\forall x \in M$.

A basic application of it is to be given under the lines below. Let the triplet $(M, \leq ; \varphi)$ be introduced as before. Call $z \in M$, $(\leq, \varphi)$-*maximal* when: $z \leq w \in M$ imply $\varphi(z) = \varphi(w)$. The following 1976 Brezis-Browder ordering principle [2] (in short: BB) about the existence of such points, is available.

Theorem 3.2. *Let $(M, \leq)$ be sequentially inductive and φ be $(\leq)$-decreasing, bounded from below. Then, for each $u \in M$ there exists a $(\leq, \varphi)$-maximal $v \in M$ with $u \leq v$.*

Proof. Assume this is not true; then, there must be some $u \in M$ such that (c04) holds with $M_u := M(u, \leq)$ in place of M. But then, the preceding statement yields $\inf[\varphi(M_u)] = -\infty$; in contradiction with the choice of φ. ∎

Note that the boundedness from below condition is not essential for the conclusion above; see Cârjă, Necula and Vrabie [4, Ch 2, Sect 2.1] for details. Moreover (cf. Zhu and Li [15]), $(R, \geq)$ may be substituted by a separable ordered structure $(P, \leq)$ without altering the conclusion above; see also Turinici [13].

For the moment, BB follows from AOP, via (S-U). A direct deduction of this is available, by the construction (c02). In fact, (a01)+(a02) hold, from the monotonicity of φ. On the other hand, (a03) holds too, by the boundedness from below of φ; and, finally, (a05) is obtainable from both these. Summing up, Theorem 1.1 is applicable to our data. This, and the observation that $(\leq, \varphi)$-maximal is identical with $(\leq, F)$-semi-maximal (in the context of (c02)) ends the argument.

Finally, note that BB (hence, a fortiori, AOP) includes EVP. Further aspects may be found in Hyers, Isac and Rassias [6, Ch 5].

4. (BB-LC) IMPLIES (DC)

By the developments above, we have the (chain of) implications (DC) $\Longrightarrow$ (AOP) $\Longrightarrow$ (S-U) $\Longrightarrow$ (BB). So, it is natural asking whether these may be reversed. The natural setting for solving such a problem is (ZF)(=the Zermelo-Fraenkel system) without (AC); referred to in the following as the *reduced* Zermelo-Fraenkel system.

Let X be a nonempty set; and $(\leq)$ be an order (i.e.: antisymmetric quasi-order) over it. We say that $(\leq)$ has the *inf-lattice* property, provided: $x \wedge y := \inf(x, y)$ exists, for all $x, y \in X$. Further, call $z \in X$, $(\leq)$-*maximal* if $X(z, \leq) = \{z\}$; the class of all these points will be denoted as $\max(X, \leq)$. Accordingly, $(\leq)$ is called a *Zorn order* when $\max(X, \leq)$ is nonempty and *cofinal* in X [for each $u \in X$, there exists a $(\leq)$-maximal $v \in X$ with $u \leq v$].

Now, the statement below is a particular case of BB:

Theorem 4.1. *Let the partially ordered structure $(X, \leq)$ be such that*

(d01) *$(X, \leq)$ is sequentially complete*

(d02) *$(\leq)$ has the inf-lattice property.*

In addition, suppose that there exists a function $\varphi : X \to R_+$ with *(d03) φ is strictly decreasing and $\varphi(X)$ is countable.*

Then, $(\leq)$ is a Zorn order.

We shall refer to it as: the lattice-countable version of BB (in short: BB-Lc). Clearly, (BB) $\Longrightarrow$ (BB-Lc); since (by (d03)) $x \leq y$ and $\varphi(x) = \varphi(y)$ imply $x = y$. The remarkable fact to be added is that this last principle yields (DC); and so, it completes the circle between all these results.

Proposition 4.1. *We have (in the reduced Zermelo-Fraenkel system) (BB-Lc) $\Longrightarrow$ (DC). So (by the above) the maximal/unboundedness results (AOP), (S-U) and (BB) are all equivalent with (DC); hence, mutually equivalent.*

Proof. Let M be some nonempty set; and $\mathcal{R}$ stand for some relation over M with the property (b01). Fix in the following $a \in M$; as well as some $b \in M(a, \mathcal{R})$. For each $n \geq 2$ in N (= the set of natural numbers), let $N(n, >) := \{0, ..., n-1\}$ stand for the initial segment determined by n; and X_n denote the class of all finite sequences $x : N(n, >) \to M$ with: $x(0) = a$, $x(1) = b$ and $x(m)\mathcal{R}x(m+1)$ for $0 \leq m \leq n-2$. In this case, $N(n, >)$ is just $\mathrm{Dom}(x)$ (the *domain* of x); and $n = \mathrm{card}(N(n, >))$ will be referred to as the *order* of x [denoted as $\omega(x)$]. Put $X = \cup\{X_n; n \geq 2\}$. Let $\leq$ stand for the partial order (on X)

(d04) $x \leq y$ iff $\mathrm{Dom}(x) \subseteq \mathrm{Dom}(y)$ and $x = y|_{\mathrm{Dom}(x)}$;

and $\prec$ denote its associated strict order. All we have to prove is that $(X, \leq)$ has strictly ascending infinite sequences.

(A) Let $x, y \in X$ be arbitrary fixed. Denote

$$K(x, y) := \{n \in \mathrm{Dom}(x) \cap \mathrm{Dom}(y); x(n) \neq y(n)\}.$$

If x and y are *comparable* (i.e.: either $x \leq y$ or $y \leq x$; written as: $x <> y$) then $K(x, y) = \emptyset$. Conversely, if $K(x, y) = \emptyset$, then $x \leq y$ if $\mathrm{Dom}(x) \subseteq \mathrm{Dom}(y)$ and $y \leq x$ if $\mathrm{Dom}(y) \subseteq \mathrm{Dom}(x)$; hence $x <> y$. Summing up, we have the characterization

$$(x, y \in X):\ x <> y \text{ if and only if } K(x, y) = \emptyset.$$

The negation of this property means: x and y are not comparable (denoted as: $x||y$). By the property above, it means: $K(x, y) \neq \emptyset$. Note that, in such a case, $k(x, y) := \min(K(x, y))$ is well defined; and $N(k(x, y), >)$ it is the largest initial interval of $\mathrm{Dom}(x) \cap \mathrm{Dom}(y)$ where x and y are identical.

Lemma 4.1. *The order $(\leq)$ has the inf-lattice property. Moreover, the map $x \mapsto \omega(x)$ is strictly increasing: $x \prec y$ implies $\omega(x) < \omega(y)$.*

Proof. Let $x, y \in X$ be arbitrary fixed. The case $x <> y$ is clear; so, without loss, one may assume that $x||y$. Note that, by the remark above, $K(x, y) \neq \emptyset$ and $k := k(x, y)$ exists. Let the finite sequence $z \in X_k$ be introduced as $z = x|_{N(k,>)} = y|_{N(k,>)}$. For the moment $z \leq x$ and $z \leq y$. Suppose that $w \in X_h$ fulfills the same properties. Then, the restrictions of x and y to $N(h, >)$ are identical; wherefrom (see above) $h \leq k$ and $w \leq z$; which tells us that $z = x \wedge y$. The last part is obvious. ∎

(B) Denote $\varphi(x) = 3^{-\omega(x)}$, $x \in X$. Clearly, $\varphi(X) = \{3^{-n}; n \geq 2\}$; hence, φ has countably many strictly positive values. Moreover, if $x, y \in X$ are such that $x \prec y$, then (by Lemma 4.1) $\varphi(x) > \varphi(y)$; which tells us that φ is strictly decreasing.

(C) From (b01), $\max(X, \leq) = \emptyset$; i.e.: for each $x \in X$ there exists $y \in X$ with $x \prec y$. This, along with (BB-Lc), tells us that $(X, \leq)$ is not sequentially complete: there exists at least one ascending sequence (x_n) in X which is not bounded above. By this last property, $B(k) := \{n \in N(k, <); x_k \prec x_n\}$ is nonempty, for all $k \in N$. Define $g(n) = \min[B(n)]$, $n \in N$; hence $n < g(n)$ and $x_n \prec x_{g(n)}$, for all n. The iterative process $[p(0) = g(0), p(n+1) = g(p(n)), n \geq 0]$ is therefore constructible without any use of (DC); and gives a strictly ascending sequence of ranks $(p(n))$. So, $(y_n := x_{p(n)}; n \in N)$ is a subsequence of (x_n) with $y_n \prec y_m$ (hence $\omega(y_n) < \omega(y_m)$) for $n < m$; moreover, as $\omega(y_0) > 1$, we must have $\omega(y_n) > n + 1$, for all n. But then, the sequence $(c_n = y_n(n); n \in N)$ is well defined (in M) with $c_0 = a$ and $c_n \mathcal{R} c_{n+1}$, for all n. The proof is thereby complete. ∎

In particular, when the specific assumptions (d02) and (d03) (the second half) are ignored in Theorem 4.1, Proposition 4.1 is comparable with a statement in Brunner [3]. Further aspects may be found in Turinici [13].

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