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THE LEFT EXACT REFLECTOR FUNCTOR

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Abstract In the category of locally convex Hausdorff spaces we study some classes of left exact reflector functors. We indicate some relationships that exists between such functors, and we examine the properties of these functors. Examples are constructed.

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1. INTRODUCTION

In the category $\mathcal{C}_2\mathcal{V}$ of the locally convex Hausdorff spaces we examine the left exact reflector functor. The structure of the paper is the following. Section 2 deals with the factorization structure $(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$ and Section 3 with the left exactitude of the reflector functor for c-reflective subcategories. In Section 4 we examine left exact completion functors, we construct a class of completion functors and we prove they are left exact (Theorem 4.4). In Section 5 those reflective subcategories for which the replique of any object is obtained in two steps: by weakening the topology and then by applying a certain completion, are examined. The main result is Theorem 5.7. Theorem 6.3 of Section 6, generalizes a well-known assertion for c-reflective subcategories.

As basic references concerning the locally convex spaces see [13], [15], [17], [19], concerning the factorization structures see [2], [3], [11], relevant to the c-reflective subcategories see [1], [14], and for the semireflexive subcategories see [4], [5], [7], [8], [9], [12], [18], [16].

In the category $\mathcal{C}_2\mathcal{V}$ we consider the following factorization structures:

- $(\mathcal{E}pi, \mathcal{M}_f)$=(the class of epimorphisms, the class of strict monomorphisms),
- $(\mathcal{E}_u, \mathcal{M}_p)$=(the class of universal epimorphisms, the class of exact monomorphisms)=(the class of surjective applications, the class of topological embeddings),
- $(\mathcal{E}_p, \mathcal{M}_u)$=(the class of exact epimorphisms, the class of universal monomorphisms) [6],
- $(\mathcal{E}_f, \mathcal{M}ono)$=(the class of strict epimorphisms, the class of monomorphisms).

We consider the following subcategories:

- Π, the subcategory of complete spaces with weak topology [13],
- $\mathcal{S}$, the subcategory of spaces with weak topology [13],
- $s\mathcal{N}$, the subcategory of strict nuclear spaces [9],
- $\mathcal{N}$, the subcategory of nuclear spaces [15],
- $\mathcal{S}c$, the subcategory of Schwartz spaces [16],
- Γ_0, the subcategory of complete spaces [17],
- $q\Gamma_0$, the subcategory of quasicomplete spaces [19],
- $s\mathcal{R}$, the subcategory of semireflexive spaces [13],
- $i\mathcal{R}$, the subcategory of inductive semireflexive spaces [7],
- $\mathcal{M}$ the subcategory of spaces with Mackey topology [17].

The last subcategory is coreflective and the others are reflective.

2. FACTORIZATION STRUCTURE $(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$

2.1. Definition [2]. *Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms of the category $\mathcal{C}$. The composition $\mathcal{A} \circ \mathcal{B}$ is the class of all morphisms that have the form fg with $f \in \mathcal{A}$ and $g \in \mathcal{B}$ when this composition exists.*

2.2. Definition [2]. *Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms of the category $\mathcal{C}$. The class $\mathcal{A}$ is called $\mathcal{B}$-hereditary if from the fact that $fg \in \mathcal{A}$, $f \in \mathcal{B}$ it follows that $g \in \mathcal{A}$. Dual notion: class $\mathcal{B}$-cohereditary.*

2.3. Lemma. *The class $\mathcal{M}_u$ is an $\mathcal{E}pi$-cohereditary.*

Proof. Let $fg \in \mathcal{M}_u$ and $g \in \mathcal{E}pi$. Then the square

$$\begin{tikzcd} \cdot \arrow[r, "fg"] \arrow[d, "g"'] & \cdot \arrow[d, "1"] \\ \cdot \arrow[r, "f"] & \cdot \end{tikzcd}$$

Figure 2.1

$fg = 1(fg)$ is a pullback. So $f \in \mathcal{M}_u$. ∎

2.4. In class $\mathbb{R}$ of non-zero reflective subcategories of the category $\mathcal{C}_2\mathcal{V}$ we assume that $\mathcal{R}_1 \leq \mathcal{R}_2$ iff $\mathcal{R}_1 \subset \mathcal{R}_2$. In the class of the right factorization structures we consider the order: $(\mathcal{P}_1, \mathcal{I}_1) \leq (\mathcal{P}_2, \mathcal{I}_2)$ iff $\mathcal{P}_1 \subset \mathcal{P}_2$.

Let Π be the subcategory of the complete spaces with weak topology. The subcategory Π is the first element of the lattice $\mathbb{R}$. Let $\mathcal{R} \in \mathbb{R}$. Denote by $r^X : X \longrightarrow rX$ the $\mathcal{R}$-replique and by $\pi^X : X \longrightarrow \pi X$ the Π-replique of the object X of the category $\mathcal{C}_2\mathcal{V}$. Since $\Pi \subset \mathcal{R}$, we have

$$\pi^X = v^X r^X$$

for some morphism v^X. We put

$$\mathcal{U} = \{r^X \mid X \in\mid \mathcal{C}_2\mathcal{V} \mid\}, \quad \mathcal{V} = \{v^X \mid X \in\mid \mathcal{C}_2\mathcal{V} \mid\}.$$

In the theory of categories the operations $(^{\urcorner}, ^{\llcorner})$ which allows us to construct the factorization structures $(\mathcal{V}^{\urcorner}, \mathcal{V}^{\urcorner\llcorner})$ and $(\mathcal{U}^{\llcorner\urcorner}, \mathcal{U}^{\llcorner})$ that are well known. We put:

$$(\mathcal{P}'', \mathcal{I}'') = (\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R})) = (\mathcal{V}^{\urcorner}, \mathcal{V}^{\urcorner\llcorner}),$$

$$(\mathcal{P}', \mathcal{I}') = (\mathcal{P}'(\mathcal{R}), \mathcal{I}'(\mathcal{R})) = (\mathcal{U}^{\llcorner\urcorner}, \mathcal{U}^{\llcorner}).$$

Let $\mathbb{B}$ be the lattice of all factorization structures in category $\mathcal{C}_2\mathcal{V}$ and $\mathbb{B}_u$ the subclass of the factorization structures $(\mathcal{P}, \mathcal{I})$ with the following properties:

a) $\mathcal{E}_p \subset \mathcal{P}$;

b) the class $\mathcal{P}$ is $\mathcal{M}_u$-hereditary.

Theorem [2]. *The mapping*

$$\mathcal{R} \longmapsto (\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$$

establishes a one-to-one correspondence between the lattices $\mathbb{R}$ *and* $\mathbb{B}_u$.

2.5. For any reflective subcategory $\mathcal{R}$ with the reflector functor $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ we denote by $\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi \quad \mathcal{C}_2\mathcal{V} \mid r(e) \in \mathcal{I}so\}$. Dual notion: for coreflective subcategory $\mathcal{K}$ with coreflector functor $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ we denote by $\mu\mathcal{K} = \{m \in \mathcal{M}ono \quad \mathcal{C}_2\mathcal{V} \mid k(m) \in \mathcal{I}so\}$.

Theorem [2]. *Let* $\mathcal{R}$ *be a non-zero subcategory of the category* $\mathcal{C}_2\mathcal{V}$. *Then:*

1. $\mathcal{P}''(\mathcal{R}) = (\varepsilon\mathcal{R}) \circ \mathcal{E}_p$, $\mathcal{I}''(\mathcal{R}) = (\varepsilon\mathcal{R})^{\perp} \cap \mathcal{M}_u$.

2. The monomorphism $m : X \longrightarrow Y$ *belongs of the class* $\mathcal{I}''(\mathcal{R})$ *iff* m *is a universal mono* ($m \in \mathcal{M}_u$) *and* $r^Y m = r(m) r^X$ *is a pullback, where* $r^X : X \longrightarrow rX$ *and* $r^Y : Y \longrightarrow rY$ *are the* $\mathcal{R}$*-replique of the objects* X *and* Y.

We mention only the $(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$-factorization of arbitrary morphism.

Let $f : X \longrightarrow Y \in \mathcal{C}_2\mathcal{V}$,

$$f = me \tag{1}$$

his $(\mathcal{E}_p, \mathcal{M}_u)$-factorized, and

$$m = m'm'', \tag{2}$$

the $((\varepsilon\mathcal{R}), (\varepsilon\mathcal{R})^{\perp})$-factorization of morphism m. Then

$$f = m'(m''e) \tag{3}$$

is the $(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$-factorization of morphism m. So $e \in \mathcal{E}_p$ and $m'' \in \varepsilon\mathcal{R}$, that $m'e \in \varepsilon\mathcal{R} \circ \mathcal{E}_p$. Next, $m \in \mathcal{M}_u$ and in the equality (2) $m'' \in \mathcal{E}pi$. So $m' \in \mathcal{M}_u$. Hence $m' \in (\varepsilon\mathcal{R})^{\perp} \cap \mathcal{M}_u$.

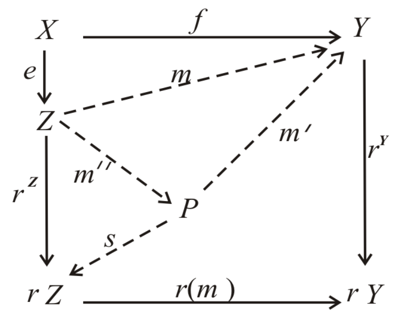

Figure 2.2

3. THE *C*-REFLECTIVE SUBCATEGORIES

3.1. Definition [1]. *The reflective subcategory $\mathcal{R}$ of category $\mathcal{C}_2\mathcal{V}$ is called c-reflective if it contains the subcategory $\mathcal{S}$ of the spaces with weak topology and the reflector functor $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ is left exact.*

3.2. Theorem [1]. *Let $\mathcal{R}$ be a non-zero reflective subcategory of $\mathcal{C}_2\mathcal{V}$. The following assertions are equivalent:*

1. $\mathcal{R}$ is a c-reflective subcategory.

2. $\mathcal{S} \subset \mathcal{R}$ and $r(\mathcal{M}_f) \subset \mathcal{M}_f$.

3. $\mathcal{S} \subset \mathcal{R}$ and $r(\mathcal{M}_p) \subset \mathcal{M}_p$.

4. There is a coreflective subcategory $\mathcal{K}$ with the coreflector functor $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ so that:

a) $kr \sim k$;

b) $rk \sim r$.

5. There is a coreflective subcategory $\mathcal{K}$ so that:

$$\varepsilon\mathcal{R} = \mu\mathcal{K}.$$

3.3. Definition [1]. *The pair of subcategories $(\mathcal{K}, \mathcal{R})$ of the category $\mathcal{C}_2\mathcal{V}$ that satisfies the equivalent conditions of the former theorem is called a pair of conjugated subcategories: $\mathcal{K}$ is called conjugated coreflective subcategory of the reflective subcategory $\mathcal{R}$ and conversely.*

The pair $(\mathcal{M}, \mathcal{S})$ is the smallest pair of conjugated subcategories and $(\mathcal{C}_2\mathcal{V}, \mathcal{C}_2\mathcal{V})$ is the biggest. Conjugated coreflective subcategories of the reflective subcategories $\mathcal{S}c$ and $s\mathcal{N}$ were described in article [14].

3.4. Theorem *(Theorem 3.2* [11]*). Let $(\mathcal{K}, \mathcal{R})$ be a pair of conjugated subcategories in the category $\mathcal{C}_2\mathcal{V}$ and let $(\mathcal{P}, \mathcal{I})$ be a factorization structure. Then the following conditions are equivalent:*
a) $r(\mathcal{I}) \subset \mathcal{I}$;
b) $k(\mathcal{P}) \subset \mathcal{P}$.

3.5. Definition. *The subcategory $\mathcal{R}$ is called closed under the extensions if from the fact that the $X \in\mid \mathcal{R} \mid$ and $f : X \longrightarrow Y \in \mathcal{E}pi \cap \mathcal{M}_p$ it follows that $Y \in\mid \mathcal{R} \mid$.*

We also remind the following affirmations (see [4], Lemma 3.6).

3.6. Lemma. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two reflective subcategories of the category $\mathcal{C}_2\mathcal{V}$ with reflector functors $r_1 : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}_1$ and $r_2 : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}_2$ and $r_1(\mathcal{R}_2) \subset \mathcal{R}_2$. Then r_1r_2 is the reflector functor of the subcategory $\mathcal{R}_1 \cap \mathcal{R}_2 : r_1r_2 : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}_1 \cap \mathcal{R}_2$.*

3.7. Theorem. *Let $\mathcal{R}$ be a c-reflective subcategory and Γ a $\mathcal{M}_p$-reflective subcategory. Then gr is the reflector functor of the subcategory $\mathcal{R} \cap \Gamma : gr : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R} \cap \Gamma$.*

3.8. We examine the Hilbert space l_2 and his subspace E constructed from sequence $(x_i)_{i\in N}$, where $x_i \neq 0$ only for a finite number of coordinates. We are examining on the subspace E the induced topology from space l_2. Let $\mathcal{R}$ be the full subcategory of the category which is $\mathcal{C}_2\mathcal{V}$ formed from subspaces E^τ for an arbitrary cardinal τ. In such way $\mathcal{R}$ is closed under $\mathcal{M}_p$ - subspaces and products. So it is a $\mathcal{E}_u$-reflective subcategory and $\mathcal{S} \subset \mathcal{R}$. Knowing that the subspace E is dense in the space l_2 we deduce that $g_0(E) = l_2$. If l_2 would belong to the subcategory $\mathcal{R}$, we would deduce that l_2 it is a subspace of the space E^τ for some cardinal τ. Since l_2 is a Banach space, we can say that l_2 will be a subspace of the space E^n for some natural number n. This is impossible, since the space E, as well as E^n, possesses the numerable Hammel basis that l_2 does not have.

Theorem. *In the category $\mathcal{C}_2\mathcal{V}$ there are reflective subcategories $\mathcal{R}$ so that*
1. $\mathcal{S} \subset \mathcal{R}$;
2. $g_0(\mathcal{R})$ *is not included in* $\mathcal{R}$;
3. *The subcategory $\mathcal{R}$ is not c-reflective.*

4. THE COMPLETING FUNCTOR LEFT EXACT

4.1. Let $\mathbb{R}_p$ be the set of all classes of the $\mathcal{M}_p$-reflective subcategories of the category $\mathcal{C}_2\mathcal{V}$. It is the complete lattice with the first element Γ_0 and the last $\mathcal{C}_2\mathcal{V}$.

Definition. *Let τ be a cardinal. The locally convex space (E, t) is called quasicomplete (respectively τ-complete) if is closed and bounded in (E, t) set up (respectively M with $\mid \mathcal{M} \mid < \tau$) is complete. The space is called sequential if any Cauchy string converges.*

Let Γ_τ^- be the subcategory of all τ complete spaces, $c\Gamma$ - the subcategory of sequentially complete spaces.

4.2. Theorem. ([3], Theorem 3.10). 1. Γ_τ^- *is the* $\mathcal{M}_p$*-reflective subcategory:* $\Gamma_\tau^- \in \mathbb{R}_p$, $\Gamma_0 \subset \Gamma_\tau^-$.

2. *Let* $\tau \leq w$. *Then* $\Gamma_\tau^- = \mathcal{C}_2\mathcal{V}$.

3. $\Gamma_{\omega_1} = s\Gamma$ *- the subcategory of sequential spaces.*

4. $\cap\Gamma_\tau^- = q\Gamma$ *- the subcategory of quasicomplete spaces.*

5. *Let* $\alpha < \beta$ *and* $w_1 \leq \beta$. *Then* $\Gamma_\alpha^- \subset \Gamma_\beta^-$.

4.3. We construct the Γ_τ-replique of the arbitrary object (E, t). We examine the completion of this object, i.e. $(\widehat{E}, t)$ is Γ_0-replique of the object (E, t).

$$\begin{tikzcd}
(E,t) \arrow[rr, "g_0^E"] \arrow[dr, dashed, "g_\tau^E"'] & & (\hat{E},t) = g_0(E,t) \\
& g_\tau(E,t) \arrow[ur, dashed, "v^E"'] &
\end{tikzcd}$$

Figure 4.1

Let $\mathcal{M}_\tau^-$ the class of all sets $M \subset E$ where $\mid M \mid < \tau$, M is closed and bounded.

We examine the closure $\overline{M}$ of the set M in the space $(\widehat{E}, t)$.

Theorem.
$$g_\tau(E, t) = \cup\{\overline{M} \mid M \in \mathcal{M}_\tau^-\}.$$

Proof. We denote $F = \cup\{\overline{M} \mid M \in \mathcal{M}_\tau^-\}$ and we will prove that F is vector subspace of the space $\widehat{E}$. Let $x_1, x_2 \in F$. Then exist two networks $\{y_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{z_\alpha \mid \alpha \in \mathcal{A}\}$, where $y_\alpha, z_\alpha \in E$, in this way $y_\alpha \longrightarrow x_1, z_\alpha \longrightarrow x_2$ and $\mid \mathcal{A} \mid < \tau$. It is clear that the networks $\{y_\alpha + z_\alpha \mid \alpha \in \mathcal{A}\}$ converge to the point $x_1 + x_2$.

Also the network $\{ay_\alpha \mid \alpha \in \mathcal{A}\}$ converge by point ax_1, where $a \in K$.

We examine on the subspace F the topology induced from space $(\widehat{E}, t)$ and we will prove that (F, t) is a τ-complete space.

Indeed, let $\{x_\alpha \mid \alpha \in \mathcal{A}\}$ be a network which converges to the point $x_0 \in \widehat{E}$, $\{ay_\alpha \mid \alpha \in \mathcal{A}\}$ where $\mid \mathcal{A} \mid < \tau$ and $x_\alpha \in F$ for any $\alpha \in \mathcal{A}$. For any index $\alpha \in \mathcal{A}$ there is a network $\{x_{\alpha\beta} \mid \beta \in \mathcal{A}\}$ of elements from space E so that $x_{\alpha\beta} \longrightarrow x_\alpha$. Then the network $\{x_{\alpha\alpha} \mid \alpha \in \mathcal{A}\}$ converge to point x_0. Therefore $x_0 \in F$.

In this way we constructed the functor g_τ on the objects

$$g_\tau : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma_\tau^-.$$

To define this functor on morphisms, let $f : (E,t) \longrightarrow (F,u) \in \mathcal{C}_2\mathcal{V}$, and $\widehat{f}$ the extending of this morphism on the completion those spaces.

$$\begin{tikzcd}
(E,t) \arrow[rr, "f"] \arrow[d] & & (F,u) \arrow[d] \\
g_\tau(E,t) \arrow[rr, dashed, "\hat{f}_\tau = g_\tau(f)"] \arrow[d] & & g_\tau(F,u) \arrow[d] \\
g_0(E,t) \arrow[rr, "\hat{f} = g_0(f)"] & & g_0(F,u)
\end{tikzcd}$$

Figure 4.2

For the proof it suffices to verify of the relation $\widehat{f}(g_\tau(E,t)) \subset g_\tau(F,u)$. Let $x_0 \in g_\tau(E,t)$ then there is a network $\{x_\alpha \mid \alpha \in \mathcal{A}\}$, $\mid \mathcal{A} \mid < \tau$, of elements of the space E which converge to point x_0. Then the network $\{\widehat{f(y_\alpha)} \mid \alpha \in \mathcal{A}\}$ converges to point $\widehat{f}(x_0)$. Therefore $\widehat{f}(x_0) \in g_\tau(F,u)$. ∎

4.4. Theorem. 1. *The functor* $g_0 : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma_0$ *is left exact.*

2. *For any cardinal number* τ, *the functor* $g_\tau : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma_\tau^-$ *is left exact.*

Proof. 1. Let $f : X \longrightarrow Y \in \mathcal{M}_f$. Then in the commutative square

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, "g_0^X"'] & & Y \arrow[d, "g_0^Y"] \\
g_0X \arrow[rr, "\hat{f} = g_0(f)"] & & g_0Y
\end{tikzcd}$$

Figure 4.3

morphisms g_0^Y and f belongs to class $\mathcal{M}_p$. So that $\widehat{f}g_0^X \in \mathcal{M}_p$. Since g_0^X is an $\mathcal{E}pi$, and the class $\mathcal{M}_p$ is $\mathcal{E}pi$-cohereditary, we deduce that $\widehat{f} \in \mathcal{M}_p$. Taking into consideration that g_0^X is a complete space, and $\widehat{f} \in \mathcal{M}_p$, we deduce that $\widehat{f} \in \mathcal{M}_f$, hence $\widehat{f}$ has the closed image.

2. We examine the diagram from Figure 4.3. We have proved that $\widehat{f} \in \mathcal{M}_f$ if $f \in \mathcal{M}_f$. Let M be a closed and bordered set in space (E, t) with $| \mathcal{M} |< \tau$.

Since $(\widehat{E}, t)$ is a closed subspace in space $(\widehat{F}, u)$, it follows that the closure of the set M in the space $(\widehat{E}, t)$ is equal with its closure in the space $(\widehat{F}, u)$. But if $| M |< \tau$, we deduce that $\overline{M}$ is the closure of the set M in the space $g_\tau(F, u)$. ∎

5. SEMIREFLEXIVE SUBCATEGORIES

5.1. The lattice $\mathbb{R}$ of the non-zero subcategories of the category $\mathcal{C}_2\mathcal{V}$ is divided into three complete sublattices:

a) The sublattice $\mathbb{R}_b$ of $\mathcal{E}_u$-reflective subcategories. A subcategory $\mathcal{R}$ is $\mathcal{E}_u$-reflective if it is characterized by the fact that the $\mathcal{R}$-replique of any object of the category $\mathcal{C}_2\mathcal{V}$ is a bijection. Another characterization is:

$$\mathbb{R}_b = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \mathcal{S}\}.$$

b) The sublattice $\mathbb{R}_p$ of $\mathcal{M}_p$-reflective subcategories, the class of those reflective subcategories $\mathcal{R}$ such that the $\mathcal{R}$-replique of any object of the category $\mathcal{C}_2\mathcal{V}$ is a topological embedding:

$$\mathbb{R}_p = \{\mathcal{R} \in \mathbb{R} \mid \mathcal{R} \supset \Gamma_0\}.$$

c) $\mathbb{R}_m = (\mathbb{R} \setminus (\mathbb{R}_b \cup \mathbb{R}_p)) \cup \{\mathcal{C}_2\mathcal{V}\}$.

We mention that $\mathbb{R}_m$ is a complete lattice with the first element Π and the last element $\mathcal{C}_2\mathcal{V}$. This latice contains the semireflexive subcategories (see [4], [5], [7], [8], [9], [12], [18], [19]).

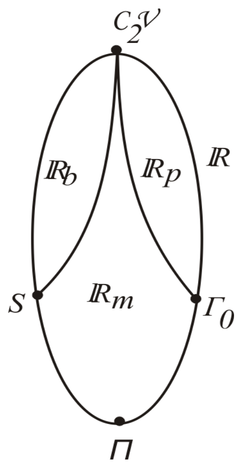

Figure 5.1

5.2. Let $\mathcal{L}$ be an element of lattice $\mathbb{R}_m$. For any object X of category $\mathcal{C}_2\mathcal{V}$ let

$$\begin{tikzcd} X \arrow[r, "b^X"] \arrow[rr, bend left, "l^X"] & bX \arrow[r, "p^X"] & lX \end{tikzcd}$$

Figure 5.2

$l^X : X \longrightarrow lX$ be the $\mathcal{L}$-replique, and $l^X = p^X b^X$ its $(\mathcal{E}_u, \mathcal{M}_p)$-factorization. We denote with $\mathcal{B} = \mathcal{B}(\mathcal{L})$ a full subcategory of the category $\mathcal{C}_2\mathcal{V}$ consisting of all the objects of bX form. We also can say $\mathcal{B}$ is a subcategory of all $\mathcal{M}_p$-subobjects of the objects $\mathcal{L}$. We can see that $\mathcal{B}$ is a $\mathcal{E}_u$-reflective subcategory, and b^X is $\mathcal{B}$-replique of the X objects. Though $\mathcal{B} \in \mathbb{R}_b$.

5.3. Let $\Gamma'' = \Gamma''(\mathcal{L})$ be a full subcategory of all objects Y of the category $\mathcal{C}_2\mathcal{V}$ with the property:

Any morphism $f : bX \longrightarrow Y$ extends through p^X $(f = gp^X)$ for some morphism g.

The subcategory Γ'' is closed under $\mathcal{M}_f$-subobjects and products. Further, $\Gamma_0 \subset \Gamma''$, though $\Gamma'' \in \mathbb{R}_p$. It is obvious that p^X is Γ''-replique of object bX.

We denote by $G(\mathcal{L})$ the class of all the $\mathcal{M}_p$-reflective subcategories for which p^X is the replique of object bX. The class $G(\mathcal{L})$ has a first element

$$\Gamma' = \Gamma'(\mathcal{L}) = \cap\{\Gamma \mid \Gamma \in G(\mathcal{L})\}.$$

Thus $G(\mathcal{L})$ is a complete lattice with first element $\Gamma'(\mathcal{L})$ and the last element $\Gamma''(\mathcal{L})$.

We can write

$$G(\mathcal{L}) = \{\Gamma \in \mathbb{R}_p \mid \Gamma'(\mathcal{L}) \subset \Gamma \subset \Gamma''(\mathcal{L})\}.$$

5.4. For any element $\Gamma \in G(\mathcal{L})$ the morphism p^X is Γ-replique of the object bX. Though if $l : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{L}$, $b : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{B}$ and $g : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma$ are the reflective functors, then

$$l = gb.$$

Theorem [20]. *Let $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ and $g : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma$ be two the reflector functors with $\mathcal{R} \in \mathbb{R}_b$ and $\Gamma \in \mathbb{R}_p$. The next affirmations are equivalent:*

1. $l = gr$.

2. $\mathcal{R} = \mathcal{B}$ and $\Gamma \in G(\mathcal{L})$.

5.5. Example. Let us examine the case $\mathcal{L} = \Pi$. Then

$$\mathcal{B}(\Pi) = S$$

and

$$\Gamma'(\Pi) = \Gamma_0.$$

Theorem. *The subcategory* $\Gamma''(\Pi)$ *contains all the normed spaces.*

Proof. Let X be a space with weak topology: $X \in\mid \mathcal{S} \mid$, and $g_0^X : X \longrightarrow g_0X$ the Γ_0-replique. Then g_0^X is also Π-replique of object X. In this case $g_0X \sim K^\tau$, where K is the field of numbers over which the vector spaces from the category $\mathcal{C}_2\mathcal{V} : K = R$ or $K = \mathbb{C}$ are examined. Let $f : X \longrightarrow Y \hookrightarrow \widehat{Y}$, where Y is a normal space, and $\widehat{Y}$ is its completion.

$$\begin{tikzcd}
X \arrow[rr, "g_0^X"] \arrow[d, "f"'] & & g_0X=K^{\mathfrak{T}} \arrow[dll, dashed, "h_1"'] \arrow[d, dashed, "h"] \\
Y \arrow[rr, "i"] & & \hat{Y}
\end{tikzcd}$$

Figure 5.3

Then

$$if = hg_0^X$$

for some morphism h, where i is a canonical embedding. Since $g_0X \sim K^\tau$, we conclude that $h(g_0X)$ is a finite dimensional subspace in $\widehat{Y}$. Then the subspace $f(X)$ of the Y space, as a finite dimensional space, is complete and

$$f = h_1 g_0^X$$

for some morphism h_1. The theorem is proved. ∎

5.6. Theorem. *Let* $\mathcal{R}$ *be a c-reflector subcategory, and* Γ *a* $\mathcal{M}_p$*-reflective subcategory with the reflector functors* $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ *and* $g : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma$.

1. $\mathcal{R} \cap \Gamma$ *is a reflective subcategory and* $gr : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R} \cap \Gamma$ *is a reflective functor.*

2. Let g *be a left exact functor. Then the functor* gr *is the same.*

Proof. 1. We examine consecutively the $\mathcal{R}$ and Γ-repliques of this arbitrary object X.

$$X \xrightarrow{r^X} rX \xrightarrow{g^{rX}} grX$$

Figure 5.4

Since the subcategory $\mathcal{R}$ is close under the extensions and $g^{rX} \in \mathcal{E}pi \cap \mathcal{M}_p$ we deduce that $grX \in\mid \mathcal{R} \cap \Gamma \mid$. Then is obviously that $g^{rX}\, r^X$ is the $(\mathcal{R} \cap \Gamma)$-replique of the object X.

2. It is obvious. ∎

5.7. Theorem. *Let $\mathcal{L} \in \mathbb{R}_m$ and the left exact functor $l : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{L}$.*
The subcategory $\mathcal{B} = \mathcal{B}(\mathcal{L})$ is c-reflective and the reflector functor $b : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{B}$ is left exact.

Proof. Let $m : X \longrightarrow Y \in \mathcal{M}_f$ and we examine the $\mathcal{B}$ and $\mathcal{L}$-replique this objects.

$$\begin{tikzcd}
X \arrow[r, "m"] \arrow[d, "b^X"] \arrow[dd, bend right=60, "l^X"'] & Y \arrow[d, "b^Y"'] \arrow[dd, bend left=60, "l^Y"] \\
bX \arrow[r, "b(m)"] \arrow[d, "p^X"] & bY \arrow[d, "p^Y"'] \\
lX \arrow[r, "l(m)"] & lY
\end{tikzcd}$$

Figure 5.5

We have

$$b(m)b^X = b^Y m \tag{1}$$

$$l(m)l^X = l^Y m \tag{2}$$

From here it follows that

$$l(m)p^X = p^Y b(m) \tag{3}$$

According of the hypothesis $l(m) \in \mathcal{M}_f \subset \mathcal{M}_p$. In such way $l(m)$, $p^X \in \mathcal{M}_p$. From equality (3) we deduce that $l(m)p^X = p^Y b(m) \in \mathcal{M}_p$. So that $b(m) \in \mathcal{M}_p$, implying that $b(m)$ is a topological embedding. The subspaces X is closed in the spaces Y. The topologies on the spaces Y and bY are compatible with one and the same duality. So that the subspaces X and bX are closed in bY ([17], Assertion II.8.). ∎

6. THE LEFT EXACT REFLECTIVE FUNCTOR

6.1. Lemma. *Let $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ be a left exact reflective functor, and $r : \mathcal{R} \longrightarrow \mathcal{C}_2\mathcal{V}$ - the inclusion functor. Then the functor $ir : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{C}_2\mathcal{V}$ is left exact.*

6.2. Theorem. *Let $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ be a left exact reflective functor. Then the functor $ir : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{C}_2\mathcal{V}$ commutes to the projective limits:*

$$ir \varprojlim\{\mathcal{S}_{\alpha\beta} \mid f_{\alpha\beta} : X_\alpha \longrightarrow X_\beta\} = \varprojlim\{ir\mathcal{S}_{\alpha\beta} \mid ir(f_{\alpha\beta}) : irX_\alpha \longrightarrow irX_\beta\}.$$

The proof relies on Theorem 1.12 of [4] the functor *ir* commute products. It remains to apply the previous lemma.

6.3. Theorem. *Let $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ be a left exact reflective functor, $\mathcal{T}$ - a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$ and $\mathcal{T} \subset \mathcal{R}$. Then $r(\mathcal{I}''(\mathcal{T})) \subset \mathcal{I}''(\mathcal{T})$.*

Proof. Let $m : X \longrightarrow Y \in \mathcal{I}''(\mathcal{T})$, $t^X : X \longrightarrow tX$ and $t^Y : Y \longrightarrow tY$ be the $\mathcal{T}$-replique of the respective objects. Then based on the theorem 2.4 the square

$$t^Y m = t(m)t^X \tag{1}$$

is copullback.

Let $r^X : X \longrightarrow rX$ and $r^Y : Y \longrightarrow rY$ the $\mathcal{R}$-replique of the respective objects. Since $\mathcal{T} \subset \mathcal{R}$, we put

$$r^Y m = r(m)r^X, \tag{2}$$

$$t^X = f^X r^X, \tag{3}$$

$$t^Y = f^Y r^Y \tag{4}$$

for some morphism f^X and f^Y. It is clear said that f^X and f^Y are the $\mathcal{T}$-repliques of the respective objects.

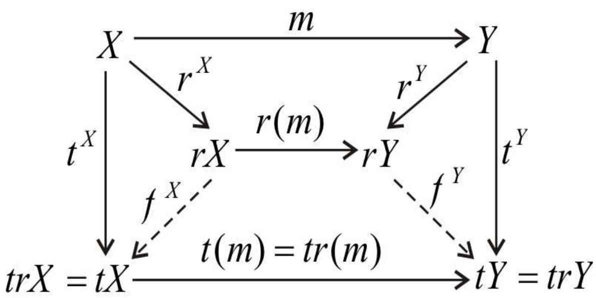

Figure 6.1

From the previous equality we have

$$t(m)f^X = f^Y r(m). \tag{5}$$

From $\mathcal{T} \subset \mathcal{R}$ we deduce that the square (5) is the image of functor r of the copullback (1). Since the functor r commutes to projective limits we deduce that the square (5) is also copullback.

Further $m \in \mathcal{I}''(\mathcal{T}) \subset \mathcal{M}_u$ and $r^Y \subset \mathcal{M}_u$. From equality (3) we have $r(m)r^X \subset \mathcal{M}_u$. Since $r^X \in \mathcal{E}pi$, and the class $\mathcal{M}_u$ is $\mathcal{E}pi$-cohereditary we deduce that $r(m) \in \mathcal{M}_u$. It remains to apply the Theorem 2.5. ∎

6.4. Remark. When $\mathcal{R}$ is c-reflective subcategory this theorem was proved in another way see the work [5], Theorem 2.7.

7. OPEN PROBLEMS

7.1. Let Γ be a $\mathcal{M}_p$-reflective subcategory. Is the reflector functor $g : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma$ left exact?

7.2. Let $\mathcal{R}_{\mathcal{E}pr}$ be the class of all reflective subcategories with the left exact reflector functor. Is the $\mathbb{R}_{\mathcal{E}pr}$ a complete lattice?

7.3. Let $\mathcal{L} \in \mathbb{R}_m$ with the left exact reflector functor $l : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$, and $\Gamma \in G(\mathcal{L})$. Is the reflector functor $g : \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma$ left exact?

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