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ON THE GENERALIZED NILPOTENT AND GENERALIZED SOLVABLE LOOPS

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Abstract The paper introduces the notions of commutator-associators of types $(\alpha, \beta, 1)$, $(\alpha, \beta, 2)$ (respect. $(\mu, 1)$, $(\mu, 2)$) of certain weight, related directly with inner mappings of loops. With the help of these notions the authors describe the (transfinite) upper series of loops, lower series of loops, derived central series of loops, of Kurosh-Chernikov's classes of loops and characterize the centrally nilpotent, centrally solvable loops, respect. Moufang loops.

Keywords: nilpotency function, upper loop, lower loop, derived series of loops, centrally nilpotent loop, centrally solvable loop, commutator-associator, Kurosh-Chernikov's classes of loops, local theorem, Moufang loop, *IP*-loop.

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1. PRELIMINARIES

We remind some notions and results from the loop theory, which can be found in [1]. The *multiplication group* $\mathfrak{C}(Q)$ of the arbitrary loop $Q = (Q, \cdot, 1)$ is generated by all mappings $R(x)$, $L(x)$, where $R(x)y = yx, L(x)y = xy$, of the loop Q. The *inner mapping group* $\mathfrak{I}(Q)$ of Q is the subgroup of $\mathfrak{C}(Q)$, generated by the mappings $T(x), R(x,y), L(x,y)$, for all x, y in Q, where

$$T(x) = L^{-1}(x)R(x), \quad R(x,y) = R^{-1}(xy)R(y)R(x),$$
$$L(x,y) = L^{-1}(xy)L(x)L(y). \tag{1.1.}$$

For the loop Q with identity 1,

$$\mathfrak{I}(Q) = \{\alpha \in \mathfrak{C}(Q) | \alpha 1 = 1\}.$$

The subloop H of the loop Q is called *normal* in Q, if

$$xH = Hx, \quad x \cdot yH = xy \cdot H, \quad H \cdot xy = Hx \cdot y, \tag{1.2}$$

or by (1.1)

$$T(x)H = H, \quad L(x,y)H = H, \quad R(x,y)H = H, \tag{1.3}$$

for every $x, y \in Q$.

We use the notation $< M >$ for the subloop of the loop Q generated by set $M \subseteq Q$.

Lemma 1.1. *Let H, K be subloops of the loop Q such that K is normal in $< H \cup K >$. Then $< H \cup K >= HK = KH$.*

Lemma 1.2. *Let H be a normal subloop of the loop Q. If H is generated as a normal subloop by set $S \subseteq Q$, then H, as a subloop, is generated by set $\{\varphi s | s \in S, \varphi \in \Im(Q)\}$, where $\Im(Q)$ is the inner mapping group of Q.*

For arbitrary elements x, y, z of the loop Q the *commutator*, (x, y), and *associator*, (x, y, z), are defined by

$$xy = (yx)[x, y], \quad xy \cdot z = (x \cdot yz)[x, y, z]. \tag{1.4}$$

The *left nucleus*, $N_\lambda(Q)$, of the loop Q is the associative subloop $N_\lambda(Q) = \{a \in Q \mid [a, x, y] = 1, \text{ for all } x, y \in Q\}$, the *middle nucleus*, $N_\mu(Q)$, of Q is the associative subloop $N_\mu(Q) = \{a \in Q \mid [x, a, y] = 1 \text{ for all } x, y \in Q\}$ and the *right nucleus*, $N_\rho(Q)$, of Q is the associative subloop $N_\rho(Q) = \{a \in Q \mid [x, y, a] = 1 \text{ for all } x, y \in Q\}$. The *nucleus*, $N(Q)$, is defined by $N(Q) = N_\lambda(Q) \cap N_\mu(Q) \cap N_\rho(Q)$ and the *centre*, $Z(Q)$, of the loop Q is the normal associative and commutative subloop, $Z(Q) = \{a \in N(Q) \mid [a, x] = 1 \text{ for all } x \in Q\}$.

The loop is *Moufang* if it satisfies the equivalent identities:

$$x(y \cdot xz) = (xy \cdot x)z; \quad (xy \cdot z)y = x(y \cdot zy); \quad xy \cdot zx = (xy \cdot z)x. \tag{1.5}$$

The Moufang loop is diassiciative, i.e. every two elements generate a subgroup.

The Moufang loop is an IP-loop. Then it satisfies the identities:

$$x^{-1} \cdot xy = y; \quad yx \cdot x^{-1} = y; \quad (xy)^{-1} = y^{-1}x^{-1}. \tag{1.6}$$

2. *F*-NILPOTENT AND *F*-SOLVABLE LOOPS

Bruck ([2], [1]) developed the theory of nilpotency for loops on the basis of the nilpotency for groups. We give the following definitions according to [1]. Let $\mathfrak{C}$ be a class of loops such that:

(a) *every subloop of a loop of $\mathfrak{C}$ is in $\mathfrak{C}$;*

(b) *every loop which is a homomorphic image of a loop of $\mathfrak{C}$ is in $\mathfrak{C}$.*

For example, $\mathfrak{C}$ may consist either of all loops or of all loops satisfying a prescribed set of identities

$$u(x_1, \ldots, x_i) = 1, \ldots, v(y_1, \ldots, u_n) = 1. \tag{2.1}$$

By a *nilpotency function*, f, for $\mathfrak{C}$ we mean a function f with the following properties:

(i) *if Q is in $\mathfrak{C}$, $f(Q)$ is a uniquely defined subloop of Q;*

ii) *if Q is in $\mathfrak{C}$ and if H is a subloop of Q, then $H \cap f(Q) \subseteq f(H)$;*

(iii) *if Q is in $\mathfrak{C}$ and if θ is a homomorphism of Q upon a loop, then $\theta f(Q) \subseteq f(\theta Q)$;*

(iv) *if Q is in $\mathfrak{C}$, if N is a normal subloop of Q and if A is the intersection of all normal subloops K of Q such that NK/K is a subloop of $f(Q/K)$, then NA/A is a subloop of $f(Q/A)$.*

For any loop Q in $\mathfrak{C}$ the *f-centre*, $Z_f(Q)$, is defined as the union of all normal subloops of Q, which is contained in $f(Q)$. For any normal subloop N of Q we define $(N, Q)_f$ as the subloop A, whose existence is guaranteed in iv). Clearly, $Z_f(Q)$ and $(N, Q)_f$ are normal subloops of Q; moreover, $(N, Q)_f$ is a subloop of N.

The (transfinite) *lower f-series*, $\{Q_\alpha\}$, of the loop Q in $\mathfrak{C}$ is defined inductively as follows: i) $Q_0 = Q$; ii) for any ordinal α, $Q_{\alpha+1} = (Q_\alpha, Q)_f$; iii) if α is a limit ordinal, Q_α is the intersection of all Q_β with $\beta < \alpha$.

The (transfinite) *upper f-series*, $\{Z_\alpha\}$, of the loop Q in $\mathfrak{C}$ is defined inductively as follows: i) $Z_0 = \{1\}$; ii) for any ordinal α $Z_{\alpha+1}$ is the unique subloop of Q such that $Z_{\alpha+1}/Z_\alpha = Z_f(Q/Z_\alpha)$; iii) if α is a limit ordinal, Z_α is the union of all Z_β with $\beta < \alpha$.

The (transfinite) *derived f-series*, $\{Q^{(\alpha)}\}$, of the loop Q in $\mathfrak{C}$ is defined inductively as follows: i) $Q^{(0)} = Q$; ii) for any ordinal α $Q^{(\alpha+1)} = (Q^{(\alpha)}, Q^{(\alpha)})_f$; iii) if α is a limit ordinal, $Q^{(\alpha)}$ is the intersection of all $Q^{(\beta)}$ with $\beta < \alpha$.

Clearly, Q_α, Z_α and $Q^{(\alpha)}$ are normal subloops of Q. From construction of Q_α, Z_α and $Q^{(\alpha)}$ it follows that if any of f-series "stopped" at any step α, then it stabilizes at this step, i.e. $Q_\alpha = Q_{\alpha+1} \Rightarrow Q_\alpha = Q_\beta, Z_\alpha = Z_{\alpha+1} \Rightarrow Z_\alpha = Z_\beta, Q^{(\alpha)} = Q^{(\alpha+1)} \Rightarrow Q^{(\alpha)} = Q^{(\beta)}$ for $\beta \geq \alpha$.

The loop Q is called *transfinitely f-nilpotent* (respect. *transfinitely upper f-nilpotent* or *transfinitely f-solvable*) if $Q_\lambda = \{1\}$ (respect. $Z_\lambda = Q$ or $Q^{(\lambda)} = \{1\}$) and *f-nilpotent* (respect. *upper f-nilpotent* or *f-solvable*) if, in addition, λ is finite. The smallest ordinal λ such that $Q_\lambda = \{1\}$ or $Q^{(\lambda)} = \{1\}$, is called *f-nilpotence class* or *f-solvability class.*

When $\mathfrak{C}$ is the class of all loops, the most important example of a nilpotency function is obtained by defining $f(Q)$ to be the centre $Z(Q)$; this leads to the notions of *centrally nilpotent loops* and *centrally solvable loops*. Such loops are studied in detail [2], [1]. Further we will also investigate such loops. If define $f(Q) = N(Q)$ then we obtain the notion of *nuclear* (or *associator*) *nilpotent loops*. Such loops are studied in [4]. Similarly, we could take $f(Q)$ to be the left, middle or right nucleus of the loop Q. Let us note that other examples of nilpotency functions specific for the theory of loops are given in [1] (see, also, [5], [6], [7]]. We also mentioned that, as proved in [8], any free loop is a transfinitely centrally nilpotent loop.

Let Q be a loop and f be a nilpotency function. We remind that the series of normal subloops $Q = C_0 \supseteq C_1 \supseteq \ldots \supseteq C_r = \{1\}$ of loop Q is called *f-nilpotent*, if

$$(C_i, Q)_f \subseteq C_{i+1} \quad \text{for all} \quad i. \tag{2.2}$$

or by property (iv) of f and definition of f-centre, equivalently,

$$C_i/C_{i+1} \subseteq Z_f(Q/C_{i+1}) \quad \text{for all} \quad i. \tag{2.3}$$

Lemma 2.1. *Let $\{C_i\}$ be a f-nilpotent series, $\{Z_i\}$ be the upper f-nilpotent series, $\{Q_i\}$ be the lower f-nilpotent series of the loop Q. Then $C_{r-i} \subseteq Z_i$, $C_i \supseteq Q_i$, for $i = 0, 1, \ldots, r$.*

Proof. We have $C_0 = Q = Q_0$. Assume that $C_i \supseteq Q_i$. By (2.2) $(C_i, Q)_f \subseteq C_{i+1}$. But then $Q_{i+1} = (Q_i, Q)_f \subseteq (C_i, Q)_f \subseteq C_{i+1}$. We assume now that $C_{r-i} \subseteq Z_i$ for a certain i. Then the loop Q/Z_i is the homomorphic image of the loop Q/C_{r-i} with kernel Z_i/C_{r-i}. But by (2.3)

$$C_{r-i-1}/C_{r-i} \subseteq Z_f(Q/C_{r-i}),$$

from where it follows that the homomorphic image of subloop C_{r-i-1}/C_{r-i} must lie in the f-centre $Z_f(Q/Z_i)$. It is clear that this image is the subloop $(C_{r-i-1} \cup Z_i)/Z_i$, while $Z_f(Q/Z_i) = Z_{i+1}/Z_i$. Consequently, $C_{r-i-1} \subseteq C_{r-i} \cup Z_i \subseteq Z_{i-1}$. ∎

Theorem 2.1. *A loop Q is f-nilpotent of class n if and only if its upper and lower f-nilpotent series have respectively the form*

$$\{1\} = Z_0 \subset Z_1 \subset \ldots \subset Z_n = Q, \quad Q = Q_0 \supset Q_1 \supset \ldots \supset Q_n = \{1\}.$$

Proof. The statement of the theorem for upper f-nilpotent series results from the definition of f-nilpotent loop. Further, if an f-nilpotent series of the length n exists, then from Lemma 2.1 it follows that the length of the upper and lower central series do not exceed n. But, as there is a term by term inclusion between the elements of these series, their lengths are equal, and the series have the indicated form. ∎

Corollary 2.1. *If a loop Q is f-nilpotent of class $\leq n$, then the inclusion $Q_i \subseteq Z_j$ is true for any naturals i, j such that $i + j = n$. Conversely, if in the loop Q the inclusion $Q_i \subseteq Z_j$ is true for some certain integers i, j, then the loop Q is f-nilpotent of class $\leq i + j$.*

Proof. If the loop Q is f-nilpotent of class $\leq n$, then $Q_i \subseteq Z_j$, by Lemma 2.1. Conversely, we assume that $Q_i \subseteq Z_1$. From the definition of the subloop Q_{i+1} it follows that $Q_{i+1} = (Q_i, Q)_f = \{1\}$ and by property (iv) of the function f the subloop Q_{i+1} is the intersection of all normal subloops K of Q, such that $Q_iK/K \subseteq f(Q/K)$. But $Z_1 = Z_f(Q)$ and $Z_f(Q) \subseteq f(Q)$. Then $Q_{i+1} = \{1\}$ and by Theorem 2.1 the loop Q is f-nilpotent of class $\leq i + 1$. Let the assertion be true for the inclusion $Q_i \subseteq Z_{j-1}$. If $Q_i \subseteq Z_j$, then $Q_i/Z_1 \subseteq Z_j/Z_1 = Z_{j-1}(Q/Z_1)$, i.e., by inductive hypothesis the loop Q/Z_1 is f-nilpotent of class $\leq i+j-1$, then the loop Q is f-nilpotent of class $\leq i+j$. ∎

Now we proceed to characterization of f-solvable loops. Remember that a loop Q is called *f-solvable of the class n* if its derived f-series have the form: $Q = Q^{(0)} \supset Q^{(1)} \supset \ldots \supset Q^{(n)} = \{1\}$, where $Q^{(i+1)} = (Q^{(i)}, Q^{(i)})_f$.

Proposition 2.1. *Let Q be an arbitrary loop and f be a nilpotency function. The loop Q will be f-solvable if and only if it has a finite series of subloops H_i such that*

$$1 = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_s = Q, \tag{2.4}$$

H_i is normal in H_{i+1} and $(H_{i+1}, H_{i+1})_f \subseteq H_i$.

Proof. If the loop Q is f-solvable, then its derived f-series have the form of (2.4). Conversely, let the loop Q possess the series (2.4) and let $\{Q^{(i)}\}$ be its derived f-series. By property (iv) of the function f the subloop $Q^{(1)}$ is the intersection of all normal subloops K of Q, such that $Q/K \subseteq f(Q/K)$. As $Q/H_{s-1} \subseteq f(Q/H_{s-1})$, then $Q^{(1)} \subseteq H_{s-1}$. We assume that the inclusion $Q^{(i)} \subseteq H_{s-i}$, $1 \leq i \leq s$ has already been established. As $H_{s-i}/H_{s-i-1} \subseteq f(H_{s-i}/H_{s-i-1})$ and as $(Q^{(i)})^{(1)} = Q^{(i+1)}$ then, again, by property (iv) of f it follows that $Q^{(i+1)} \subseteq H_{s-i-1}$. From here it follows $Q^{(s)} = 1$. ∎

Let h be a nilpotency function for $\mathfrak{C}$. Further we will assume that for any loop Q from $\mathfrak{C}$ the subloop $f(Q)$ is an abelian group. Particularly, h may be *central nilpotency function*, i. e. when $h(Q)$ is the center of loop Q.

Let Q be a loop. Similarly to groups we can prove that Q satisfies the maximum condition for its subloops when and only when all subloops of a loop Q are finitely generated: any ascending series of subloops $H_1 \subseteq H_2 \subseteq \ldots$ break, i.e, $H_n = H_{n+1} = \ldots$ for a certain n.

Corollary 2.2. *Let h be the nilpotency function, mentioned above. A loop Q is h-solvable and satisfies the maximum conditions when and only when Q has a series of subloops*

$$\{1\} = Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_n = Q \tag{2.5}$$

such that Q_i is normal in Q_{i+1} ($i = 1, \ldots, n-1$) and quotient loop Q_{i+1}/Q_i is a cyclic group.

Proof. Let the loop Q have a series of subloops (2.5). Then, according to Proposition 2.1, the loop Q is h-solvable. If H is a subloop of loop Q then crossing the sequence (2.5) with H we get the sequence of subloops

$$\{1\} = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_n = H, \text{where} \quad H_i = Q_i \cap H. \tag{2.6}$$

Using (1.3) we get that the subloop H_i is normal in H_{i+1}. Further, using the homomorphism theorems, we have

$$H_{i+1}/H_i = H_{i+1}/(H_{i+1} \cap Q_i) \cong H_{i+1}Q_i/Q_i \subseteq Q_{i+1}/Q_i.$$

The quotient loop Q_{i+1}/Q_i is a cyclic group. Hence H_{i+1}/H_i is also a cyclic group. From (2.6) it follows that the subloop H is finitely generated, then the loop Q satisfies the maximum condition for subloops.

Conversely, let the loop Q be h-solvable and satisfy the maximum condition for subloops. By Proposition 2.1, Q has a series of subloops

$$\{1\} = Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_n = Q$$

such that Q_i is normal in Q_{i+1} and the factors Q_{i+1}/Q_i are abelian groups. From the maximum condition for subloops it follows easily for the loop Q that the abelian groups Q_{i+1}/Q_i are finitely generated. Then Q_{i+1}/Q_i decompose into a direct product of cyclic groups $Q_{i+1}/Q_i = \prod_{j=1}^{k} B_j/Q_j$. Hence, the factors of sequence

$$Q_i \subseteq Q_iB_1 \subseteq \ldots \subseteq Q_i \prod_{j=1}^{k-1} B_j \subseteq Q_i \prod_{j=1}^{k} B_j$$

are cyclic groups. ∎

3. CENTRALLY NILPOTENT AND CENTRALLY SOLVABLE LOOPS

Let Q be a loop and let $a, b, c \in Q$. We denote the solution of equation $ab{\cdot}c = ax{\cdot}bc$ (respect. $c \cdot ba = cb \cdot xa$) by $\alpha(a, b, c)$ (respect. $\beta(a, b, c)$) and call it the *associator of type* α (respect. *of type* β) *of weight 1* of elements a, b, c.

The *commutator of weight 1* of elements $a, b \in Q$ (a, b) is defined by the equality $ab = b(a(a, b))$.

The last definitions will be written in the form

$$T(b)a = a(a, b),\ R(b, c)a = a\alpha(a, b, c),\ L(c, b)a = \beta(a, b, c)a, \tag{3.1}$$

if we use (1.1) (see, also, [1]).

Obviously, if Q is a commutative loop then

$$\alpha(x, y, z) = \beta(x, y, z). \tag{3.2}$$

Lemma 3.1. *Any IP-loop satisfies the identity*

$$\alpha(x, y, z)^{-1} = \beta(x^{-1}, y^{-1}, z^{-1}).$$

Proof. From the identity $xy \cdot z = x\alpha(x, y, z) \cdot yz$ by (1.6) we get $(xy \cdot z)^{-1} = (x\alpha(x, y, z) \cdot yz)^{-1}$, $z^{-1} \cdot y^{-1}x^{-1} = z^{-1}y^{-1} \cdot \alpha(x, y, z)^{-1}x^{-1}$ and from the definition of the associator of type β it follows that $\alpha(x, y, z)^{-1} = \beta(x^{-1}, y^{-1}, z^{-1})$. ∎

Lemma 3.2. *According to (1.4) any Moufang loop Q satisfies the identities*

$$[x, y, z]^{-1} = \alpha(x, z^{-1}, y^{-1}),\ [x, y, z] = \beta(x^{-1}, z, y),\ [x, y] = (x, y).$$

Proof. Let $a, b, c \in Q$. By (1.4) $ab \cdot c = (a \cdot bc)[a, b, c], (ab \cdot c)[a, b, c]^{-1} =$

$$a \cdot bc, [a, b, c]^{-1} = (ab \cdot c)^{-1}(a \cdot bc),$$

$$[a,b,c]^{-1} = (c^{-1} \cdot b^{-1}a^{-1})(a \cdot bc)$$

and by the first identity from (1.5) we have

$$a[a,b,c]^{-1} = a((c^{-1} \cdot b^{-1}a^{-1})(a \cdot bc)) =$$

$$(a(c^{-1} \cdot b^{-1}a^{-1}) \cdot a)(bc) = ((ac^{-1})(b^{-1}a^{-1} \cdot a))(bc) = (ac^{-1} \cdot b^{-1})(bc).$$

Hence $a[a,b,c]^{-1} \cdot c^{-1}b^{-1} = ac^{-1} \cdot b^{-1}$. Further,

$$(a[a,b,c]^{-1} \cdot c^{-1}b^{-1})^{-1} = (ac^{-1} \cdot b^{-1})^{-1}, bc \cdot [a,b,c]a^{-1} = b \cdot ca^{-1}.$$

Then, from here and from the definition of the associator of type α or β, it follows that $[a,b,c]^{-1} = \alpha(a,c^{-1},b^{-1})$, $\quad [a,b,c] = \beta(a^{-1},c,b)$.

Any Moufang loop is diassociative, so $[a,b] = (a,b)$. ∎

Let Q be a loop and let A, B, C be a non-empty subsets of Q. We denote
$\alpha(A,B,C) =< \alpha(a,b,c)|a \in A, b \in B, c \in C >$,
$\beta(A,B,C) =< \beta(a,b,c)|a \in A, b \in B, c \in C >$,
$[A,B,C] =< [a,b,c]|a \in A, b \in B, c \in C >$,
$(A,B) =< (a,b)|a \in A, b \in B >$, $[A,B] =< [a,b]|a \in A, b \in B >$.

Lemma 3.3. *Let N be a normal subloop of the loop Q. Then the subloop H, generated by the set $\alpha(N,Q,Q) \bigcup \beta(N,Q,Q) \bigcup (N,Q)$ is normal in Q. Particularly, if Q is a commutative loop or an IP-loop (respect. Moufang loop) then H is generated by the set $\alpha(N,Q,Q)$, equivalent, $\beta(N,Q,Q)$, or the set $\alpha(N,Q,Q) \bigcup (N,Q)$, equivalent, $\beta(N,Q,Q) \bigcup (N,Q)$(respect. the set $[N,Q,Q] \bigcup [N,Q]$.)*

Proof. Let $n \in N$, $x, y \in Q$. From (3.1) we get

$$\begin{aligned} \alpha(n,x,y) &= L^{-1}(n)R(x,y)n, \\ \beta(n,x,y) &= R^{-1}(n)L(x,y)n, \quad (n,x) = L^{-1}(n)T(x)n. \end{aligned} \tag{3.3}$$

The subloop N is normal in Q, then by (1.3), $\alpha(n,x,y)$, $\beta(n,x,y)$, $(n,x) \in N$ for all $n \in N$ and all $x, y \in Q$, i.e. $H \subseteq N$. Let $h \in H$. According to (3.1) we get $T(x)h = h(h,x) \in H$, $L(x,y)h = \beta(h,y,x) \in H$, $R(x,y)h = h\alpha(h,x,y) \in H$. Hence, by (1.3), the subloop H is normal in Q.

The statements for IP-loops and commutative loops (respect. Moufang loops), in accordance with the first case, immediately follow from Lemma 3.1 and (3.2) (respect. Lemma 3.2). ∎

Proposition 3.1. *Let Q be one of the following: (i) a loop, (ii) a commutative loop, (iii) an IP-loop, (iv) a Moufang loop. A subloop H of Q is normal in Q if and only if (i) $\alpha(H,Q,Q) \subseteq H$, $\beta(H,Q,Q) \subseteq H$, $(H,Q) \subseteq H$, either (ii) $\alpha(H,Q,Q) \subseteq H$ (equivalent $\beta(H,Q,Q) \subseteq H$), or (iii) $\alpha(H,Q,Q) \subseteq H$, $(H,Q) \subseteq H$ (equivalent, $\beta(H,Q,Q) \subseteq H$, $(H,Q) \subseteq H$), or (iv) $[H,Q,Q] \subseteq H$, $[H,Q] \subseteq H$ respectively.*

Proof. Let H be a normal subloop of Q. Then from (1.3), (3.1) and (3.2), Lemmas 3.1, 3.2 it follows that H satisfies the inclusions of the Proposition. The inverse statement of our Proposition follows from Lemma 3.3. ∎

Let H be a normal subloop of a loop Q. We denote $\mathcal{Z}_H(Q) = \{a \in Q | \alpha(a, Q, Q) \subseteq H, \beta(a, Q, Q) \subseteq H, (a, Q) \subseteq H\}$ if (i) Q is an arbitrary loop, $\mathcal{Z}_H(Q) = \{a \in Q | \alpha(a, Q, Q) \subseteq H$ (equivalent $\mathcal{Z}_H(Q) = \{a \in Q | \beta(a, Q, Q) \subseteq H\}$) if (ii) Q is a commutative loop, $\mathcal{Z}_H(Q) = \{a \in Q | \alpha(a, Q, Q) \subseteq H, (a, Q) \subseteq H\}$ if (iii) Q is an IP-loop, $\mathcal{Z}_H(Q) = \{a \in Q | [a, Q, Q] \subseteq H, [a, Q] \subseteq H\}$ if (iv) Q is a Moufang loop.

Let $E = \{1\}$ and let $\mathcal{Z}_E(Q) = \mathcal{Z}(Q)$. We prove that for all cases (i) - (iv)

$$\mathcal{Z}(Q) = Z(Q), \tag{3.4}$$

where $Z(Q)$ means the centre of loop Q. Really, from the definitions of the associators $\alpha(x, y, z)$, $\beta(x, y, z)$ and the nucleus of the loop it follows that $\alpha(a, x, y) = 1$ for all $x, y \in Q \Leftrightarrow a \in N_\lambda(Q), \beta(a, x, y) = 1$ for all $x, y \in Q \Leftrightarrow a \in N_\rho(Q)$ and $(a, N_\lambda \cap N_\rho) = 1 \Rightarrow a \in N_\mu$. From here it follows that the equality (3.4) holds in case (i). If Q is a commutative loop, then $N_\lambda = N_\rho \subseteq N_\mu$ and case (ii) follows from (i). If Q is an IP-loop then $N_\lambda = N_\mu = N_\rho$. Then the case (iii) follows from case (i). Finally, the case (iv) follows from case (iii) and Lemma 3.2.

Lemma 3.4. *Let H be a normal subloop of the loop Q. Then the set $\mathcal{Z}_H(Q)$ is a normal subloop of the loop Q and $H \subseteq \mathcal{Z}_H(Q)$. Moreover, if N is a subloop of Q and $H \subseteq N \subseteq \mathcal{Z}_H(Q)$ then N is a normal subloop of Q.*

Proof. Let $\varphi : Q \to \overline{Q} = Q/H$ be the natural homomorphism. Denote by $\overline{A}$ the image of set $A \subseteq Q$ under homomorphism φ. It is clear that $A \subseteq H$ if and only if $\overline{A} = \{1\}$. Particularly, the inclusion $\alpha(A, B, C) \subseteq H$ is equivalent to the equality $\alpha(\overline{A}, \overline{B}, \overline{C}) = \{1\}$. Hence, due to one-to-one correspondence between normal subloops of Q containing H and all normal subloops of $\overline{Q}$, to prove the Lemma it is sufficient to consider that $H = E = \{1\}$. By (3.4) $\mathcal{Z}_E(Q)$ coincides with center $Z(Q)$, of loop Q, hence $\mathcal{Z}_E(Q)$ is a normal subloop of Q. Then due to our supposition $\mathcal{Z}_H(Q)$ is also a normal subloop of Q.

Now let N be a subloop of Q and $N \subseteq \mathcal{Z}_H(Q)$. Then $\alpha(N, Q, Q) \subseteq \alpha(\mathcal{Z}_E(Q), Q, Q) = E \subseteq N$. Similarly, $\beta(N, Q, Q) = N$, $(N, Q) = N$. By Lemma 3.3, N is a normal subloop of Q. ∎

Let N be a normal subloop of the loop Q. We denote by $\mathcal{A}^N(Q)$ the subloop of Q generated by the set $\alpha(N, Q, Q) \bigcup \beta(N, Q, Q) \bigcup (N, Q)$. Particularly, if Q is a commutative loop or an IP-loop (respect. Moufang loop) then $\mathcal{A}^N(Q)$ is generated by the set $\alpha(N, Q, Q)$, equivalent, $\beta(N, Q, Q)$, or the set $\alpha(N, Q, Q) \bigcup (N, Q)$, equivalent, $\beta(N, Q, Q) \bigcup (N, Q)$ (respect. the set $[N, Q, Q] \bigcup [N, Q]$.)

The subloop $\mathcal{A}^Q(Q)$ will be called *commutator-associator subloop* of the loop Q, and sometimes is denoted by $Q^{(1)}$. For an arbitrary loop Q the subloop $\mathcal{A}^Q(Q)$

is generated by the set $\alpha(Q,Q,Q) \bigcup \beta(Q,Q,Q) \bigcup (Q,Q)$. But if Q is a commutative loop or an IP-loop (respect. Moufang loop) then $\mathcal{A}^Q(Q)$ is generated by the set $\alpha(Q,Q,Q)$, equivalent, $\beta(Q,Q,Q)$, or the set $\alpha(Q,Q,Q) \bigcup (Q,Q)$, equivalent, $\beta(Q,Q,Q) \bigcup (Q,Q)$ (respect. the set $[Q,Q,Q]$).

Lemma 3.5. *Let N be a normal subloop of the loop Q. Then the subloop $\mathcal{A}^N(Q)$ will be a normal subloop of the loop Q. Moreover, if H is a subloop of Q such that $N \supseteq H \supseteq \mathcal{A}^N(Q)$ then H will be a normal subloop of the loop Q.*

Proof. The subloop N is normal in Q, then by Proposition 3.1 $\alpha(N,Q,Q) \subseteq N$, $\beta(N,Q,Q) \subseteq N$, $(N,Q) \subseteq N$. Hence, $\mathcal{A}^N(Q) \subseteq N$. From here it follows that for a subloop H such that $N \supseteq H \supseteq \mathcal{A}^N(Q)$ we have $\alpha(H,Q,Q) \subseteq \alpha(N,Q,Q) \subseteq \mathcal{A}^N(Q) \subseteq H$. Analogically, $\beta(H,Q,Q) \subseteq H$, $(H.Q) \subseteq H$. Then by Proposition 3.1 the subloop H is normal in Q. ∎

Corollary 3.1. *The commutator-associator subloop $\mathcal{A}^Q(Q)$ of a loop Q is the least normal subloop of Q such that the quotient loop $Q/\mathcal{A}^Q(Q)$ is an abelian group.*

From the definitions of subloops $\mathcal{Z}_N$, $\mathcal{A}^N$ and Lemmas 3.4, 3.5 it follows.

Corollary 3.2. *Let N be a normal subloop of the loop Q. Then the normal subloops $\mathcal{Z}_N(Q) = \mathcal{Z}_N$, $\mathcal{A}^N(Q) = \mathcal{A}^N$ satisfy the relations*

$$\mathcal{Z}_N/N = \mathcal{Z}(Q/N), \quad N/\mathcal{A}^N \subseteq \mathcal{Z}(Q/\mathcal{A}^N),$$

$$\mathcal{Z}_{\mathcal{A}^N} \supseteq N, \quad \mathcal{A}^{\mathcal{Z}_N} \subseteq N.$$

Lemma 3.6. *Let N be a normal subloop of the loop Q and let H be the normal subloop defined in Lemma 3.3. Then, for any normal subloop K of Q, $NK/K \subseteq Z(Q/K)$ if and only if $H \subseteq K$.*

Proof. As K is a normal subloop of the loop Q then by (1.2) $(xK)(yK \cdot nK) = (xK \cdot yK)(nK)$, $(nK \cdot xK)(yK) = (nK)(xK \cdot yK)$, $nK \cdot xK = xK \cdot nK$ for all $n \in N$ and all $x, y \in Q$ when and only when $(x \cdot yn)K = xy \cdot Kn$, $(nx \cdot y)K = nK \cdot xy$, $(nx)K = x \cdot nK$ respectively. But this is equivalent to $x \cdot yn \in xy \cdot Kn$, $nx \cdot y \in nK \cdot xy$, $nx \in x \cdot nK$ or $R^{-1}(n)L(x,y)n \in K$, $L^{-1}(n)R(x,y)n \in K$, $L^{-1}(n)T(x)n \in K$. The center $Z(Q/K)$ of the loop Q/K is an abelian group, then by (3.3) $NK/K \subseteq Z(Q/K)$ when and only when $H \subseteq K$. ∎

We proceed to the description of upper central series, lower central series and derived central series. From the definition of upper f-series, (3.4) and Lemma 3.4 it follows that:

Proposition 3.2. *The transfinite upper central series $\{\mathcal{Z}_\alpha\}$ of a loop Q have the form:*
i) $\mathcal{Z}_0 = \{1\}$;
ii) for any ordinal α $\mathcal{Z}_{\alpha+1}/\mathcal{Z}_\alpha = \mathcal{Z}_\alpha(Q/\mathcal{Z}_\alpha)$;

iii) if α is a limit ordinal, $\mathcal{Z}_\alpha = \bigcup_{\beta<\alpha} \mathcal{Z}_\beta$.

Particularly, if Q is a commutative loop either an IP-loop, or a Moufang loop then the normal subloops $\mathcal{Z}_\alpha$ change

From the definitions of transfinite lower f-series and derived f-series, Lemmas 3.5, 3.6 and Corollary 3.1 it immediately follows that:

Theorem 3.1. *The transfinite lower central series $\{Q_\xi\}$ of a loop Q have the form:*

$Q_0 = Q$, $Q_1 = \mathcal{A}^Q(Q)$, $Q_{\xi+1} = \mathcal{A}^{Q_\xi}(Q)$ *for any ordinal* ξ;

$Q_\xi = \bigcap_{\eta<\xi} Q_\eta$ *if ξ is a limit ordinal and the derived central series $\{Q^{(\xi)}\}$ have the form:*

$Q^{(0)} = Q$, $Q^{(1)} = \mathcal{A}^Q(Q)$, $Q^{(\xi+1)} = \mathcal{A}^{Q^{(\xi)}}(Q^{(\xi)})$ *for any ordinal* ξ;

$Q^{(\xi)} = \cap_{\eta<\xi} Q^{(\eta)}$ *if ξ is a limit ordinal.*

If Q is a commutative loop, either an IP-loop, or a Moufang loop then the normal subloops change in accordance with the definition of the subloop $\mathcal{A}^N(Q)$, where N is a normal subloop of Q.

Now we define the *commutator-associator of type* $(\alpha, \beta, 1)$ *of weight n* inductively:

1) any associators of the form $\alpha(x, y, z)$, $\beta(x, y, z)$ and any commutator (x, y) are commutator-associator of the type $(\alpha, \beta, 1)$ of weight 1;

2) if a is a commutator-associator of the type $(\alpha, \beta, 1)$ of weight $n - 1$, then $\alpha(a, x, y)$, $\beta(a, x, y)$, (a, x), where $x, y \in Q$, are a commutator-associator of the type $(\alpha, \beta, 1)$ of the weight n.

If only the associators of type α (respect. β) and the commutators (x, y) participate in the definition then we get *commutator-associators of type* $(\alpha, 1)$ (respect. $(\beta, 1)$).

We also define by induction the *commutator-associator of type* $(\alpha, \beta, 2)$ *of weight* n:

1) any associator of form $\alpha(x, y, z)$, $\beta(x, y, z)$ and any commutator (x, y) is the commutator-associator of type $(\alpha, \beta, 2)$ of weight 1;

2) if a, b, c are commutator-associator of type $(\alpha, \beta, 2)$ of weight $n-1$, then $\alpha(a, b, c)$, $\beta(a, b, c)$, (a, b) are commutator-associators of type $(\alpha, \beta, 2)$ of weight n.

If only associators of type α (respect. β) and commutators (a, b) participate in the definition then we get *commutator-associators of type* $(\alpha, 2)$ (respect. $(\beta, 2)$).

We define by induction the *commutator-associator of type* $(\mu, 1)$ *of weight n* :

1) any associator of form $[x, y, z]$ and any commutator $[x, y]$ are commutator-associators of type $(\mu, 1)$ of weight 1;

2) if a is a commutator-associator of type $(\mu, 1))$ of weight $n - 1$, then $[a, x, y]$, $[a, x]$ are commutator-associators of type $(\mu, 1)$ of weight n.

We define by induction the *commutator-associator of type* $(\mu, 2)$ *of weight n*:

1) any associator of form $[x, y, z]$ and any commutator $[x, y]$ are commutator-associators of type $(\mu, 2)$ of weight 1;

2) if a, b, c are commutator-associators of type $(\mu, 2)$) of weight $n-1$, then $[a, b, c]$, $[a, b]$ are commutator-associators of type $(\mu, 2)$ of weight n.

We assume that in the loop Q all commutator-associators of weight n of one of the aforementioned types, for example $(\alpha, \beta, 1)$, are equal to unit 1. Then we say that the loop Q satisfies the *commutator-associator identities* of type $(\alpha, \beta, 1)$ of weight n.

Further we denote by $W_n(\alpha, \beta, 1)$ the set of all commutator-associators of type $(\alpha, \beta, 1)$ of weight n, by $W_n(\alpha, 1)$ the set of all commutator-associators of type $(\alpha, 1)$ of weight n, and so on. Reinforce Theorem 2.1 and Proposition 2.1 in case of centrally nilpotent loops and centrally solvable loops using the Proposition 3.2 and Theorem 3.1.

Theorem 3.2. *For a loop Q the following statements are equivalent:*

1) the loop Q is centrally nilpotent of class n;
2) the upper central nilpotent series of Q have the form

$$E = \mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \ldots \subset \mathcal{Z}_{n-1} \subset \mathcal{Z}_n = Q,$$

where $E = \{1\}$, $\mathcal{Z}_{i+1}/\mathcal{Z}_i = \mathcal{Z}(Q/\mathcal{Z}_i)$, $i = 0, \ldots, n-1$, and

$$\mathcal{Z}_i = \{a \in Q \,|\, w(a, x_1, \ldots, x_j) = 1 \forall w \in W_i(\alpha, \beta, 1), \forall x_1, \ldots, x_j \in Q\}. \tag{3.5}$$

3) the lower central nilpotent series of Q have the form

$$Q = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \ldots \supset \mathcal{A}_{n-1} \supset \mathcal{A}_n = \{1\},$$

where $\mathcal{A}_{i+1} = \mathcal{A}^{\mathcal{A}_i}(Q)$ for $i = 0, 1, \ldots, n-1$ and the normal subloop $\mathcal{A}_i$ coincides with the subloop A_i generated by all commutator-associators $w \in W(\alpha, \beta, 1)$.

Particularly, it is sufficient to consider in items 2), 3) $w \in W_i(\alpha, 1)$ or $w \in W_i(\beta, 1)$ if Q is a commutative loop or an IP-loop, $w \in W_i(\mu, 1)$ if Q is a Moufang loop.

Proof. The equivalence of items 1) - 3) follows from Theorem 2.1 and Proposition 2.1.

By definition, $\mathcal{Z}_1 = \{a \in Q \,|\, \alpha(a, x, y) = 1, \beta(a, x, y) = 1, (a, x) = 1, \forall x, y \in Q\}$. From $\mathcal{Z}_{i+1}/\mathcal{Z}_i = \mathcal{Z}(Q/\mathcal{Z}_i)$ it follows that $\mathcal{Z}_{i+1}/\mathcal{Z}_i = \{a \in Q \,|\, \alpha(a, x, y) \subseteq \mathcal{Z}_i, \beta(a, x, y) \subseteq \mathcal{Z}_i, (a, x) \subseteq \mathcal{Z}_i, \forall x, y \in Q\}$. From here, by induction, (3.5) follows easily.

From (3.5) it follows that all commutator-associators of type $(\alpha, \beta, 1)$ of weight $n-1$ belong to the normal subloop $\mathcal{Z}_1$, which according to (3.4) coincides with centre $Z(Q)$ of loop Q. Then the subloop A_{n-1}, generated by these commutator-associators, is normal in Q.

We assume that the subloop A_{i+1} is normal in Q and we consider the natural homomorphism $\varphi : Q \to Q/A_{i+1}$. The subloop A_i/A_{i+1} belongs to the centre $Z(Q/A_{i+1})$, then it is normal in Q/A_{i+1}. The inverse image of A_i/A_{i+1} under homomorphism φ is A_i. Hence, the subloop A_i is normal in Q. ∎

Corollary 3.3. *According to Theorem 3.1 let* $\{\mathcal{A}_\xi\}$*, where* $\mathcal{A}_{\xi+1} = \mathcal{A}^{\mathcal{A}_\xi}(Q)$*, be the (transfinite) lower central series of a loop* Q*. Then for any natural number* n *the normal subloop* $\mathcal{A}_n$ *of the series* $\{\mathcal{A}_\xi\}$ *coincides with the subloop* A_n *of the loop* Q*, generated by all commutator-associators of type* $(\alpha, \beta, 1)$ *of weight* n *and the quotient loop* $Q/\mathcal{A}_n$ *is centrally nilpotent of class* $\leq n$*.*

Proof. From Lemma 1.2, (1.3), (3.1) it follows that the subloop A_n is normal in Q. By definition, $\mathcal{A}_1 = \mathcal{A}^Q(Q)$ and from Lemma 3.5 it follows that $\mathcal{A}_1$ is the subloop of Q generated by all commutator-associators of type $(\alpha, \beta, 1)$ of weight 1. Hence $\mathcal{A}_1$ is the normal subloop of Q generated by all commutator-associators of type $(\alpha, \beta, 1)$ of weight 1, i.e. $\mathcal{A}_1 = A_1$.

We consider the normal subloops $\mathcal{A}_{n+1}$ and A_{n+1}. By construction, $\mathcal{A}_{n+1}$ is the subloop of Q generated by the set $\alpha(\mathcal{A}_n, Q, Q) \bigcup \beta(\mathcal{A}_n, Q, Q) \bigcup (\mathcal{A}_n, Q)$. By the inductive hypothesis, $\mathcal{A}_n = A_n$. Then the set $\alpha(A_n, Q, Q) \bigcup \beta(A_n, Q, Q) \bigcup (A_n, Q)$ contains all commutator-associators of type $(\alpha, \beta, 1)$ of weight $n + 1$. Hence
$\mathcal{A}_{n+1} \supseteq A_{n+1}$ because $\mathcal{A}_{n+1}$ is a normal subloop in Q.

Taking this into consideration, we consider the quotient loop Q/A_{n+1}. In this loop all commutator-associators of type $(\alpha, \beta, 1)$ of weight $n + 1$ are equal to unit. Then by (3.4) all commutator-associators of weight n will be in the centre of the loop Q/A_{n+1}. Consequently, $\mathcal{A}_n/A_{n+1} = A_n/A_{n+1} \subseteq \mathcal{Z}(Q/A_{n+1})$ as the centre of any loop is a normal subloop. Proceeding to inverse images we get $\alpha(\mathcal{A}_n, Q, Q)$, $\beta(\mathcal{A}_n, Q, Q)$, $(\mathcal{A}_n, Q) \subseteq A_{n+1}$. Hence $\mathcal{A}_{n+1} \subseteq A_{n+1}$. Consequently, $A_{n+1} = \mathcal{A}_{n+1}$.

From $\mathcal{A}_n = A_n$ it follows that any commutator-associators of type $(\alpha, \beta, 1)$ of weight n of $Q/\mathcal{A}_n$ is equal to unit. Then by Theorem 3.2 the loop $Q/\mathcal{A}_n$ is centrally nilpotent of class $\leq n$. ∎

Corollary 3.4. *A loop* Q *is centrally nilpotent of class* n *when and only when it satisfies all commutator-associators identities* $w(x_1, \ldots, x_j) = 1$ *of type* $(\alpha, \beta, 1)$ *and weight* n*,* $w \in W_n(\alpha, \beta, 1)$*, but does not satisfy at least one identity* $v(x_1, \ldots, x_l) = 1$ *of type* $(\alpha, \beta, 1)$ *of weight* $n - 1$*,* $v \in W_{n-1}(\alpha, \beta, 1)$*.*

Particularly, if Q *is a commutative loop or an IP-loop, then it is sufficient to consider* $w \in W_n(\alpha, 1) \bigcup W_n(\beta, 1)$*,* $v \in W_{n-1}(\alpha, 1) \bigcup W_{n-1}(\beta, 1)$ *and* $w \in W_n(\mu, 1)$*,* $v \in W_{n-1}(\mu, 1)$ *for Moufang loops.*

The statements that follow from the equivalence of items 1), 3) of Theorem 3.2, are true.

Corollary 3.5. *Let* Q *be a loop with upper central series* $\{\mathcal{Z}_\alpha\}$ *described in Proposition 3.2. Then for any natural number* n *the member* $\{\mathcal{Z}_n\}$ *is a centrally nilpotent loop of class* $\leq n$*.*

The statement follows from (3.5).

Corollary 3.6. *The set of all centrally nilpotent loops of class* $\leq n$ *form a variety.*

Now we proceed to characterize the centrally solvable loops. Remind that $\mathcal{A}^{Q}(Q)$ denotes the commutator-associator of a loop Q, defined in Lemma 3.6. From the definition of derived f-series and Lemma 3.5 or Lemma 3.6 it follows.

Lemma 3.7. *The derived central series of a loop Q, denoted by $\{\mathcal{Q}^{\alpha}\}$, have the form:*
1) $\mathcal{Q}^{(0)} = Q$, $\mathcal{Q}^{(1)} = \mathcal{A}^{Q}(Q)$;
2) for any ordinal α, $\mathcal{Q}^{(\alpha+1)} = \mathcal{A}^{\mathcal{Q}^{(\alpha)}}(\mathcal{Q}^{(\alpha)}) = (\mathcal{Q}^{(\alpha)})^{(1)}$;
3) if α is a limit ordinal, then $\mathcal{Q}^{(\alpha)} = \bigcap_{\beta<\alpha} \mathcal{Q}^{(\beta)}$.

From Lemma 3.5 it follows that $\mathcal{Q}^{(\alpha+1)}$ is a normal subloop of $\mathcal{Q}^{(\alpha)}$ and quotient loop $\mathcal{Q}^{(\alpha)}/\mathcal{Q}^{(\alpha+1)}$ is an abelian group.

Theorem 3.3. *For any loop Q the following statements are equivalent:*

1) the loop Q is centrally solvable;
2) for a natural n the derived central series of the loop Q have the form:

$$Q = \mathcal{Q}^{(0)} \supset \mathcal{Q}^{(1)} \supset \ldots \supset \mathcal{Q}^{(s-1)} \supset \mathcal{Q}^{(s)} = \{1\}; \tag{3.6}$$

3) the loop Q has a finite series of subloops H_i such that

$$1 = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_t = Q,$$

H_i is normal in H_{i+1} and the quotient loop H_{i+1}/H_i is an abelian group;
4) the loop Q satisfies all commutator-associator identities $w(x_1,\ldots,x_j) = 1$ of type $(\alpha,\beta,2)$ of weight n, $w \in W_n(\alpha,\beta,2)$.

Particularly, if Q is a commutative loop or an IP-loop, then it is sufficient to considered item 2) $w \in W_s(\alpha,2) \bigcup W_s(\beta,2)$, and $w \in W_s(\mu,2)$, $v \in W_{s-1}(\mu,2)$ for Moufang loops.

Proof. We note that the last statement follows from Lemma 3.1 and (3.2). The equivalence of items 1), 2) is the definition of centrally solvable loops. Follows from definition of f-solvable loop and Lemma 3.7.

2) $\Rightarrow$ 4). We assume that the s-commutator-associator subloop $\mathcal{Q}^{(s)}$ of series (3.6) is equal to one. Since the $(s-1)$-commutator-associator subloop of the loop $\mathcal{Q}/\mathcal{Q}^{(s-1)}$ is equal to one then by inductive hypothesis we can consider that $w_{s-1}(g_1,\ldots g_i) \in \mathcal{Q}^{(s)}$ for any $w_{s-1} \in W_{s-1}(\alpha,\beta,2)$ and any $g_1,\ldots,g_i \in Q$. From here and from commutativity and associativity of the loop $\mathcal{Q}^{(s)}$ it follows that $w_s(g_1,\ldots g_j) = 1$ for any $w_s \in W_s(\alpha,\beta,2)$ and any $g_1,\ldots,g_j \in Q$.

4) $\Rightarrow$ 2). Conversely, let the loop Q satisfy the identities $\{w_s = 1 | \forall w_s \in W_s(\alpha,\beta,2)\}$. Let $H =< w_{s-1}(g_1,\ldots,g_i) \,|\, \forall w_{s-1} \in W_{s-1}(\alpha,\beta,2), \forall g_1,\ldots\ldots,g_i \in Q >$. Since $w_s = 1$, then according to the definition of commutator-associators w_s and (3.4) it follows that H is a normal, commutative and associative subloop. The quotient loop Q/H satisfies the identities $w_{s-1} = 1$. Hence $\mathcal{Q}^{(s-1)} \subseteq H$ that implies $\mathcal{Q}^{(s)} = 1$. ∎

We remind that for centrally solvable loops the smallest natural n such that $\mathfrak{Q}^{(s)} = 1$ is called *central solvability class.* From item 4) of Theorem 3.3 it follows that the set of all central solvable loops of class $\leq s$ form a variety. Then by Birkhoff's Theorem it follows that this class is closed with respect to taking of subloops, homomorphism images and finite direct products.

Corollary 3.7. *Let n be a natural number and let* $\{\mathfrak{Q}^{(\alpha)}\}$ *be the derived central series of a loop Q, defined by Lemma 3.7. Then the quotient loop* $\mathfrak{Q}^{(s)}/\mathfrak{Q}^{(s+1)}$ *is a centrally solvable loop of class* $\leq s+1$.

The statement follows from Theorem 3.3.

4. LOCAL THEOREM FOR THE KUROSH-CHERNIKOV'S CLASS OF LOOPS

By a *linear system* Σ of a loop Q we mean a (non-empty) set of subloops of Q, simply ordered by inclusion, such that:

(i) Σ contains 1 and Q;

(ii) the intersection of any non-empty set of elements of Σ is an element of Σ and

(iii) the union of any non-empty set of elements of Σ is an element of Σ (*completeness*).

By a *jump* of a linear system Σ of Q we mean an ordered pair H, K of elements of Σ such that $H \subset K$, $H \neq K$ and such that there exists no element M of Σ satisfying $H \subset M \subset K$, $M \neq H$, $M \neq K$. A linear system Σ is called *normal* (by the terminology in [1] *chief system*) of Q if each subloop of Σ is normal in Q. Also Σ is called a *subnormal system* (by the terminology in [1] *normal system*) if, for each jump H, K of Σ, H is normal in K.

If a subnormal (respect. normal) system contains another subnormal (respect. normal) system, then the first one is called the *refinement* of the second one.

Every finite loop, obviously, has a subnormal (respect. normal) system. Using the transfinite induction it's not difficult to prove that the subnormal (respect. normal) system also has any infinite loop.

Now let f be any nilpotency function for $\mathfrak{C}$. By an *f-nilpotent* (respect. *f-solvable*) *system*, Σ, of a loop Q in $\mathfrak{C}$ we mean a normal (respect. subnormal) system of Q such that, for each jump H, K of Σ, $(K, Q)_f \subseteq H$ or by property (iv) of f and definition of f-centre, equivalently, $K/H \subseteq Z_f(Q/H)$ (respect. $(K, K)_f \subseteq H$).

A linear system Σ of Q is called *descending* (respect. *ascending*) *series* if every non-empty subset of Σ has a maximal (respect. minimal) element.

For any nilpotency function f of a class of loops $\mathfrak{C}$ we define classes of loops, similar to the classes of Kurosh-Chernikov groups [9], [10], preserving the same name and, in contrast to [1], the same notations. For the class $\mathfrak{C}$ we get the following Kurosh-Chernikov's classes of loops:

RN_f: the loop has f-solvable subnormal system,

RN_f^*: the loop has f-solvable ascending subnormal system,

$\overline{RN}_f$: any subnormal system of loops can be refined to f-solvable subnormal system,

RI_f: the loop has f-solvable normal system,

RI_f^*: the loop has f-solvable ascending normal system,

$\overline{RI}_f$: any normal system of loops can be refined to f-solvable normal system,

Z_f: the loop has f-nilpotent normal system,

ZD_f: the loop has f-nilpotent descending normal system,

ZA_f: the loop has f-nilpotent ascending normal system,

$\overline{Z}_f$: any normal system of loop can be refined to f-nilpotent normal system.

According to [1] a nilpotency function f will be said to have *word type* if there exists at least one non-empty set

$$W = \{w_1(x_1, x_2, \dots, x_i), \dots, w_n(y_1, y_2, \dots, y_j)\} \tag{4.1}$$

of loop words such that, for Q in $\mathfrak{C}$, the loop $f(Q)$ is characterized as follows: The element a of Q is in $f(Q)$ if only if $w_j(a, x_2, \dots, x_{i_j}) = 1$ for every w_j in W and all $x_2, \dots, x_{i_j}$ in Q. It is readily verified that every nilpotency function discussed above has a word type. For example, a suitable set of words for central nilpotency consists of $[x, y]$, $\alpha(x, y, z)$, $\beta(x, y, z)$.

If (P) is a property of loops, a loop Q is said to have the property (P) *locally* if every finitely generated subloop of Q has the property (P). And (P) is a *local* property of loops if every loop, which has (P) locally also has (P). Further it is said that for a property of a loop (and respectively for the class of loops) the *local theorem* is true, if every loop, having this property locally, has it itself. In [1] by methods of loops it is proved, rather in a lengthy way, that if the nilpotency function f has the loops word type, then for Z_f-loops and RI_f-loops (SI_f-loops by the terminology in [1]) the local theorem is true. Here's the question if the local theorem is true for RN_f-loops (SN_f-loops by the terminology in [1]). The answer will be 'yes'. By Malcev's methods from [11] we will prove the fairness of the local theorem for Kurosh-Chernikov's classes of loops. But first we'll remind some necessary notions and results, which we can find, for instance, in [10], [11].

The system M_i ($i \in I$) of subsets of the set M is called its *local covering*, if any element from M is in some M_i and any two sets M_i, M_j are in some third set M_k.

Let M be a set. The function of n variables, defined on M and taking the value from the set $\{true, false\}$ (respect. from M), is called the *n- ary predicate* (respect. *n-ary operation*) on M. The set M with predicates $P_\alpha^{n_\alpha}$ and operations $f_\beta^{m_\beta}$ (n_α, m_β are their arity) marked in it is called the *algebraic system*.

The alphabet of *predicate calculus* language (*PC*) consists of objective and predicative variables, brackets and the following logical symbols (the first four are called *connections* and the last - *quantifiers*): $\wedge$ - and, $\vee$ - or, $\neg$ - non, $\to$ - imply, $=$ - equal, $\forall$ - for any, $\exists$ - exists. For example, the class of loops $\mathfrak{C}$, defined by the identities (2.1), in the ordinary signature of the loops theory $\cdot, /, \backslash, 1$, where $xy = z \leftrightarrow z/y = x \leftrightarrow x\backslash z = y$, can be given by the following axioms of *PC*:

a) the definition of the unit element: $(\forall x)(x1 = 1x = x)$;

b) the definition of the quasigroup:
$(\forall x)(\forall y)(x(x\backslash y) = x\backslash xy = y \wedge yx/x = (y/x)x = y)$;

c) the definition of the class $\mathfrak{C}$: $(\forall x_1)\dots(\forall x_i)\dots(\forall y_1)\dots(\forall y_n)\,(u(x_1,\dots\dots,x_i) = 1 \wedge \dots \wedge v(y_1,\dots,y_n) = 1)$.

Any formula of *PC* can be canonically rewritten into an equivalent *prenex formula*, where the quantifiers precede all other logical symbols. The prenex formula is called *objectively-universal*, if it doesn't contain the quantifiers $\exists$, for objective variables. The formula *PC* is called *quasiuniversal*, if it comes out of the objectively-universal formulas without free objective variables by using only logical operators with the following quantification $\forall$ on the free predicative variables.

Theorem 4.1. *(Malcev A. I.) If the quasiuniversal formula F is true on subsystems M_i, locally covering the algebraic system M, then F is also true on M.*

We use a method of Malcev [10], [11] to prove the following:

Theorem 4.2. *Let $\mathfrak{C}$ be the class of loops, defined by identities (2.1) and let f be a nilpotency function of word type (4.1). Then for the properties RN_f, RI_f, Z_f, $\overline{RN}_f$, $\overline{RI}_f$, $\overline{Z}$, $\overline{N}_f$ of loops of the class $\mathfrak{C}$ the local theorem is true.*

Proof. Let us turn from the language of subnormal systems to the language of predicates. For this we will connect the predicate P on Q with every subnormal system Σ of subloops of the loop Q, setting $P(x,y) = true$ if and only if in the system Σ there is a subloop, which contains x and doest's contain y. The predicate P satisfies the following universal axioms (the quantifiers are omitted):

1) $\neg P(x,x)$;

2) $P(x,y) \wedge P(y,z) \rightarrow P(x,z)$;

3) $P(x,z) \wedge \neg P(y,z) \rightarrow P(x,y)$;

4) $P(x,z) \wedge P(y,z) \rightarrow P(x\backslash y,z) \wedge P(x/y,z)$;

5) $x \neq 1 \rightarrow P(1,x)$;

6) $P(x,y) \wedge P(x,z) \rightarrow P(T(y)x,x) \wedge P(L(y,z)x,x) \wedge P(R(y,z)x,x)$.

Let us designate $A_y = \{x | x \in Q, P(x,y) = true\}, y \neq 1$. It is easy to verify that

$$A_y = \bigcup_{y \notin A \in \Sigma}, A = \bigcap_{y \notin A} A_y (A \in \Sigma), A \neq Q,$$

i.e. $A_y \in \Sigma$ and every $A \in \Sigma, A \neq Q$, is presented in a form of the intersection of the loops A_y.

Conversely, let there be a predicate P with the properties 1) - 6) in Q. The sets A_y form the system of subloops, linearly ordered by inclusion. If we add to this system of the loop Q, and also the unions and intersections of any subsystems of subloops, then we will receive a subnormal system of subloops in the loop Q. From 1) - 3) it follows easily that the indicated jumps from Σ to P and from P to Σ are mutually converse, and that's why they realize the required transference in PC language.

The main notions of the theory of Kurosh-Chernikov's classes after the transference in language will look as follows:

7) f-solvability of the system: $(x_1 \neq 1) \wedge (y_1 \neq 1) \wedge \neg P(x_1, x_2) \wedge \neg P(x_2, x_1) \wedge \ldots \wedge \neg P(x_1, x_i) \wedge \neg P(x_i, x_1) \wedge \ldots \wedge \neg P(y_1, y_2) \wedge \neg P(y_2, y_1) \wedge \ldots \wedge \neg P(y_1, y_j) \wedge \neg P(y_j, y_1) \rightarrow P(w_1(x_1, x_2, \ldots, x_i), x_1) \wedge \ldots \wedge P(w_n(y_1, y_2, \ldots\ldots, y_j), y_1)$;

8) the normality of the system: $P(x, y) \rightarrow P(T(u)x, y) \wedge P(L(u, v)x, y) \wedge P(R(u, v)x, y)$;

9) f-nilpotency of the system:

$$(x_1 \neq 1) \wedge (y_1 \neq 1) \rightarrow P(w_1(x_1, x_2, \ldots\ldots, x_i), x_1) \wedge \ldots \wedge P(w_n(y_1, y_2, \ldots, y_j), y_1).$$

Let us also characterize the "trinomial" of systems $1 \subseteq A \subseteq Q$ in PC language. Obviously, for this we have to add to the axioms 1) - 5) the axiom:

10) the trinomiality of the system: $x \neq 1 \wedge P(x, y) \rightarrow \neg P(y, x)$.

Now it's clear that the properties RN_f, RI_f, Z_f are written by the formulas:

$RN_f : (\exists P)(\forall x)(\forall y)(\forall z)((1) \wedge (2) \wedge (3) \wedge (4) \wedge (5) \wedge (7))$,
$RI_f : (\exists P)(\forall x)(\forall y)(\forall z)((1) \wedge (2) \wedge (3) \wedge (4) \wedge (5) \wedge (7) \wedge (8))$,
$Z_f : (\exists P)(\forall x)(\forall y)(\forall z)((1) \wedge (2) \wedge (3) \wedge (4) \wedge (5) \wedge (9))$,

where $(1), (2), \ldots$ are the expressions from $1), 2), \ldots$, and the properties $\overline{RN}_f, \overline{RI}_f, \overline{Z}_f, \overline{N}_f$, containing the property of refinement, are written by the formulas:

$\overline{RN}_f : (\forall P)((P-\text{ subn.}) \rightarrow (\exists G)((G- f\text{-solv. subn.}) \wedge (\forall u)(\forall v)(P(u, v) \rightarrow G(u, v))))$,
$\overline{RI}_f : (\forall P)((P-\text{ norm.}) \rightarrow (\exists G)((G- f\text{-solv. norm.}) \wedge (\forall u)(\forall v)(P(u, v) \rightarrow G(u, v))))$,
$\overline{Z}_f : (\forall P)((P-\text{ norm.}) \rightarrow (\exists G)((G- f\text{-nilpot.}) \wedge (\forall u)(\forall v)(P(u, v) \rightarrow G(u, v))))$,
$\overline{N}_f : (\forall P)((P-\text{trinomial}) \rightarrow (\exists G)((G-\text{ subn.}) \wedge (\forall u)(\forall v)(P(u, v) \rightarrow G(u, v))))$,
where the abridged notes must be changed by the evident combinations of formulas 1) – 10). Because all the seven properties are quasiuniversal, we will use the Theorem 4.1 to complete the proof of Theorem 4.2. ∎

For the properties $RN^*_f, RI^*_f, ZA_f, ZD_f, N_f$, which are not covered by Theorem 4.1, the local theorem is not true even in case of groups [10].

Defining some Kurosh-Chernikov's classes of loops the notions of ascending (or descending) f-nilpotent normal system, and also of ascending f-solvable normal (or subnormal) system were used. Let us make this fact more concrete. Let us introduce new notions following mainly the ideas from [1]. Note that the terminology introduced here differs from the one of [1]. Let f be some nilpotency function for $\mathfrak{C}$ and let the loop Q be in $\mathfrak{C}$. The ascending f-nilpotent (respect. f-solvable) normal series, $\{Z_\alpha\}$, (respect. $\{H_\alpha\}$) of Q is defined inductively as follows: 1) $Z_0 = H_0 = 1$; 2) for any ordinal α $Z_{\alpha+1}/Z_\alpha \subseteq Z_f(Q/Z_\alpha)$ or, equivalently, $(Z_{\alpha+1}, Q)_f \subseteq Z_\alpha$ (respect. $(H_{\alpha+1}, H_{\alpha+1})_f \subseteq H_\alpha$); 3) if α is a limit ordinal, $Z_\alpha = \bigcup_{\beta<\alpha} Z_\beta$ (respect. $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$). Then the loop Q is a ZA_f-loop (respect. RI_f-loop) if for some ordinal α $Z_\alpha = Q$ (respect. $H_\alpha = Q$). If for a finite ordinal n $Z_n = Q$, then Q is called *f-nilpotent*. The smallest n with this property is called *f-nilpotence class*.

The series $\{Z_\alpha\}$ is called *upper f-nilpotent series* if in 2) $Z_{\alpha+1}/Z_\alpha = Z_f(Q/Z_\alpha)$ or, equivalently, $(Z_{\alpha+1}, Q)_f = Z_\alpha$. The descending f-nilpotent normal series. $\{Q_\alpha\}$, of Q is defined inductively as follows: 1) $Q_0 = Q$; 2) for any ordinal α, $Q_{\alpha+1} \subseteq (Q_\alpha, Q)_f$; 3) if α is a limit ordinal, $Q_\alpha = \cap_{\beta<\alpha} Q_\beta$. Then the loop Q is ZD_f-loop if for some ordinal α $Q_\alpha = 1$. If in 2) $Q_{\alpha+1} = (Q_\alpha, Q)_f$, then $\{Q_\alpha\}$ is called *lower f-nilpotent series*. Similarly we define the *f-derived series* $\{Q^{(\alpha)}\}$ of Q inductively: $Q_0 = Q$; for any ordinal α, $Q^{(\alpha+1)} = (Q^{(\alpha)}, Q^{(\alpha)})_f$; if α is a limit ordinal, $Q^{(\alpha)}$ is the intersection of all $Q^{(\beta)}$ with $\beta < \alpha$. The loop Q will be called *f-solvable* if for a finite ordinal n $Q^{(n)} = 1$. The smallest integer n with such a property will be called the *class of f-solvability*.

Proposition 4.1. *Let $\mathfrak{C}$ be a class of loops satisfying (a), (b) and let f be any nilpotency function for $\mathfrak{C}$. Then for the loops from $\mathfrak{C}$ the properties RN_f, RI_f, RN^*_f, RI^*_f, Z_f, ZA_f, N_f, $\overline{N}_f$ are transferred to the subloops, and the properties RN^*_f, RI^*_f, $\overline{RN}_f$, $\overline{RI}_f$, Z_f, ZA_f, N_f, $\overline{N}_f$ are transferred to the homomorphic images from $\mathfrak{C}$.*

Proof. Let Q be a loop from $\mathfrak{C}$ with a subnormal (respect. normal) system $\Sigma = \{Q_\alpha\}$ and let H be a subloop of Q. Let us denote $\sigma = \{H_\alpha\}$, where $H_\alpha = H \cap Q_\alpha$. The system σ will be the ordered system of subloops of the loop H, possibly with repetitions, and $1 \in \sigma$, $H \in \sigma$. From the completeness of the system Σ it follows easily that the system σ is complete.

If $H_\beta = H_{\beta+1} = \ldots$ in σ, then due to the completeness of the system σ, the intersections of all $H_\tau \in \sigma$ with the condition $H_\tau > H_\beta$ and the union of all $H_\eta \in \sigma$ with the condition $H_\eta < H_\beta$ are contained in σ. That's why interchanging every such set of subloops $\{H_\beta, H_{\beta+1}, \ldots\}$ in one subloop, we don't create new jumps. Note that if the system Σ is an ascending series, then the system σ is the same.

If the subloop H of the loop Q contains the subloop K, then the intersection with H of the subnormal (respect. normal) system Σ of the loop Q, which goes through K, after removal of repetitions gives a subnormal (respect. normal) system σ of the loop H, which contains the subloop K.

For any jump H_α, $H_{\alpha+1}$ of the system σ there is a corresponding jump Q_α, $Q_{\alpha+1}$ of the system Σ and by (1.3) from the normality of the subloop Q_α in $Q_{\alpha+1}$ (respect. in Q) it follows that the subloop H_α in $H_{\alpha+1}$ (respect. in H) is normal. According to the homomorphism theorems we obtain:

$$H_{\alpha+1}/H_\alpha = H_{\alpha+1}/(H_{\alpha+1} \cap Q_\alpha) \cong H_{\alpha+1}Q_\alpha/Q_\alpha.$$

However $H_{\alpha+1}Q_\alpha \subseteq Q_{\alpha+1}$, that's why the quotient loop $H_{\alpha+1}H_\alpha$ is isomorphic to the subloop of the quotient loop $Q_{\alpha+1}Q_\alpha$.

Let Σ be a f-solvable system. Then $Q_\alpha = (Q_{\alpha+1}, Q_{\alpha+1})_f$ or, by property (iv) of f, $Q_{\alpha+1}/Q_\alpha \subseteq f(Q_{\alpha+1}/Q_\alpha)$. Since the subloop $H_{\alpha+1}/H_\alpha$ is isomorphic to a subloop of the loop $Q_{\alpha+1}/Q_\alpha$, then by property (ii) of f $H_{\alpha+1}/H_\alpha \subseteq f(H_{\alpha+1}/H_\alpha)$. Therefore σ will be a f-solvable system. Similarly it is proved that if Σ is a f-nilpotent system, then σ will also be a f-nilpotent system.

Now from the different combinations of the aforementioned statements the proof of the first part of the Proposition follows.

Now let H be a normal subloop of the loop Q with a subnormal (respect. normal) system Σ. Let us put $\overline{H}_\alpha = Q_\alpha H/H$. By Lemma 1.1 $Q_\alpha H$ is a subloop. If the subloop Q_α is normal in $Q_{\alpha+1}$ (respect. in Q), then according to the homomorphism theorems, we have the normality of the subloop $\overline{Q}_\alpha$ in $\overline{Q}_{\alpha+1}$ (respect. in $\overline{Q}$). Moreover, we obtain

$$\begin{aligned}\overline{Q}_{\alpha+1}/\overline{Q}_\alpha &\cong Q_{\alpha+1}H/Q_\alpha H \cong Q_{\alpha+1}/(Q_\alpha(Q_{\alpha+1}\cap H))\\ &\cong (Q_{\alpha+1}/Q_\alpha)/((Q_\alpha(Q_{\alpha+1}\cap H))/Q_\alpha).\end{aligned}$$

This means that the quotient loop $\overline{Q}_{\alpha+1}/\overline{Q}_\alpha$ is a homomorphic image of the quotient loop $Q_{\alpha+1}/Q_\alpha$. If $\{Q_\alpha\}$ is an ascending series, then $\{\overline{Q}_\alpha\}$ will be the same. Then, by property (iii) of f we can easily prove the second part of the Proposition in cases RN^*_f, RI^*_f, Z_f, ZA_f.

Now assume that the loop Q has the property $\overline{RI}_f$ and let us consider the subnormal system σ in quotient loop Q/H. The subnormal system σ corresponds to a subnormal system, which goes through H. Refine it to a f-solvable system and turn to the quotient loop Q/H. As a result we get the f-solvability of the subnormal system, obtained by refining the system σ. Therefore the property $\overline{RI}_f$ is transferred to the quotient loop. Similarly, the cases $\overline{RN}_f$, $\overline{Z}_f$, N_f, $\overline{N}_f$ are also proved. ∎

Let us consider the set

$$W = \{w_s(x_{i_1}, x_{i_2}, \dots, x_{i_s}) | s = 1, \dots, t\} \tag{4.2}$$

of loop words. We define the *loop words of type* $(W,1)$ *of weight* n in the loop Q inductively:

1) the elements of the loop Q are the loop words of type $(W,1)$ of weight 0;

2) the elements $w_s(u_{i_1}, u_{i_2}, \dots, u_{i_s})$, where $u_{i_j} \in Q, j = 1,2,\dots,s, s = 1,\dots,t$, are the loop words of type $(W,1)$ of weight 1;

3) if a_{i_1} is a loop word of type $(W,1)$ of weight $n-1$, then $w_s(a_{i_1}, u_{i_2}, \dots, u_{i_s})$, where $u_{i_j} \in Q, j = 2,\dots,s, s = 1,\dots,t$, is the loop word of type $(W,1)$ of weight n.

Similarly *the loop words of type* $(W,2)$ *of weight* n are defined. The condition 3) should be only replaced by:

3') if a_{i_j}, where $j = 1,2,\dots,s, s = 1,\dots,t$ are the loop words of type $(W,2)$ of weight $n-1$, then $w_s(a_{i_1}, a_{i_2}, \dots, a_{i_s})$ are the loop word of type $(W,2)$ of weight n.

If in the loop Q all the loop words of type $(W,1)$ (respect. $(W,2)$) of weight n are equal to the unit, then we'll say, that the *identity of type* $(W,1)$ *(respect.* $(W,2)$*) of weight* n is fulfilled in Q.

Let f be a nilpotency function of word type for $\mathfrak{C}$, defined by the set of the loop words (4.2). If $Q \in \mathfrak{C}$, then by the definition of nilpotency function of word type, the set $f(Q) = \{a \in Q \,|\, w_j(a, x_2, \ldots, x_{i(j)}) = 1\, \forall x_2, \ldots, x_{j(i)} \in Q, \forall w_j \in W\}$ is a subloop of Q. Further we will assume that the subloop $f(Q)$ is normal in Q. For this it is enough to assume that the set W would contain the loop words $\alpha(x, y, z)$, $\beta(x, y, z)$, (x, y). In this case, the subloop $f(Q)$ belongs to the centre $Z(Q)$ of the loop Q and, therefore, is normal in Q.

Proposition 4.2. *Let f be a nilpotency function of word type for a class $\mathfrak{C}$, defined by set (4.2) of loop words W and such, that if a loop Q is in $\mathfrak{C}$, then the subloop $f(Q)$ is normal in Q, and let $\{Q_i\}$ (respect. $\{Q^{(i)}\}$) be the lower f-nilpotent series (respect. f-derived series) of loop Q. Then the subloop Q_n (respect. $Q^{(n)}$) coincides with the smallest normal subloop A_n (respect. B_n) of Q containing all loop words of type $(W, 1)$ (respect. $(W, 2)$) of weight n $\{a_n\}$ (respect. $\{b_n\}$), where $n = 1, 2, \ldots$.*

Proof. If f has a word type and K is a normal subloop of the loop Q in $\mathfrak{C}$, then the natural homomorphism $x \to xK$ maps $f(Q)$ into a subloop of $f(Q/K)$; consequently, the set of all elements of form $w_j(k, x_2, \ldots, x_{i(j)})$, where $w_j \in \mathfrak{C}$, $k \in K$ and $x_2, \ldots, x_{i(j)} \in Q$, must be a subset of K. Considering next the homomorphism $x \to x(K, Q)_f$, we can see that the smallest normal subloop of Q containing the set of elements just described must be $(K, Q)_f$. Then from the definition of the element a_n it follows that $a_n \in Q_n$, and then $A_n \subseteq Q_n$.

Let $w_j \in W$, $x_2, \ldots, x_{i(j)} \in Q$. Then $w_j(a_n, x_2, \ldots, x_{i(j)}) \in A_{n+1}$. Let us consider the homomorphism $Q \to Q/A_{n+1} = \overline{Q}$. In the loop $\overline{Q}$ the equality $w_j(\overline{k}, \overline{x}_2, \ldots, \overline{x}_{i(j)}) = \overline{1}$ is true for all loop words $w_j \in W$. Then $\overline{a}_n \in f(\overline{Q})$. As it was assumed the subloop $f(\overline{Q})$ is normal in $\overline{Q}$ and $\overline{a}_n \in \overline{A}_n$. Then $\overline{A}_n \subseteq f(\overline{Q})$. Further we have $A_1 = Q_1$. Suppose that $A_n = Q_n$. Then $Q_n/A_{n+1} = A_n/A_{n+1} \subseteq f(Q/A_{n+1})$ and since $A_{n+1} \subseteq Q_{n+1}$, then from property (iv) of the definition of the nilpotency function it follows that $A_{n+1} = Q_{n+1}$. Similarly, it is proved, that $B_{n+1} = Q^{(n+1)}$. ∎

Corollary 4.1. *Under the hypothesis of Proposition 4.2 let a set of loop words W contain the words (x, y), $\alpha(x, y, z)$, $\beta(x, y, z)$, defined in (3.1). Then the subloop Q_n (respect. $Q^{(n)}$) of the lower f-nilpotent series (respect. f-derived series) of the loop Q coincides with the subloop of the loop Q, generated by all loop words of type $(W, 1)$ (respect. $(W, 2)$) of multiplcity n.*

Proof. This is a consequence of Proposition 4.2 and Lemma 3.2. ∎

Now let f be a nilpotency function of word type defined by the set of the loop words (4.2). It is clear, that any locally f-nilpotent loop belongs to the class of Z_f-loops, because the local theorem is true for this class. Also it is clear that the converse inclusion is not true. However such a statement, generalizing the corresponding result of Malcev for the ZA-loops is true [12]. Note, that here a similar reasoning as in [12] is used.

Proposition 4.3. *Every ZA_f-loop is locally f-nilpotent.*

Proof. Let us prove it by induction using the length of the ascending f-nilpotent series. Let

$$\{1\} = Z_0 \subset Z_1 \subset \ldots \subset Z_\alpha \subset \ldots \subset Z_\gamma = Q \tag{4.3}$$

be the ascending f-nilpotent series of the loop Q and let the statement be true for the loops with lesser length of the central series. Let us consider the finite set of non-identity elements

$$a_1, a_2, \ldots, a_n \tag{4.4}$$

from Q. If γ is a limit ordinal, then all these elements are contained in some subloop Z_δ ($\delta < \gamma$) and due to the induction they generate the f-nilpotent subloop in it.

Let γ be non-limit ordinal. Then there is such a limit ordinal δ and such a natural number k, that $\gamma = \delta + k$. Let us take all the loop words of type $(W, 1)$ of weight k generated from the elements of the set (4.4). There is a finite number of such loop words and by definition of the series (4.3) all of them are in the subloop Z_δ, and therefore, since the number δ is a limit ordinal, they are in some Z_η, $\eta < \delta$. Let

$$H =< Z_\eta, a_1, a_2, \ldots, a_n > .$$

In H the ascending f-nilpotent series can be constructed as follows. The beginning of this series will be the segment of the series (4.4) up to Z_η inclusively. Further there is the subloop $\overline{Z}_{\eta+1}$, which corresponds to the f-centre of the quotient loop H/Z_η, then the subloop $\overline{Z}_{\eta+2}$, which corresponds to the f-centre of the quotient loop $H/\overline{Z}_{\eta+1}$, etc. Since the subloop $\overline{Z}_{\eta+1}$ contains all loop words of type $(W, 1)$ of weight $k-1$ of the elements of the set (4.4), then the subloop $\overline{Z}_{\eta+2}$ contains all the loop words of type $(W, 1)$ of weight $k-2$ of the elements from (4.4), etc. Thus, this f-nilpotent series reaches H no more than in k steps, that is, it turns into the ascending f-nilpotent series, whose length doesn't exceed $\eta + k$. Since δ is a limit ordinal, and $\eta < \delta$, then $\eta + k$ is strictly less than $\delta + k$ and hence more less than γ. Hence by inductive hypothesis we obtain the f-nilpotency of the subloop $< a_1, a_2, \ldots, a_n >$. This completes the proof of the Proposition. ∎

For the ZA-groups the local theorem is not true. This follows from the existence of locally finite p-groups without the centre. The local theorem is not true even for such close to abelian groups as commutative Moufang loops. It follows from the following reasons. Such periodic loops are locally finite and locally centrally nilpotent, and there are commutative Moufang loops of the exponent three without the centre [1]. However the Proposition 4.4 is true, and also the Proposition 4.5, which generalizes the corresponding statement for the ZA-groups [12]. Note, that the Proposition 4.4 also follows from the Theorem 4.2, but in contrast to the Theorem 4.2, when we prove it, we don't use the local theorem of Malcev (Theorem 4.1).

Let $\mathfrak{C}$ be a class of loops satisfying (a), (b) and let h be a nilpotence function for $\mathfrak{C}$ satisfying one of the conditions: 1) for the loop Q from $\mathfrak{C}$ the subloop $h(Q)$ is defined as the centre $Z(Q)$ of the loop Q; 2) if $\mathfrak{C}$ is the class of Moufang loop, then for $Q \in \mathfrak{C}$ the subloop $h(Q)$ is defined as the left nucleus $N_l(Q)$ of the loop Q. By [1] the subloop $h(Q)$ is normal in Q.

We will prove the following statements for the function with the condition 2), as the proof and in case 1) almost identically.

Lemma 4.1. *Every non-trivial normal subloop H of a ZA_h-loop Q has an intersection with its centre, which is different from the unit subgroup.*

Proof. By repeating word by word the reasoning of the proof of Theorem 1.2 from [1], pag. 97, (or the Theorem 2.2) and by only using the simple transfinite induction the following statement can be proved.

Let f be a nilpotence function. A loop Q then and only then there is a ZA_f-loop, if continuing its upper f-nilpotent series (may be transfinitely) we reach the loop Q.

Then let

$$\{1\} = Z_0 \subset Z_1 \subset \ldots \subset Z_\alpha \subset \ldots \subset Z_\gamma = Q \tag{4.5}$$

be the upper h-nilpotent series of loop Q and H_α be the first member of this series, which has the non-unity intersection H_μ. Since the ordinal number μ is a non-limit, then the member $Z_{\mu-1}$ of the series is considered. The intersection $H_{\mu-1}$ is equal to unit according to the assumption about the intersection H_μ. If all the associators $\alpha(n, x, y)$, where $n \in H_\mu$, $x, y \in Q$, are equal to the unit, then $H_\mu \subseteq Z_1$ and the statement is proved.

Let us consider the second case. Assume, for example, $\alpha(n, x, y) \neq 1$ for some $a, b, \in Q, n \in H_\mu$. The subloop H is normal in Q, thus by (3.1) and (1.3) $\alpha(n, a, b) = L^{-1}(n)R(a, b)n \in Z_\mu$. From the relations $n \in Z_\mu$ and $Z_\mu/Z_{\mu-1} = Z_h(Q/Z_{\mu-1}$ it follows, that $\alpha(n, x, y) \in Z_{\mu-1}$. But this contradicts the fact, that $Z_{\mu-1} \cap H = \{1\}$. Consequently, the second case is impossible. ∎

Corollary 4.2. *In a ZA_h-loop any minimal normal subloop lies in its h-centre.*

Proposition 4.4. *Any locally h-nilpotent loop Q is a Z_h-loop.*

Proof. Assume, that the minimal normal subloop H of the loop Q is not contained in its h-centre. Then there are such elements $n \in H$ and $a, b \in Q$, that $\alpha(n, a, b) \neq 1$. Assume, for example, that $c = \alpha(n, a, b) \neq 1$. According to (3.1) $R(a, b)n = n\alpha(n, a, b)$, then by (1.3) $c \in H$. Due to the minimality of the normal subloop H, it is generated due to (1.3) by the set $\{\varphi c \,|\, \varphi \in \Im(Q)\}$. Then in $\Im(Q)$ there are such mappings $\varphi_1, \ldots, \varphi_r$, that n lies in the subloop, generated by the elements

$$\varphi_1 c, \ldots, \varphi_r c. \qquad (4.6).$$

By [1] any inner mapping of the loop Q is a product of a finite number of mappings of the type $R^{\pm 1}(a, b), L^{\pm 1}(a, b), T^{\pm 1}$, where $a, b \in Q$. Let $v_1, v_2, \ldots, v_s \in Q$ be the elements, which were used in the definitions of the mappings $\varphi_1, \ldots, \varphi_r$ (they are finite number). We denote by K the subloop, generated by the elements $n, a, b, v_1, \dots, v_s$ (according to the supposition it is centrally nilpotent), and by M the subloop, generated by the set $\{\psi n | \psi \in \Im(K)\}$. By property (iv) of the definition of the nilpotence function the subloop $(M, K)_h$, clearly, contains the element c. Since the subloop M is normal in K, then all the elements (4.6) are included as well as the element n in $(M, K)_h$. Thus $M \subseteq (M, K)_h$. By (3.1) from the normality of the subloop M in K it follows that $(M, K)_h \subseteq M$, and then $M = (M, K)_h$. By Zorn's Lemma there is a maximal normal in K subloop S from M, which doesn't contain the element n. Then M/S is a minimal normal subloop of the quotient loop K/S. The loop K is h-nilpotent, then, according to the Corollary 4.2, $M/S \subseteq Z_h(K/S)$ or $(M, K)_h \subseteq S$. Since $S \neq M$, we get $(M, K)_h \neq M$. This contradiction demonstrates the fairness of the statement proved. ∎

We will need below the following result:

Lemma 4.2. *The loop Q is a ZA_f-loop if and only if for any of its element a and any sequens of elements $x_1, x_2, \dots, x_k, \dots$ there is such a number n, that at least one loop word of type $(W, 1)$ of weight n is equal to unit.*

Proof. Let (4.5) be ascending f-nilpotent series of the ZA_f-loop Q. If none of the elements

$$u(n) = w_j(a, x_1, x_2, \dots, x_{i(n)}), n = 1, 2, \dots, \tag{4.7}$$

is equal to unit, then there is such a number α, $\alpha \geq 0$, that the subloop Z_α doesn't have any the elements, say, $u(n)$, while the subloop $Z_{\alpha+1}$ has at least one of the elements, say, $u(r)$. Then due to the relation $Z_{\alpha+1}/Z_\alpha = Z_f(Q/Z_\alpha)$, $u(r+1) = w_s(u(r), x_{i(r)+1}, \dots, x_{(i(r)+t)}) \in Z_\alpha$, but this contradicts the choice of α.

Let us prove now, that if in the loop Q for any elements a and $x_1, x_2, \dots$ $\dots, x_k, \dots$ at least one of the element (4.7) is equal to unit, then Q has the nontrivial f-centre. In fact, if the element a of Q doesn't lie in the f-centre, then there will be such a word $w_j \in W$ and such elements $x_2, \dots, x_{i(j)} \in Q$, that $u_j = w_j(a, x_2, \dots, x_{i(j)}) \neq 1$. If u_j doesn't lie in the f-centre $Z_f(Q)$, then there will be such a word $w_k \in W$ and such elements $x_{i(j)+1}, \dots, x_{i(k)} \in Q$, that $w_k(u_j, x_{i(j)+1}, \dots, x_{i(k)}) \neq 1$. This process cannot continue infinitely, because we would come into contradiction with the assumption about the loop Q. Hence some element $1 \neq w_k(a, x_{i(2)}, \dots, x_{i(k)}) \in Z_f(Q)$. Then in the loop $Q/Z_f(Q) = \overline{Q}$ the equality $w_k(\overline{a}, \overline{x}_{i(2)}, \dots, \overline{x}_{i(k)}) = \overline{1}$ is fulfilled. Consequently, in $Q/Z_f(Q)$ the condition of lemma is realized. Therefore the loop $Q/Z_f(Q)$ is a ZA_f-loop, and then the loop Q will also be a ZA_f-loop. ∎

By using the above Lemma, we prove the following.

Proposition 4.5. *Let f be a nilpotency function of word type, defined by set of loop words (4.2). Then any loop, all the countable subloops of which are ZA_f-loops, will be the ZA_f-loop itself.*

Proof. If the loop Q is not a ZA_f-loop, then by the Lemma 4.2, there will be such elements $a, x_1, \dots, x_k, \dots$ in it, that none of the elements (4.7) will be not equal to unit. The subloop $< a, x_1, \dots, x_k, \dots >$ is countable, but again by the Lemma 4.2 is not a ZA_f-loop. We've get the contradiction, thus the loop Q is a $ZA - f$-loop. ∎

Note, that similarly to groups [10] the statement may be proved: *the extension of the locally finite loop by means of locally finite loop is locally finite itself.* It is easy to see, that the union of the ascending series of locally finite loops is locally finite itself. The periodic abelian group is locally finite. If under the *periodic loop* we understand the loop, in which any of its elements generates a finite subloop, then from this we easily get.

Proposition 4.6. *Every periodic RN_h^*-loop is locally finite.*

The local finiteness of any periodic h-solvable loop, and also the local finiteness of any periodic locally h-solvable loop follow from this statement.

By the condition of the local finiteness the classes RN_h-loops, $\overline{RN_h}$-loops coincide with the locally h-solvable loops and, consequently, coincide with one another. In fact, since the local theorem is true for them, then by Corollary 2.2 if Q is locally finite RN_h-loop, all the subloops are h-solvable, and thus it is locally h-solvable.

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PARAMETER EVALUATION FOR BIOCONCENTRATION MODEL, USING CELLULAR EXCLUSION METHOD

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Abstract We consider a class of biological phenomena described by ordinary differential equations (ODEs) for which certain physical parameters are unknown and very difficult (if not impossible) to be measured. Focusing on the problem of determining "the best fitting parameters" for given observable data, we report on a method for finding the set of parameters that minimizes a cost function associated to the problem. This technique uses the "*Cellular Exclusion Algorithm*" [3] as a fast locating tool of its critical points. We also emphasize how a carefully chosen dominant function can improve the efficiency and reduce the cost of such method.

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1. A BIOCONCENTRATION MODEL

1.1. INTRODUCTION

We consider an environment contaminated by a certain substance (a metal, by example) with uniform concentration c_w. The environment is populated by N species $S_1, S_2, \ldots, S_N$. Each species S_r is characterized by a concentration C_r of the contaminant, for $r = 1, \ldots, N$. The toxic substance flows from the environment into the species and circulates between species, obeying the following linear transfer rules: the uptake flux from environment into the species S_r is $k_r^u c_w$, the release flux for the species S_r is $k_r^e C_r$ (for $r = 1, 2, \ldots, N$) and the uptake flux from species S_p ($p = 1, 2, \ldots, r-1$) into the species S_r ($r = 2, 3, \ldots, N$) is $k_{rp}^u C_p$. It is assumed that there is no uptake flux form species S_p into the species S_r for $p > r$. Such models can be found in [1], [2], for example.