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ON METHODS OF DERIVING THE PSEUDO-MINOR GROUPS OF W_Q-SYMMETRY

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Abstract In this paper the properties and the general structure of pseudo-minor groups of W_q-symmetry are analyzed. A method for deriving the pseudo-minor groups of W_q-symmetry from its generating groups and initial group definition is developed.

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1. INTRODUCTION

The groups of the classical symmetry and of generalized symmetries (antisymmetry, l-multiple antisymmetry, P-symmetry, polychromatic symmetry, complex symmetry, W-symmetry and others) have wide applications in physics, crystallography, geometry, biology and other fields of contemporary science and arts (see references from [7]-[10]).

The bases of general theory of recently generalized symmetries have been developed in the last two decades [3], [4]. Some aspects of the theory of pseudo-minor groups of W-symmetry were studied in [1]-[6]. In the present paper, first of all we briefly analyze the properties and the general structure of pseudo-minor groups of W_q-symmetry. In the last section we have developed a method for deriving the pseudo-minor groups of W_q-symmetry from its generating groups and initial group definition.

2. ELEMENTS OF GENERAL THEORY OF DISCRETE GROUPS OF W_Q-SYMMETRY

First we recall some aspects of the general theory of groups of W_q-symmetry.

Let us have the group G and the set $N = \{1, 2, ..., m\}$, whose elements represent non-geometrical features. We fix a transitive group of permutations P onto N. We construct the Cartesian product W of isomorphic copies of the group P which are indexed by elements of G, i.e. $W = P^{g_1} \times P^{g_2} \times ... \times P^{g_n} \times ... = \overline{\prod}_{g_i \in G} P^{g_i}$.

Let us consider groups G, P and $W = \overline{\prod}_{g_i \in G} P^{g_i}$ (where $P^{g_i} \cong P$), the isomorphic injection $\varphi : G \to AutW$ defined by the rule $\varphi(g) = \overleftarrow{g}$, where the automorphism $\overleftarrow{g}$ makes the left g-translation of the components in $w \in W$ (i.e $\overleftarrow{g}$: $w \mapsto w^g, w = \langle p^{g_1}, p^{g_2}, ...\rangle$, and $w^g = \langle p^{gg_1}, p^{gg_2}, ...\rangle$), and also the homomorphism $\tau : G \to \Phi \leq AutW$, with the kernel $Ker\tau = H_1$, where $\tau(g) = \overrightarrow{\tau_g}$ and $\overrightarrow{\tau_g}\ (w) = gwg^{-1}$.

The set A of all pairs wg, where $w \in W$ and $g \in G$, forms a group with the operation $w_i g_i * w_j g_j = w_k g_k$, where $g_k = g_i g_j$, $w_k = w_i^{g_j}\ \overrightarrow{\tau}_{g_i}\ (w_j)$, $w_i^{g_j}(g_s) = w_i(g_j g_s)$ and $\overrightarrow{\tau}_{g_i}\ (w_j) = g_i w_j g_i^{-1}$, called standard crossed wreath product of group P with the group G, accompanied with the left translation and the homomorphism τ, i.e. $A = P\ \overrightarrow{\imath}_{\tau H(\Phi)} \overleftarrow{G}$ [3].

Let M_1 be a point of general position of a geometrical figure F with the discrete symmetry group G. Acting with group G on the point M_1, we get the entire system of G-equivalent points, i.e. $g_j(M_1) = M_j \in F$, where $g_j \in G$. Assigning to each point of the geometrical figure F at least one "index" from the set N, consequently, we obtain "indexed" geometrical figure $F^{(N)}$.

The transformation of W_q-symmetry is defined as an isometric mapping $g^{(w)} = wg$ of the figure $F^{(N)}$ onto itself, in which the geometrical component g operates both on points M_k and on "indices" by the given rule independent of the points, but the permutation p^{g_k} ("g_k-component" in w) is only a supplementary permutation of "indices" at the point M_k to map $F^{(N)}$ onto itself [3].

The set $G^{(W_q)}$ of transformations of W_q-symmetry of the given "indexed" geometrical figure $F^{(N)}$ forms a group $G^{(W_q)}$ with the operation:

$$g_i^{(w_i)} \cdot g_j^{(w_j)} = g_k^{(w_k)}, \tag{1}$$

where $g_k = g_i g_j$, $w_k = w_i^{g_j}\ \overrightarrow{\tau}_{g_i}\ (w_j)$, $w_i^{g_j}(g_k) = w_i(g_j g_k)$, and $\overrightarrow{\tau}_{g_i}\ (w_j) = g_i w_j g_i^{-1}$. $G^{(W_q)}$ is called group of W_q-symmetry of "indexed" figure $F^{(N)}$.

Let $G^{(W_q)}$ be a group of W_q-symmetry. We note by W' the set of all w, which enter as components in $g^{(w)}$ from $G^{(W_q)}$. The set W', in general, is not a group, but W' always contains the unit w_o of the group W and verifies the condition $w_0 \subseteq W' \subseteq W$. If $W' = W$, the group $G^{(W_q)}$ is defined as a group of complete W_q-symmetry, and if $W' \neq W$, then $G^{(W_q)}$ is called group of incomplete W_q-symmetry. The group G is called generating group for $G^{(W_q)}$ ($G^{(W_q)} = G$, if $W' = w_0$), and the set of all groups of W_q-symmetry with the same generating group - family groups.

Let $G^{(W_q)}$ be a group of complete W_q-symmetry. It easy to verify, that $H = G^{(W_q)} \cap G$ is a symmetry subgroup of the group $G^{(W_q)}$, and $V = G^{(W_q)} \cap W$ represents the subgroup of W-identical transformations of the group $G^{(W_q)}$. The group $G^{(W_q)}$ is called senior, minor or middle, if $V = W$, $V = w_0$ or $w_0 \prec V \prec W$, respectively.

Let $G^{(W_q)}$ be a group of incomplete W_q-symmetry with generating group G and subset W' ($w_0 \subset V \subset W$). Obviously, that $G^{(W_q)} \cap W' = G^{(W_q)} \cap W = V$. If W' is a non-trivial subgroup of W, then the group $G^{(W_q)}$ is called W'-semi-senior, W'-semi-minor or (W', V)-semi-middle according to the cases when $V = W'$, $w_0 = V \prec W'$ or $w_0 \prec V \prec W'$. If W' is not a group ($w_0 \subset V \subset W$), then it is not possible the case when $V = W'$, because V is a subgroup, but W' is not a subgroup in W. The group $G^{(W_q)}$ is called W'-pseudo-minor or (W', V)-pseudo-middle, if $V = w_0$ or $w_0 \prec V \subset W'$, respectively.

So, in any family of $G^{(W_q)}$ groups of W_q-symmetry can be one of the following types of groups: generating ($G^{(W_q)} = G, W' = w_0$); senior ($w_0 \prec V = W' = W$); minor ($w_0 = V \prec W' = W$); middle ($w_0 \prec V \prec W' = W$); semi-senior ($w_0 \prec V = W' \prec W$); semi-minor ($w_0 = V \prec W' \prec W$); semi-middle ($w_0 \prec V \prec W' \prec W$); pseudo-minor ($w_0 \prec V \subset W' \subset W, W' \nprec W$); pseudo-middle ($w_0 \prec V \subset W' \subset W, W' \nprec W$).

The groups $G^{(W_q)}$ of W_q-symmetry are subgroups of senior group of the same family, so are subgroups of the standard crossed wreath product of group $W = \overline{\prod}_{g_i \in G} P^{g_i}$ with generating group G, accompanied with the left translation and the homomorphism τ: $G^{(W_q)} \leq P \overrightarrow{\iota}_{\tau H(\Phi)} \overleftarrow{G}$.

3. ABOUT GENERAL STRUCTURE OF PSEUDO-MINOR GROUPS OF W_Q-SYMMETRY

Next we analyze the necessary and sufficient conditions for a group to be pseudo-minor group of W_q-symmetry, and some properties that are checked by these groups.

Theorem 3.1. [6] *For $G^{(W_q)}$ to be W'-pseudo-minor group of W_q-symmetry with generating group G and initial group of permutation P and kernel H_1 of the accompanying homomorphism $\tau : G \to AutW$ it is necessary and sufficient to verify the following conditions:*

1) $G^{(W_q)}$ is a subgroup of major group $\widetilde{G^{(W_q)}}$ of the same family and the same category, i.e. $G^{(W_q)} \prec \widetilde{G^{(W_q)}} = P \overrightarrow{\iota}_{\tau H_1(\Phi)} \overleftarrow{G} = [P, H_1, \overrightarrow{\Phi}, \overleftarrow{G}]$;

2) in $G^{(W_q)}$ to be available the rule of multiplication of the elements

$$g_i w_i * g_j w_j = g_i g_j w_i^{g_j} w_j, \tag{2}$$

where $g_k = g_i g_j$, $w_k = w_i^{g_j} \overrightarrow{\tau}_{g_i} (w_j)$, $w_i^{g_j}(g_k) = w_i(g_j g_k)$, and $\overrightarrow{\tau}_{g_i} (w_j) = g_i w_j g_i^{-1}$;

3)$G^{(W_q)}$ consists of such transformations as $g^{(w)} = wg$, those components g and w form respectively, the generating group G and the subset with unit W' which is not a subgroup of group $W = P^{g_1} \times P^{g_2} \times ... \times P^{g_n} \times ...$, i.e. $G = \{g | g^{(w)} \in G^{(W_q)}\}$ and $W' = \{w | g^{(w)} \in G^{(W_q)}\} \subset W$, where W';

4) the application φ of the group $G^{(W_q)}$ onto the group G according to the rule $\varphi[g^{(w)}] = g$ is isomorphic.

Theorem 3.2. [6] *Any W'-pseudo-minor group of W_q-symmetry $G^{(W_q)}$ with generating group G, the kernel H_1 of the accompanying homomorphism $\tau : G \to AutW$ and symmetry subgroup H, contains as a subgroup a group of W_p-symmetry $H_1^{(W_p)}$ of the family with the generating group H_1 and the symmetry subgroup H', where $H' = H_1 \cap H$.*

Proof. Let $G^{(W_q)}$ be a W'-pseudo-minor group of W_q-symmetry, with the generating group G, the kernel H_1 of the accompanying homomorphism $\tau : G \to AutW$ and symmetry subgroup H.

Then, for the rule (2) of multiplication of elements from the group $G^{(W_q)}$ two cases are highlighted (depending on the action of automorphism $\overrightarrow{\tau}_{g_i}$):

$$\overrightarrow{\tau}_{g_i}(w_j) = w_j, \tag{3}$$

for all elements $w_j \in W$, where $g_i \in H_1$;

$$\overrightarrow{\tau}_{g_i}(w_j) \neq w_j, \tag{4}$$

for some elements $w_j \in W$, where $g_i \in G \setminus H_1$.

The case (3) represents the rule of multiplication

$$g_i^{(w_i)} \circ g_j^{(w_j)} = g_k^{(w_k)}, \tag{5}$$

where $w_k = w_i^{g_j} w_j$ and $g_k = g_i g_j$, because all automorphisms $\overrightarrow{\tau}_{g_i}$ for $g_i \in H_1$ coincide with identical automorphism of the group W. The totality of all transformations $g^{(w)} \in G^{(W_q)}$, where $g \in H_1$, forms the group $H_1^{(W_p)}$ with the rule of multiplication (5), and the totality of permutations of "indices", as components of transformations $g^{(w)}$ from $H_1^{(W_p)}$, forms the subset with unit $W_1 = \{w | g^{(w)} \in H_1^{(W_p)}\}$ of the group W, which is included in W': $w_0 \subset W_1 \subset W' \subset W$. From the fact that, the unit 1 of the group G ($1 \in H_1$) is combined in pairs only with element $w_0 \in W$ (because, $V = G^{(W_q)} \cap W = G^{(W_q)} \cap W' = w_0$), result, that $H_1^{(W_p)} \cap W_1 = w_0$. Obviously, the symmetry subgroup H' of the group $H_1^{(W_p)}$ coincide with the intersection of generating group H_1 (subgroup of the group $H_1^{(W_p)}$) and the symmetry subgroup H of the group $G^{(W_q)}$, i.e. $H' = H_1 \cap H$. ∎

Theorem 3.3. *Any W'-pseudo-minor group of W_q-symmetry $G^{(W_q)}$, with generating group G, the kernel H_1 of the accompanying homomorphism $\tau : G \to AutW$ and the symmetry subgroup H, contains as a subgroup a group $G_1^{(W'')}$ of $\overline{P}$-symmetry [2, 10], with generating group G_1, the same kernel H_1 of the accompanying homomorphism τ and with set $W'' = \{w | g^{(w)} \in G_1^{(W'')}\}$, where $G_1 \leq G$, and $W'' = W' \cap DiagW$.*

The proof of this Theorem is similar with the proof of Theorem 3.2 (depending on the action of the automorphism $\overleftarrow{g}_j$ concerning to the elements w from W in the multiplication rule from $G^{(W_q)}$).

4. ONE METHOD FOR DERIVING THE PSEUDO-MINOR GROUPS OF W_Q-SYMMETRY

For the development of the mathematical theory of the groups of W_q-symmetry in the paper [11] were defined and studied the crossed quasi-homomorphisms of the arbitrary group G onto subsets W' with unit of the group W. We will comment some properties of theirs.

Let us consider groups G, P and $W = \overline{\prod}_{g_i \in G} P^{g_i}$ (where $P^{g_i} \cong P$), the isomorphic injection $\varphi : G \to AutW$ defined by the rule $\varphi(g) = \overleftarrow{g}$, where the automorphism $\overleftarrow{g}$ makes the left g-translation of the components in $w \in W$, and also the homomorphism $\tau : G \to \Phi \leq AutW$, where $\tau(g) = \overrightarrow{\tau}_g$ and $\overrightarrow{\tau}_g\,(w) = gwg^{-1}$.

Definition 4.1. *The mapping α of the group G onto the subset W' of the group W defined by the rule $\alpha(g) = w$ is called a crossed quasi-homomorphism, accompanied by exact left translation of the components and by homomorphism τ of the right conjugation, if for any g_i and g_j, from G, of the fact that $\alpha(g_i) = w_i$ and $\alpha(g_j) = w_j$ results $\alpha(g_i g_j) = w_i^{g_j}\, \overrightarrow{\tau}_{g_i}\,(w_j) = w_k$, where $w_i, w_j, w_k \in W'$, $w_i^{g_j}(g_k) = w_i(g_j g_k)$, and $\overrightarrow{\tau}_{g_i}\,(w_j) = g_i w_j g_i^{-1}$.*

We note that the case of $\overrightarrow{\tau}_g = \overrightarrow{1}$ (where $\overrightarrow{1}$ is the identical automorphism of the group W and $g \in G$) the crossed quasi-homomorphism α is an ordinary exact natural left quasi-homomorphism. If $w^g = w$ (for all $g \in G$ and $w \in \alpha(G)$), then α is the right quasi-homomorphism accompanied by homomorphism τ of right conjugation [10].

The crossed quasi-homomorphism has the following properties:

1) At the crossed quasi-homomorphism $\alpha : G \to W$, the image of the unit from G is the unit of the group W;

2) The kernel of the crossed quasi-homomorphism $\alpha : G \to W$ is a subgroup H' of the group G: $Ker\alpha = H' < G$, which in general is not a normal divisor;

3) In order that at the crossed quasi-homomorphism α with the kernel $Ker\alpha = H'$ of the group G onto subset W' from W, to each left cosset gH' to correspond only one element w from W', is necessary and sufficient, as for any element $h \in H'$ and $w \in W'$ to check equality $w^h = w$;

4) If the kernel H' of the crossed quasi-homomorphism $\alpha : G \to W$ is a subgroup of the kernel H of the accompanying homomorphism τ, then, at the application α to each right cosset $H'g$ of the group G in relation with the subgroup H' is put into correspondence one and just one element w from the group W;

5) If the kernel H' of the crossed quasi-homomorphism $\alpha : G \to W$ is a normal divisor in G ($H' \lhd G$) and it is included in the kernel H of the accompanying homomorphism τ ($H' \leq H$), then: a) to each left cosset gH' the application α is put into correspondence only one element w from W; b) for any $h \in H'$ and $w \in \alpha(G)$ occurs the equality $w^h = w$; c) according to homomorphism τ the elements $h \in H'$

correspond only those automorphisms $\overrightarrow{\tau}_h$, which action on the $\alpha(G)$ coincide with the action identical automorphism i of the group W; d) according to homomorphism τ, all elements gh from the fixed cosset gH' correspond only those automorphisms $\overrightarrow{\tau}_h$, which action on the $\alpha(G)$ is the same;

6) If the group G at the crossed quasi-homomorphism α is applied onto the subset W' of the group W, then W' is not always a subgroup;

7) The restriction on the H of the crossed quasi-homomorphism α with the kernel $Ker\alpha = H'$ of the group G onto subset W' of the group W, accompanied by the left direct translation and by homomorphism τ of the right conjugation with the kernel $Ker\tau = H$, is a natural left quasi-homomorphism ($Ker\varphi = 1$). Moreover, $\alpha(H) = W^*$ and $w_0 \subseteq W^* \subseteq W'$.

Next we will describe a method for deriving of pseudo-minor groups of W_q-symmetry, exposed in the following theorem:

Theorem 4.1. *For deriving all pseudo-minor group $G^{(W_q)}$ of W_q-symmetry with generating group G, the kernel H_1 of the accompanying homomorphism $\tau : G \to AutW$ and group $W = \overline{\prod}_{g_i \in G} P^{g_i}$ of multi-component "permutations" it is necessary and sufficient:*

1) to find in the group W any subset with unit W' (which is not subgroup), and, in the group G such subgroups H, which have the index equal with the power of subset W';

2) to construct the crossed quasi-homomorphism α of the group G onto the subset with the unit W' with the kernel $Ker\alpha = H$ by the rule $\alpha(Hg) = w$, which restriction onto the subgroup $H_1 = Ker\tau$ is an exact natural left quasi-homomorphism with the kernel $H' = H_1 \cap H$;

3) to determine as components of the transformation $g^{(w)} = wg$ the elements of the group G and the subset W' that correspond to each other by α;

4) to introduce into the set of all these pairs the operation:

$$g_i^{(w_i)} * g_j^{(w_j)} = g_k^{(w_k)}, \tag{6}$$

where $g_k = g_i g_j$, $w_k = w_i^{g_j} \overrightarrow{\tau}_{g_i}(w_j)$, $w_i^{g_j}(g_s) = w_i(g_j g_s)$, and $\overrightarrow{\tau}_{g_i}(w_j) = g_i w_j g_i^{-1}$.

Proof. Necessity. Let us have W'-pseudo-minor group $G^{(W_q)}$ of W_q-symmetry, with the generating group G, the kernel H_1 of the accompanying homomorphism $\tau : G \to AutW$ and the symmetry subgroup H. The elements of the group $G^{(W_q)}$ are multiply by the rule (6), and the intersection of the groups $G^{(W_q)}$ and $W = \overline{\prod}_{g_i \in G} P^{g_i}$ is equal with unit w_0 of the group W. The unit w_0 of the group W coincides with intersection of group $G^{(W_q)}$ with subset $W' = \{w | g^{(w)} \in G^{(W_q)}\}$, and verifies the condition $w_0 \subset W' \subset W$.

Let's determine the applications λ and σ of the group $G^{(W_q)}$ respectively, onto its generating group G and the subset W' with the unit (which is not subgroup) defined by the rules $\lambda[g^{(w)}] = g$ and $\sigma[g^{(w)}] = w$, where g and w are components of the transformation $g^{(w)} \in G^{(W_q)}$. Evidently, that λ is an isomorphism. Therefore there exists the inverse isomorphism λ^{-1} of the group G onto $G^{(W_q)}$ according to the rule $\lambda^{-1}(g) = g^{(w)}$.

Now we shall prove that σ is a crossed quasi-homomorphism with the kernel H of the group $G^{(W_q)}$ onto W'. Indeed, let it be $\sigma[g_i^{(w_i)}] = w_i$ and $\sigma[g_j^{(w_j)}] = w_j$. Then, according to the multiplication operation from $G^{(W_q)}$, we have that $\sigma[g_i^{(w_i)} * g_j^{(w_j)}] = \sigma[g_k^{(w_k)}] = w_k$, where $g_k = g_i g_j$, and $w_k = w_i^{g_j} \overrightarrow{\tau}_{g_i}(w_j)$. We mention that $w_i^{g_i}$ it obtained through the left g_j-translation of components, and $\overrightarrow{\tau}_{g_i}(w_j) = g_i w_j g_i^{-1}$. Consequently, the automorphisms $\overleftarrow{g}_j$, which determine the left g_j-translation of components in every w from W', is emphasized at the isomorphic inclusion φ of the group G in the group $AutW$ by the rule $\varphi(g_j) = \overleftarrow{\varphi}_{g_j} \equiv \overleftarrow{g}_j$, and the automorphisms $\overrightarrow{\tau}_{g_i}$ - at the accompanying homomorphism $\tau : G \to AutW$ by the rule $\tau(g_i) = \overrightarrow{\tau}_{g_i}$. Therefore, the kernel $Ker\sigma = H$, where $H = G^{(W_q)} \cap G$ is the symmetry subgroup of the group $G^{(W_q)}$.

The subgroup $H_1^{(W_p)}$ of W_p-symmetry is applied by the σ onto the totality $W_1 = \{w | g^w \in H_1^{(W_p)}\}$, which verify the following condition $w_0 \subseteq W_1 \subset W'$. It is trivial to verify that the restriction of the σ onto the $H_1^{(W_p)}$ is an exact natural left quasi-homomorphism, the kernel H' of which coincide with the intersection $H' = H_1 \cap H$.

As a mapping α of the group G onto W' we take the result of performing successive of applications λ^{-1} and σ defined by the rule $\alpha(g) = \sigma\lambda^{-1}(g) = \sigma[g^{(w)}] = w$, where g and w are components of the transformation $g^{(w)}$ from $G^{(W_q)}$. As a kernel $Ker\alpha$ is the subgroup H, because λ^{-1} is an isomorphism and the kernel of application σ is the subgroup H.

Since W'-pseudo-minor group $G^{(W_q)}$ of W_q-symmetry consists only of such elements $g^{(w)} = wg$, where geometrical components g are different for different elements $g^{(w)}$ and which determine the group G, but the components w from the subset W' with the unit of the group W that is not subgroup, then for $G^{(W_q)}$ are checked all the conditions described in theorem.

Sufficiency. Now we prove that the set of al transformations, obtained according to the steps described in the theorem from the given groups G and W of multi-component "permutations" always is a W'-pseudo-minor group of W_q-symmetry with generating group G and the kernel H_1 of the accompanying homomorphism, where $H_1 \lhd G$.

We consider that for the given groups G and W there exists the crossed quasi-homomorphism α of the group G onto the subset W' with unit (where $W' \not< W$) of the group W defined by the rule $\alpha(g_i) = w_i$, which restriction on the invariant subgroup H_1 from the group G is an exact natural left quasi-homomorphism. Then according to the definition of the crossed quasi-homomorphism we will have that $\alpha(g_i g_j) = w_i^{g_j} \overrightarrow{\tau}_{g_i}(w_j)$, where the automorphism $\overleftarrow{g}_j = \varphi(g_j)$ acts on the elements w_i

through the left g_j-translation of its components, and the automorphism $\overrightarrow{\tau}_{g_i}$ acts on the w_j in form of the right conjugation.

We form the set $G^{(W_q)}$ of these pairs $wg = g^{(w)}$, where $\alpha(g) = w$ and in this set we introduce the operation (6). The set closing $G^{(W_q)}$ toward given operation results from the fact that G is a group, and α is a crossed quasi-homomorphism. Indeed, for any elements g_i and g_j from the group G we have $\alpha(g_i) = w_i$, $\alpha(g_j) = w_j$ and $\alpha(g_i g_j) = w_i^{g_j} \overrightarrow{\tau}_{g_i}(w_j) = w_k = \alpha(g_k)$, where $w_i, w_j, w_k \in W'$.

The associativity of the operation and the existence in the set $G^{(W_q)}$ of the unit element $1^{(w_0)} = w_0 1$ is trivial to verify, this follow from the fact that, the action of the automorphisms $\overleftarrow{g}$ and $\overrightarrow{\tau}_g$ onto the elements w from the group W is commutative.

Let be $g^{(w)} \in G^{(W_q)}$, then $g^{(w)} = wg$, where $g \in G$, $w \in W'$ and $w = \alpha(g)$. Because G is a group, then $g^{-1} \in G$. Let be $\alpha(g^{-1}) = w^* \in W'$, so $w^* g^{-1} = g^{-1(w^*)} \in G^{(W_q)}$. On the one hand $\alpha(gg^{-1}) = \alpha(1) = w_0$, and on the other hand $\alpha(gg^{-1}) = w^{g^{-1}} \overrightarrow{\tau}_g(w^*)$. Therefore, $w^{g^{-1}} \overrightarrow{\tau}_g(w^*) = w_0$, so $\overrightarrow{\tau}_g(w^*) = (w^{-1})^{g^{-1}}$, and $w^* = \overrightarrow{\tau}_{g^{-1}}((w^{-1})^{g^{-1}}) = [\overrightarrow{\tau}_{g^{-1}}(w^{-1})]^{g^{-1}}$. We obtain, that $\overrightarrow{\tau}_g^{-1}((w^{-1})^{g^{-1}})g^{-1} = [wg]^{-1} \in G^{(W_q)}$. Consequently, $G^{(W_q)}$ is a group of W_q-symmetry.

From the fact, that α applies the group G onto the subset W' with unit from W, results, that $W' = \{w | g^{(w)} \in G^{(W_q)}\}$. The intersection V of the obtained group $G^{(W_q)}$ with the subset W' is equal with the unit of the group W, because the unit 1 of the group G according to the application α it will apply only on the unit w_0 of the group W. The group $G^{(W_q)}$ is isomorphe with his generating group G, because the kernel V of the homomorphism $\lambda : G^{(W_q)} \to G$ by the rule $\lambda[g^{(w)}] = g$ is unit. Therefore, the obtained group $G^{(W_q)}$ is W'-pseudo-minor group of W_q-symmetry. ∎

Remark 4.1. *For deriving pseudo-minor groups of W_q-symmetry is practical to use the following algorithm:*

1) Having groups G and P to find $W = \overline{\prod}_{g_i \in G} P^{g_i}$ of isomorphic copies of the group P which are indexed by elements of G and to praise in the group W the subgroup $W'' = DiagW \cong P$;

2) To construct the isomorphism $\varphi : G \to AutW$ defined by the rule $\varphi(g) = \overleftarrow{g}$, where $\overleftarrow{g}$ makes left g-translation of components in each w of the group W and to describe in details the action of the automorphism $\overleftarrow{g}$ on the factors P^{g_i} at standard of the permutations of these factors;

3) To find in G all possible real normal subgroups H_1 ($H_1 \lhd G$) and real subgroups G_1 ($H_1 < G_1 < G$), in the group H_1 all possible real subgroup H ($H < H_1$), and in W - all possible subsets with unit W' which are not subgroups and that verify the conditions: $G_1/H \cong \Phi < AutW$, $[G|H] = |W'|$, $H_1 \cap H = H'$, $[H_1|H'] = |W_1|$, where $w_0 \subseteq W_1 \subset W'$, $G_1 \cap H = H''$, $W' \cap DiagW = W''$ and $[G_1|H''] = |W''|$;

4) To construct a homomorphism τ with the kernel $Ker\tau = H_1$ of the group G onto the group AutW, where $\tau(g) = \overrightarrow{\tau}_g$ and $\overrightarrow{\tau}_g\ (w) = gwg^{-1}$;

5) To decompose the group G in right cosset compared with the subgroup H and to establish a reciprocal correspondence α between this decomposition and the subset W' (preserving: a) the correspondence between elements of the subgroups H_1 and W_1, obtained in the result of the exact natural left quasi-homomorphism φ with the kernel H'; b) the correspondence between elements of the subgroups G_1 and W'', obtained in the result of the right quasi-homomorphism β with the kernel H'' of the group G_1 onto W''), that as application of group G onto W' would be exact natural left quasi-homomorphism φ with the kernel H.

For the presentation of the pseudo-minor groups $G^{(W_q)}$ of W_q-symmetry it is practical to use the following complex symbol (with more terms):

$$[P, H_1, \overrightarrow{\Phi}, G]/(W'|W|W''; H_1|H', G_1/H_1/H'')H$$

which contains enough information about the group structure:

1) $[P, H_1, \overrightarrow{\Phi}, G]$ - the symbol of the senior group of the subfamily with the generating group G and the kernel H_1 of the accompanying homomorphism $\tau : G \to AutW$, $\overrightarrow{\Phi} = Im\tau$, and P is the initial group of permutations; 2) W' - the subset of "permutations", which enter as components in $g^{(w)} \in G^{(W_q)}$, where $w_0 \subset W' \subseteq W = \overline{\prod_{g_i \in G} P^{g_i}}$ and $W' \nless W$; 3) W - the set of "permutations", which enter as components in $g^{(w)}$ of the group $H_1^{(W_p)}$ of W_p-symmetry, which includes in $G^{(W_q)}$ as a subgroup; 4) W'' - the set of "permutations", which enter as components in $g^{(w)}$ of the group $G_1^{(W'')}$ of $\overline{P}$-symmetry, which includes in $G^{(W_q)}$ as a subgroup; 5) H_1 - generating group, and H' - the symmetry subgroup of the group $H_1^{(W_p)}$ of W_p-symmetry, where $H' = H_1 \cap H$; 6) G_1 - generating group, H_1 - the kernel of the accompanying homomorphism, and H'' - the symmetry subgroup of the group $G_1^{(W'')}$ of $\overline{P}$-symmetry, where $G_1 \prec G$, $H'' = H \cap G_1$; 7) H - the symmetry subgroup for the group $G^{(W_q)}$ ($H = G^{(W_q)} \cap G$).

5. CONCLUSIONS

In this paper we presented the necessary and sufficient conditions for deriving all pseudo-minor groups $G^{(W_q)}$ of W_q-symmetry from its generating groups G of classical symmetry and initial group definition P.

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