A SPECTRAL REGULARIZATION METHOD FOR A HEAT EQUATION BACKWARD IN TIME ON THE PLANE
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Abstract For a two-dimensional heat conduction problem on the plane, we consider its initial boundary value problem and the related inverse problem of determining the initial temperature distribution from transient temperature measurements. The stability for this inverse problem and the error analysis for the truncation method are presented. Some quite sharp error estimates between the approximate solution and exact solution are provided.

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1. INTRODUCTION

Let T be a positive number. We consider the problem of finding the temperature \( u(x, y, t), \ (x, y, t) \in \mathbb{R}^2 \times [0, T], \) such that

\[
\begin{aligned}
&u_{xx} + u_{yy} = u_t, \quad (x, y, t) \in \mathbb{R} \times \mathbb{R} \times (0, T), \\
u(x, y, T) = \varphi(x, y), \quad (x, y) \in \mathbb{R}^2,
\end{aligned}
\]

(1)

where \( \varphi(x, y) \) is given. The problem is called the backward heat problem (BHP), the backward Cauchy problem or the final value problem.

It is known in general that the backward problem is ill-posed, i.e., a solution does not always exist, and in the case of existence, it does not depend continuously on the given datum. In fact, from a small noise contaminated physical measurement, the corresponding solutions may have a large error. This makes the numerical computation difficult. Hence, a regularization is in order.

The backward heat problem has been studied by many authors in recent years. In a few words, we mention Ames and Payne [1], Lattes and Lions [15], Showalter [20], who approximated the BHP by quasi-reversibility method; Tautenhahn and Schroter [22] who established an optimal error estimate for a BHP; Seidman [19] who estab-
lished an optimal filtering method; and Hao [14] who studied a modification method. We also refer to various other works of Chu-Li Fu et al [11, 12, 4, 25], Campbell et al [3], Lien et al [24], Murniz et al [16], Dokuchaev et al [7], D. Gilliam et al [13] and H.W. Engl et al [9] et al. Recently, the 1-D version of backward heat in an infinite strip has been considered in [12, 14].

Although there are many works on the one dimensional case, the literature on the two dimensional case of the backward heat problem in unbounded region is quite scarce. The discrete form of 2-D version is studied in [17]. In the present paper, a spectral regularization method is used for solving 2-D backward heat problem, which will improve the stability results in some related paper. Under some suitable conditions on the exact solution \( u \), we shall introduce the error estimate of order \( \epsilon^m (m > 0) \).

This is a significant improvement in comparison with [11, 14, 18, 21, 22, 24, 25]. Some comments on the usefulness of this method are given in some Remarks. This method has been also successfully applied to some inverse heat conduction problems [2, 5].

This paper is organized as follows: In section 2, we give some auxiliary results. In section 3, we give the regularization solution by using the truncated method and give the stable error estimates between the regularization solution and the exact solution.

2. SOME AUXILIARY RESULTS

Let \( \hat{g}(\xi, \eta) \) denote the Fourier transform of function \( g \in L^2(R^2) \) defined formally by

\[
\hat{g}(\xi, \eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) e^{-i(\xi x + \eta y)} \, dx \, dy.
\]  

(2)

Let \( H^1(R^2) = W^{1,2}(R^2) \), \( H^2(R^2) = W^{2,2}(R^2) \) be the Sobolev spaces which are defined by

\[
H^s(R^2) = \{ g \in L^2(R^2), (1 + \xi^2 + \eta^2)^{s/2} \hat{g}(\xi, \eta) \in L^2(R^2) \}.
\]

We denote by \( ||.|| \), \( ||.||_{H^1}, ||.||_{H^2} \) the norms in \( L^2(R^2) \), \( H^1(R^2) \), \( H^2(R^2) \) respectively, namely

\[
||g||_{H^s} = ||(1 + \xi^2 + \eta^2)^{s/2} \hat{g}(\xi, \eta)||^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\hat{g}(\xi, \eta)|^2 (1 + \xi^2 + \eta^2)^s \, d\xi \, d\eta.
\]

When \( s = 0 \), \( ||.||_{H^0} \) is the \( L^2(R^2) \)-norm.

As a solution of problem (1) we understand a function \( u(x, y, t) \) satisfying (1) in the classical sense and for every fixed \( t \in [0, T] \), the function \( u(x, y, t) \in L^2(R^2) \). In this class of functions, if the solution of problem (1) exists, then it must be unique (See [10]).

**Theorem 2.1.** Let \( \varphi \in L^2(R^2) \). Assume that \( u \in C([0, T], H^2(R^2)) \cap C^1([0, T], L^2(R^2)) \) be a solution of the equation

\[
\hat{u}(\xi, \eta, t) = e^{(T-t)(\xi^2+\eta^2)} \hat{\varphi}(\xi, \eta).
\]  

(3)
Then \( u, u_{xx}, u_{yy} \in C([0, T], L^2(\mathbb{R}^2)) \) and \( u \) is a solution of the heat equation (1) where the main equation holds in \( C([0, T], L^2(\mathbb{R}^2)) \).

**Proof.** We take \( t = T \) in the equation

\[
\hat{u}(\xi, \eta, t) = e^{(T-t)(\xi^2 + \eta^2)}\hat{\varphi}(\xi, \eta), \quad 0 \leq t \leq T
\]

and we have immediately \( \hat{u}(\xi, \eta, T) = \hat{\varphi}(\xi, \eta) \). Therefore, we get \( u(x, y, T) = \varphi(x, y) \) in \( L^2(\mathbb{R}^2) \). Then, by multiplying the above equation with \( e^{t(\xi^2 + \eta^2)} \) we obtain

\[
e^{t(\xi^2 + \eta^2)}\hat{u}(\xi, \eta, t) = e^{T(\xi^2 + \eta^2)}\hat{\varphi}(\xi, \eta), \quad t \in [0, T].
\]

By differentiating the latter equation w.r.t. the time variable \( t \) we get

\[
e^{t(\xi^2 + \eta^2)}\left((\xi^2 + \eta^2)\hat{u}(\xi, \eta, t) + \frac{d}{dt}\hat{u}(\xi, \eta, t)\right) = 0
\]

namely

\[
(\xi^2 + \eta^2)\hat{u}(\xi, \eta, t) + \frac{d}{dt}\hat{u}(\xi, \eta, t) = 0, \quad t \in [0, T].
\]

Since \( u \in C([0, T], H^2(\mathbb{R}^2)) \cap C^1([0, T], L^2(\mathbb{R}^2)) \) we have

\[
(\xi^2 + \eta^2)\hat{u}(\xi, \eta, t) = u_{xx}(\xi, \eta) + u_{yy}(\xi, \eta)
\]

and \( \frac{d}{dt}\hat{u}(\xi, \eta, t) \) belongs to \( C([0, T], L^2(\mathbb{R}^2)) \). The latter equation means

\[
u_t - u_{xx} - u_{yy} = 0
\]

in the sense of \( C([0, T], L^2(\mathbb{R}^2)) \).

**Theorem 2.2.** The problem (1) has a unique solution \( u \) if and only if

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2T(\xi^2 + \eta^2)}|\hat{\varphi}(\xi, \eta)|^2 d\xi d\eta < \infty.
\]

**Proof.** Assume that Problem (1) has an exact solution

\[
u \in C([0, T], H^2(\mathbb{R}^2)) \cap C^1([0, T], L^2(\mathbb{R}^2)),
\]

then \( u \) can be formulated in the frequency domain

\[
u(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{(T-t)(\xi^2 + \eta^2)}\hat{\varphi}(\xi, \eta)e^{i(\xi x + \eta y)} d\xi d\eta.
\]
This implies that
\[ \hat{u}(x, y, 0) = e^{T(x^2 + y^2)} \hat{\varphi}(\xi, \eta). \] (6)

Then
\[
\begin{align*}
||u(., ., 0)||^2 & = \int\int |\hat{u}(x, y, 0)|^2 d\xi d\eta \\
& = \int\int e^{2T(x^2 + y^2)}|\hat{\varphi}(\xi, \eta)|^2 d\xi d\eta < \infty.
\end{align*}
\]
Assume that (5) holds. Then, we define \( v(x, y) \) by
\[
v(x, y) = \int\int e^{T(x^2 + y^2)} e^{i(\xi x + \eta y)} \hat{\varphi}(\xi, \eta) d\xi d\eta.
\] (7)

Consider the problem
\[
\begin{cases}
 u_{xx} + u_{yy} = u_t, \\
 u(x, y, 0) = v(x, y), \quad (x, y) \in \mathbb{R}^2.
\end{cases}
\] (8)

It is clear to see that (8) is the direct problem so it has a unique solution \( u \) (See [10]). We have
\[
u(x, y) = \int\int e^{T(x^2 + y^2)} e^{i(\xi x + \eta y)} \hat{\varphi}(\xi, \eta) d\xi d\eta.
\] (9)

Let \( t = T \) in (9), we have
\[
u(x, y, T) = \int\int e^{-T(x^2 + y^2)} e^{i(\xi x + \eta y)} e^{T(x^2 + y^2)} \hat{\varphi}(\xi, \eta) d\xi d\eta
\]
\[= \varphi(x, y). \]

Hence, \( u \) is the unique solution of (1).

3. REGULARIZATION BY TRUNCATION METHOD

A natural way to stabilize the problem is to eliminate all high frequencies from the solution and instead consider (1) only for \( \xi, \eta \in [-\beta(\epsilon), \beta(\epsilon)] \). For short, we denote \( \beta(\epsilon) \) by \( \beta \). Define the Fourier regularization solution of problem (1) corresponding to the final value \( \varphi \) and \( \varphi_\epsilon \) as follows
\[
u^\epsilon(x, y, t) = \frac{1}{2\pi} \int_{-\beta}^{+\beta} \int_{-\beta}^{+\beta} e^{T(\xi^2 + \eta^2)} \hat{\varphi}(\xi, \eta) e^{i(\xi x + \eta y)} d\xi d\eta,
\] (10)
and

$$U^\epsilon(x, y, t) = \frac{1}{2\pi} \int_{-\beta}^{+\beta} \int_{-\beta}^{+\beta} e^{(T-t)(\xi^2 + \eta^2)} \hat{\varphi}(\xi, \eta) e^{i(\xi x + \eta y)} d\xi d\eta,$$  \hspace{1cm} (11)

where $\beta(\epsilon)$ is a positive constant which will be selected appropriately as regularization parameter such that $\lim_{\epsilon \to 0} \beta(\epsilon) = \infty$. We now study the properties of (10) considered as an approximation to (1), i.e., we will give some new stability estimates. This regularization method is rather simple and convenient for dealing with some ill-posed problems. However, as far as we know, there are few results of Fourier method for treating 2-D backward heat equation until now. The present paper is devoted to establishing such a method for problem (1.1).

**Theorem 3.1.** The solution $u^\epsilon$ given in (10) depends continuously on $\varphi$ in $L^2(R^2)$. Furthermore, we have

$$\|u^\epsilon(.,., t) - U^\epsilon(.,., t)\| \leq e^{(T-t)|\psi^2}| \epsilon.$$  \hspace{1cm} (12)

**Proof.** Let $u^\epsilon$ and $w^\epsilon$ be two solutions of (10) corresponding to the final values $\varphi$ and $\phi$. From (10), we have

$$u^\epsilon(x, y, t) = \frac{1}{2\pi} \int_{-\beta}^{+\beta} \int_{-\beta}^{+\beta} e^{(T-t)(\xi^2 + \eta^2)} \hat{\varphi}(\xi, \eta) e^{i(\xi x + \eta y)} d\xi d\eta \hspace{0.5cm} 0 \leq t \leq T, \hspace{1cm} (13)$$

$$w^\epsilon(x, y, t) = \frac{1}{2\pi} \int_{-\beta}^{+\beta} \int_{-\beta}^{+\beta} e^{(T-t)(\xi^2 + \eta^2)} \hat{\phi}(\xi, \eta) e^{i(\xi x + \eta y)} d\xi d\eta \hspace{0.5cm} 0 \leq t \leq T. \hspace{1cm} (14)$$

By using Parseval equality, we obtain

$$\|u^\epsilon(.,., t) - w^\epsilon(.,., t)\|^2 = \int_{-\beta}^{+\beta} \int_{-\beta}^{+\beta} e^{2(T-t)(\xi^2 + \eta^2)} |\hat{\varphi}(\xi, \eta) - \hat{\phi}(\xi, \eta)|^2 d\xi d\eta$$

$$\leq e^{2(T-t)|\psi^2} \int_{-\beta}^{+\beta} \int_{-\beta}^{+\beta} |\hat{\varphi}(\xi, \eta) - \hat{\phi}(\xi, \eta)|^2 d\xi d\eta \hspace{1cm} (15)$$

$$\leq e^{2(T-t)|\psi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\varphi(\xi, \eta) - \phi(\xi, \eta)|^2 d\xi d\eta$$

$$\leq e^{2(T-t)|\psi^2} \|\varphi - \phi\|^2.$$

From (15) and the condition $\|\varphi^\epsilon - \varphi\| \leq \epsilon$, we have

$$\|u^\epsilon(.,., t) - U^\epsilon(.,., t)\| \leq e^{(T-t)|\psi^2}| \epsilon.$$  \hspace{1cm} (16)
This completes the proof of the theorem.

**Theorem 3.2.** Let $A$ be a positive number such that $\|u(\cdot, 0)\| \leq A$. If $\beta = \sqrt{1/T \ln(\frac{1}{\epsilon})}$, then the following convergence estimate holds for every $t \in [0, T]$:

$$\|u^\epsilon(\cdot, \cdot, t) - u(\cdot, \cdot, t)\| \leq (4A + 1) e^{\frac{t}{\epsilon}}.$$  \hfill (17)

**Proof.** Denote $F(\xi, \eta) = e^{(T-t)(\xi^2 + \eta^2)} \hat{\varphi}(\xi, \eta)$. Then

$$u(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\xi, \eta) e^{i(\xi x + \eta y)} d\xi d\eta$$

and

$$u^\epsilon(x, y, t) = \frac{1}{2\pi} \int_{-\beta}^{+\beta} \int_{-\beta}^{+\beta} F(\xi, \eta) e^{i(\xi x + \eta y)} d\xi d\eta.$$  

Hence

$$u(x, y, t) - u^\epsilon(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\xi, \eta) e^{i(\xi x + \eta y)} d\xi d\eta$$

$$+ \frac{1}{2\pi} \int_{-\beta}^{+\beta} \int_{-\beta}^{+\beta} F(\xi, \eta) e^{i(\xi x + \eta y)} d\xi d\eta.$$

Therefore, by using Parseval equality, we obtain

$$\|u(\cdot, \cdot, t) - u^\epsilon(\cdot, \cdot, t)\|^2 = \|\hat{u}(x, y, t) - \hat{u}^\epsilon(x, y, t)\|^2$$

$$\leq 4 \int_{-\infty}^{+\infty} |F(\xi, \eta)|^2 d\xi d\eta + 4 \int_{+\beta}^{+\infty} |F(\xi, \eta)|^2 d\xi d\eta$$

$$+ 4 \int_{-\beta}^{-\infty} |F(\xi, \eta)|^2 d\xi d\eta + 4 \int_{-\beta}^{+\infty} |F(\xi, \eta)|^2 d\xi d\eta.$$
We have
\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi, \eta)|^2 \, d\xi \, d\eta = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2(T-t)(\xi^2 + \eta^2)} |\hat{\phi}(\xi, \eta)|^2 \, d\xi \, d\eta
\]
\[
\leq e^{-2\beta^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2T(\xi^2 + \eta^2)} |\hat{\phi}(\xi, \eta)|^2 \, d\xi \, d\eta
\]
\[
\leq e^{-2\beta^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2T(\xi^2 + \eta^2)} |\hat{\phi}(\xi, \eta)|^2 \, d\xi \, d\eta
\]
\[
\leq e^{-2\beta^2} \|u(., 0)\|^2.
\]

In a similar way, we also get the following results
\[
\int_{+\beta}^{+\infty} \int_{+\beta}^{+\infty} |F(\xi, \eta)|^2 \, d\xi \, d\eta \leq e^{-2\beta^2} \|u(., 0)\|^2
\]
\[
\int_{-\beta}^{-\beta} \int_{-\beta}^{-\beta} |F(\xi, \eta)|^2 \, d\xi \, d\eta \leq e^{-2\beta^2} \|u(., 0)\|^2
\]
\[
\int_{-\beta}^{-\beta} \int_{+\beta}^{+\infty} |F(\xi, \eta)|^2 \, d\xi \, d\eta \leq e^{-2\beta^2} \|u(., 0)\|^2
\]
\[
\int_{+\beta}^{+\infty} \int_{-\beta}^{-\beta} |F(\xi, \eta)|^2 \, d\xi \, d\eta \leq e^{-2\beta^2} \|u(., 0)\|^2.
\]

Hence, we obtain
\[
\|u(., t) - u^\epsilon(., t)\|^2 \leq 16e^{-2\beta^2} \|u(., 0)\|^2. \tag{18}
\]

Combining (16) and (18) then
\[
\|U^\epsilon(., t) - u(., t)\| \leq \|U^\epsilon(., t) - u^\epsilon(., t)\| + \|u^\epsilon(., t) - u(., t)\|
\]
\[
\leq 4e^{-\beta^2} A_1 + e^{T-t} e^2 \epsilon.
\]

From \(\beta = \sqrt{\frac{1}{T} \ln(\frac{1}{\epsilon})}\), then the following convergence estimate holds
\[
\|U^\epsilon(., t) - u(., t)\| \leq e^T (4A_1 + 1),
\]
q.e.d.

**Remark 3.1.** From Theorem 3.2, we find that \(U^\epsilon\) is an approximation of the exact solution \(u\). The approximation error depends continuously on the measurement error.
for fixed $0 < t \leq T$. However, as $t \to 0$ the accuracy of regularized solution becomes progressively lower. This is a common thing in the theory of ill-posed problems, if we do not have additional conditions on the smoothness of the solution. To retain the continuous dependence of the solution at $t = 0$, we introduce a stronger a priori assumption.

**Theorem 3.3.** Assume that there exists the positive numbers $q, B$ such that

$$
||u(\ldots,t)||_{\mathcal{H}^q(\mathbb{R}^2)} < B.
$$

Let $\beta = \sqrt{\frac{1}{1+\alpha} \ln \left( \frac{1}{\epsilon} \right)}$ for $\alpha > 0$. Then the following convergence estimate holds

$$
||U^\epsilon(\ldots,t) - u(\ldots,t)|| \leq B \left( \frac{1}{T+\alpha} \ln \left( \frac{1}{\epsilon} \right) \right)^{\frac{2}{q}} + \epsilon^{\frac{1}{2q}},
$$

for every $t \in [0, T]$.

**Proof.** Using (18), we have

$$
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi,\eta)|^2 d\xi d\eta = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2(\xi^2+\eta^2)(\xi^2+\eta^2)} |\hat{\varphi}(\xi,\eta)|^2 d\xi d\eta
$$

$$
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\xi^2 + \eta^2)^{-q} (\xi^2 + \eta^2)^{q} e^{2(\xi^2+\eta^2)(\xi^2+\eta^2)} |\hat{\varphi}(\xi,\eta)|^2 d\xi d\eta
$$

$$
\leq \frac{1}{\beta^q} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\xi^2 + \eta^2)^{q} |\hat{u}(\xi,\eta,t)|^2 d\xi d\eta
$$

$$
\leq \frac{1}{\beta^q} ||u(\xi,\eta,t)||_{\mathcal{H}^q}^2.
$$

In a similar way, we also get the following results

$$
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi,\eta)|^2 d\xi d\eta \leq \frac{1}{\beta^q} ||u(\xi,\eta,t)||_{\mathcal{H}^q}^2
$$

$$
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi,\eta)|^2 d\xi d\eta \leq \frac{1}{\beta^q} ||u(\xi,\eta,t)||_{\mathcal{H}^q}^2
$$

$$
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi,\eta)|^2 d\xi d\eta \leq \frac{1}{\beta^q} ||u(\xi,\eta,t)||_{\mathcal{H}^q}^2.
$$
Hence, we obtain
\[ \|u(x, y, t) - u^\epsilon(x, y, t)\|^2 \leq \frac{1}{\beta^2} \|u(\xi, \eta, t)\|^2_{H^q}. \] (20)

By combining (16) and (20)
\[ \|U^\epsilon(\ldots, t) - u(\ldots, t)\| \leq \|U^\epsilon(\ldots, t) - u^\epsilon(\ldots, t)\| + \|u^\epsilon(\ldots, t) - u(\ldots, t)\| \leq 4\beta^{-q/2}B + e^{(T-t)\alpha^2}\epsilon. \]

From
\[ \beta = \sqrt{\frac{1}{T + \alpha} \ln\left(\frac{1}{\epsilon}\right)} \]
then the following convergence estimate holds
\[ \|U^\epsilon(\ldots, t) - u(\ldots, t)\| \leq 4\left(\frac{1}{T + \alpha} \ln\left(\frac{1}{\epsilon}\right)\right)^{-\frac{q}{2}} B + e^{(T-t)\alpha}. \]

**Remark 3.2.**
1. The error estimate of this theorem is the same as that contained in the results (obtained for other domains in plane) by Chu-Li Fu et al [11, 12] and D.N. Hao et al [14].
2. Since (19), the first term of the right hand side of (19) is the logarithmic form, and the second term is a power, so the order of (19) is also logarithmic order. Suppose that \(E_\epsilon = \|U^\epsilon - u\|\) be the error of the exact solution and the approximate solution. In most of results concerning the backward heat, then optimal error between is of the logarithmic form. It means that
\[ E_\epsilon \leq C\left(\ln\left(\frac{T}{\epsilon}\right)\right)^{-q} \]
where \(q > 0\). The error order of logarithmic form is investigated in many recent papers, such as [5, 6, 11, 12, 14]. This often occurs in the boundary error estimate for ill-posed problems. To retain the Hölder order in \([0, T]\), we prove the following Theorem with a different assumption.

**Theorem 3.4.** Assume that there exist the positive numbers \(\gamma, C\) such that
\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2\gamma(\xi^2 + \eta^2)}|\hat{u}(\xi, \eta, t)|^2 d\xi d\eta < C^2. \] (21)

If \(\beta = \sqrt{\frac{1}{T + \gamma} \ln\left(\frac{1}{\epsilon}\right)}\), then the following convergence estimate holds
\[ \|U^\epsilon(\ldots, t) - u(\ldots, t)\| \leq \left(4C + e^{\frac{T\alpha}{\gamma}}\right) e^{\frac{T\alpha}{\gamma}} \] (22)
for every $t \in [0, T]$.

Proof. We have

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi, \eta)|^2 d\xi d\eta = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2(T-t)(\xi^2+\eta^2)} |\hat{\varphi}(\xi, \eta)|^2 d\xi d\eta$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2\gamma(\xi^2+\eta^2)} e^{2(T-t)(\xi^2+\eta^2)} |\hat{\varphi}(\xi, \eta)|^2 d\xi d\eta$$

$$\leq e^{-2\mu^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2\gamma(\xi^2+\eta^2)} |\hat{u}(\xi, \eta, t)|^2 d\xi d\eta$$

$$\leq e^{-2\mu^2} C^2.$$

In a similar way, we also get the following results

$$\int_{+\beta}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi, \eta)|^2 d\xi d\eta \leq e^{-2\mu^2} C^2,$$

$$\int_{-\beta}^{-\infty} \int_{-\infty}^{+\infty} |F(\xi, \eta)|^2 d\xi d\eta \leq e^{-2\mu^2} C^2,$$

$$\int_{-\beta}^{+\infty} \int_{-\infty}^{+\infty} |F(\xi, \eta)|^2 d\xi d\eta \leq e^{-2\mu^2} C^2.$$

Thus, we have

$$\|u(t, \ldots, t) - u^\epsilon(t, \ldots, t)\|^2 \leq 16 e^{-2\mu^2} C^2.$$  

Combining (16) and (23), we get

$$\|U^\epsilon(t, \ldots, t) - u(t, \ldots, t)\| \leq \|U^\epsilon(t, \ldots, t) - u^\epsilon(t, \ldots, t)\| + \|u^\epsilon(t, \ldots, t) - u(t, \ldots, t)\|$$

$$\leq 4e^{-\mu^2} C + e^{(T-t)\mu^2} \epsilon.$$

From

$$\beta = \left[ \frac{1}{\sqrt{T + \gamma \ln(\frac{1}{\epsilon})}} \right]$$

the following convergence estimate holds

$$\|U^\epsilon(t, \ldots, t) - u(t, \ldots, t)\| \leq 4e^{\frac{T}{\gamma}} C + e^{\frac{T}{\gamma \mu^2}} \left( 4C + e^{\frac{1}{\gamma \mu^2}} \right).$$
Remark 3.3. 1. Since the exact solution \( u \) is unknown, we can see that the condition (21) is not verifiable. Hence, to check it, we should replace it by the conditions of \( \varphi \). Thus, we note that

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2\gamma(\xi^2+\eta^2)} |\hat{u}(\xi,\eta,t)|^2 d\xi d\eta = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2(T+\gamma-t)(\xi^2+\eta^2)} |\hat{\varphi}(\xi,\eta,t)|^2 d\xi d\eta. \tag{24}
\]

Hence, we can replace (21) by the different condition

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2(T+\gamma)(\xi^2+\eta^2)} |\hat{\varphi}(\xi,\eta,t)|^2 d\xi d\eta < C^2.
\]

2. Notice that the error (22) (\( \beta > 0 \)) is of Hölder type for all \( t \in [0,T] \). It is easy to see that the convergence rate of \( e^\alpha \), (\( 0 < p \)) is greater than that of the logarithmic order \((\ln(1/\epsilon))^{-q}(q > 0)\) when \( \epsilon \to 0 \). Comparing (22) with the results in some related papers, we can see that the method in this paper gives a better approximation. This proves that our method is effective.

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