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ON THE GENERALIZED NILPOTENT AND GENERALIZED SOLVABLE LOOPS -II

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Abstract This paper studies the Frattini subloops of any loop and proves the equivalence of the following statements for a locally nilpotent loop Q: 1) Q is finitely generated; 2) Q satisfies the maximum condition for subloops; 3) Q satisfies the maximum condition for normal subloops. It also proves that the variety generated by a centrally nilpotent Moufang loop (or centrally nilpotent A-loop) is finitely based.

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This paper is a natural continuation of paper "On the generalized nilpotent and generalized solvable loops" published in Romai J., 7, 1(2011), 39 - 63 [4] (the paper was divided in two only to meet the requirement on the maximum number of pages allowed). We present the entire material in an unified manner, hence the numbering of the sections continues that of the first part of the work.

This paper studies the Frattini subloops of any loop and with the help of these results proves the equivalence of the following statements for a locally nilpotent loop Q: 1) Q is finitely generated; 2) Q satisfies the maximum condition for subloops; 3) Q satisfies the maximum condition for normal subloops. It also proves that the variety generated by a centrally nilpotent Moufang loop (or centrally nilpotent A-loop) is finitely based.

5. THE FRATTINI SUBLOOP AND MAXIMUM CONDITION FOR SUBLOOPS

An element a of a loop Q is a *non-generator* of Q if, for every subset T of Q, $< a, T >= Q$ implies $< T >= Q$. It is easy to see that the non-generators of Q form a subloop, $\varphi(Q)$, of Q. This is the *Frattini subloop*.

Lemma 5.1 [1]. *Let Q be any loop. If Q has at least one maximal proper subloop, then $\varphi(Q)$ is the intersection of all maximal proper subloops of Q. In the contrary case, $\varphi(Q) = Q$.*

For any arbitrary loop Q, let Q' denote the *commutator-associator subloop*. This is the subloop generated by all commutators (x, y) and all associators $\alpha(x, y, z)$, $\beta(x, y, z)$ of loop Q. By (1.4) and (3.1) of [4] it is readily verified that the loop Q' is normal in Q.

Lemma 5.2. *Let Q be any centrally nilpotent loop. If K is its subloop with condition $KQ' = Q$, then $K = Q$. Particularly, $Q' \subseteq \varphi(Q)$.*

Proof. Let

$$1 \subset Z_1 \subset \ldots \subset Z_n = Q$$

be the upper centrally nilpotent series of loop Q and let $HZ_{i-1} \neq Q$, $HZ_i = Q$ where H is the maximal subloop of Q. From the maximality of subloop H we have $Z_{i-1} \subseteq H$, $Z_{i-1} \subseteq H \cap Z_i$. Then

$$(H/Z_{i-1})(Z_i/Z_{i-1}) \cong Z_i/Z_{i-1} = Z(Q/Z_{i-1}).$$

Now for an arbitrary loop G and its subloop L we suppose that $LZ(G) = G$ where $Z(G)$ is the centre of G. By (1.3) of [4] it is easy to see that L is normal in G. Hence the subloop H/Z_{i-1} is normal in Q/Z_{i-1}. As $Z_{i-1} \subseteq H$, then the inverse image of subloop H/Z_{i-1} under the homomorphism $Q \to Q/Z_{i-1}$ will be H. Therefore the subloop H is normal in Q. We have $Q = HZ_i$, then

$$Q/H \cong Z_i/(Z_i \cap H).$$

Now have $Z_{i-1} \subseteq Z_i \cap H$, then Q/H will be an abelian group. Hence H contain the commutator-associator subloop Q' of Q. Consequently, $Q' \subseteq \varphi(Q)$ and by Lemma 5.1 $K = Q$ follows from the relation $Q'K = Q$. ∎

Lemma 5.3. *Any maximal subloop H of locally centrally nilpotent loop Q is normal in Q.*

Proof. Let us suppose the contrary, i.e., that H is not normal in Q. Then by (1.4) of [4] there exists such an element $h \in H$, that $L(a, b)h \notin H$, or $R(u, v)h \notin H$, or $T(w)h \notin H$ for certain $a, b, u, v, w \in Q$. To be more clear let us suppose that $L(a, b)h \notin H$. Then by (3.1) [4] $\beta(h, b, a)h \notin H$ or $c = \beta(h, b, a) \notin H$. Taking into account the maximum of H we have $< c, H >= Q$. Let h, a, b be expressed from c and $u_1, \dots, u_k$ from H and let $\overline{H} =< u_1, \dots, u_k >, \overline{Q} =< c, u_1, \dots, u_k >$. It is obvious that $c \notin \overline{Q}'$, where $\overline{Q}'$ means commutator-associator subloop of $\overline{Q}$.

By Zorn's Lemma, the set of all subloops of $\overline{Q}$, which contain $\overline{H}$ but not c has at least one maximal element, say M. By Lemma 5.2 the maximal subloops contains the commutator-associator subloops, then $c \in M$. We have obtained a contradiction. Consequently, the subloop H is normal in Q. ∎

Corollary 5.4. *The Frattini subloop in locally centrally nilpotent loops contains the commutator-associator subloop.*

Proof. If H is a maximal subloop of centrally nilpotent loop Q, then by Lemma 5.3 it is normal in Q. The quotient loop Q/H is generated by a single element and is centrally nilpotent, as the quotient loop of locally centrally nilpotent loop is locally centrally nilpotent itself. Loop Q/H doesn't have its own non-unitary subloops. Consequently, Q/H is a cyclic group of simple order. Then $Q' \subseteq H$ and by Lemma 5.1 $Q' \subseteq \varphi(Q)$. ∎

It is known that for every group the Frattini subgroup is normal. In [2] loops with non-normal Frattini subloops are constructed. However, the following result holds.

Proposition 5.5. *Let the loop Q be locally centrally nilpotent. Then the Frattini subloop $\varphi(Q)$ is normal in Q and quotient loop $Q/\varphi(Q)$ is an abelian group. Hence either $\varphi(Q) = Q$ or $Q/\varphi(Q)$ is a subdirect product of cyclic groups of prime order.*

Proof. We need only to consider the case that $\varphi(Q) \neq Q$. By Lemma 5.3 the subloop $\varphi(Q)$ is normal in Q and by Corollary 5.4 the quotient loop $Q/\varphi(Q)$ is an abelian group. Let group $Q/\varphi(Q) = G$ be different from unitary element. Obviously, $\varphi(G) = 1$. In this case G is a subdirect product of cyclic groups of prime order. ∎

We note that Proposition 5.5 generalized the respectively result for ZA-loops from [1] in view of following statement: *every ZA-loop is locally centrally nilpotent* (Proposition 4.6).

Theorem 5.6. *For any locally centrally nilpotent loop Q the following statements are equivalent:*

1) the loop Q is finitely generated;

2) the loop Q satisfies the maximum condition for subloops;

3) the loop Q satisfies the maximum condition for normal subloops;

4) the loop Q has a central series, whose factors are cyclic groups of prime or infinite order.

Proof. Let the loop Q be finitely generated and let H be a subloop of Q. Since Q is finitely generated, then Q has a maximal proper subloop and, by Lemma 5.1, $\varphi(Q) \neq Q$. Let $h_\alpha \in H$, $\alpha \in I$, and let Q_α be a maximal (proper) subloop of Q such that $h_\alpha \notin Q_\alpha$. By Lemma 5.3, Q_α is normal in Q and, by Corollary 5.4, $Q' \subseteq Q_\alpha$. Then the quotient loop Q/Q_α is a non-unitary cyclic group of prime order.

Let $H_\alpha = Q_\alpha \cap H$. By the homomorphism theorems $HQ_\alpha/Q_\alpha \cong H/(H \cap Q_\alpha) = H/H_\alpha$, therefore H/H_α is isomorphic with a subgroup of cyclic group of prime order and as $h_\alpha \notin H_\alpha$, then H/H_α will be a non-unitary cyclic group of prime order. Hence H_α is a proper maximal subloop of Q. Consequently, $\varphi(H) \neq H$.

We have $H_\alpha = Q_\alpha \cap H$ for $\alpha \in I$, then $\cap H_\alpha = \cap Q_\alpha \cap H$ for all $\alpha \in I$. Hence $\varphi(Q) \cap H \subseteq \varphi(H)$. Since the quotient loop $Q/\varphi(Q)$ is finitely generated, then from the relation $H/(H \cap \varphi(Q)) \cong H\varphi(Q)/\varphi(Q)$ it follows that $H/(H \cap \varphi(Q))$ is also finitely generated. From inclusion $\varphi(Q) \cap H \subseteq \varphi(H)$ it follows that $H/\varphi(H)$ is a homomorphic image of $H/(\varphi(Q) \cap H)$, hence $H/\varphi(H)$ is finitely generated. It follows from here that the subloop H is finitely generated. Therefore 1) implies 2). The implication 2) $\to$ 3) is obvious.

Assume now that the loop Q satisfies the maximum condition for normal subloops. As Q is a locally centrally nilpotent loop, then by Theorem 4.2 or Proposition 4.9 Q has a centrally nilpotent system, which in view of the maximum condition for normal subloops is a descending series. If Q' is a commutator-associator subloop, then it is clear that $Q' \neq Q$. Easily, the abelian group Q/Q' satisfy the maximum condition for subgroups, thus Q/Q' is finite generated. Therefore it exist a finite generated subloop L of Q such that $LQ' = Q$. By Corollary 5.4 $L = Q$. Hence 3) implies 1).

The implication 4) $\to$ 1) is proved similarly.

1) $\to$ 4). Any member $\mathcal{A}_i(Q)$ of the transfinite lower central series of Q is finite generated by the implications 1) $\leftrightarrow$ 2). Then, from inclusion $\mathcal{A}_i(Q)/\mathcal{A}_{i+1}(Q) \subseteq \mathcal{Z}(Q/\mathcal{A}_{i+1}(Q))$, it follows that $\mathcal{A}_i(Q)/\mathcal{A}_{i+1}(Q)$ is a finite generated abelian group; consequently, it is a direct product of cyclic groups of prime and/or infinite order

$$\mathcal{A}_i(Q)/\mathcal{A}_{i+1}(Q) = \prod_{j=1}^{n} B_j/\mathcal{A}_{i+1}(Q).$$

Therefore, the factors of the series

$$\prod_{j=1}^{n} B_j \cdot \mathcal{A}_{j+1}(Q) \supseteq B_n \cdot \mathcal{A}_{i+1}(Q) \supseteq \mathcal{A}_{i+1}(Q)$$

are cyclic groups of finite or infinite order. By the hypothesis the finitely generated loop Q is nilpotent, therefore by Theorem 2.2 $\mathcal{A}_k(Q) = \{1\}$ for a certain k.

As the refinement of the central series is also a central series, when making the refinements laid down hereinbefore, we obtain that in loop Q the condition 4) is true. ∎

Corollary 5.7. *Every subloop of a centrally nilpotent finitely generated loop is finitely generated.*

Consider the property $\mathcal{L}$ of the loop G (Lagrange's Theorem): if H is a subloop of a subloop K of the finite loop G, the order of H divides the order of K.

According to [1], Theorem 2.2, a necessary and sufficient condition that a finite loop G have property $\mathcal{L}$ is that G posses a chain of subloops $G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n = 1$ such that G_i is normal in G_{i-1} and G_{i-1}/G_i has property $\mathcal{L}$ for $i = 1, 2, \ldots, n$.

From items 1), 4) of Theorem 5.6 it follows.

Corollary 5.8. *In a centrally nilpotent finite loop Q all maximal chains of subloops have the same length, equal to the number of prime factors (not necessarily different) of the order of loop Q.*

6. LOWER, UPPER CENTRAL SERIES OF MOUFANG LOOPS AND A-LOOPS

According to Proposition 3.10 and Theorem 3.11 let

$$1 \subseteq \mathcal{Z}_1(Q) \subseteq \ldots \subseteq \mathcal{Z}_{k-2}(Q) \subseteq \mathcal{Z}_{k-1}(Q) \subseteq \mathcal{Z}_k(Q) \subseteq \ldots, \tag{6.1}$$

$$Q \supseteq \mathcal{A}_1(Q) \supseteq \ldots \supseteq \mathcal{A}_k(Q) \supseteq \mathcal{A}_{k+1}(Q) \supseteq \mathcal{A}_{k+2}(Q) \supseteq \ldots, \tag{6.2}$$

where

$$\mathcal{Z}_{\alpha+1}(Q)/\mathcal{Z}_\alpha(Q) = \mathcal{Z}(Q/\mathcal{Z}_\alpha(Q)), \quad \mathcal{A}_{\xi+1}(Q) = \mathcal{A}^{\mathcal{A}_\xi}(Q)$$

be the (transfinite) upper and lower central series of a loop Q. Let k be a natural number and let $\mathcal{A}_i(Q) = \mathcal{A}_i$. We consider the quotient loop $Q/\mathcal{A}_{k+2} = G$ and assume that the image of series (6.2) under homomorphism $\varphi : Q \to Q/\mathcal{A}_{k+2}$ has the form

$$Q/\mathcal{A}_{k+2} \supset \mathcal{A}_1/\mathcal{A}_{k+2} \supset \mathcal{A}_{k+1}/\mathcal{A}_{k+2} \supset \mathcal{A}_{k+2}/\mathcal{A}_{k+2} = 1.$$

Let $a \in \mathcal{A}_{k+2}$ and $x, y \in Q$. Then, by Corollary 3.13, $\alpha(a, x, y), \beta(a, x, y), (a, x) \in \mathcal{A}_{k+1}$ and, by (1.3), $\alpha(a, x, y)\mathcal{A}_{k+2}, \beta(a, x, y)\mathcal{A}_{k+2}, (a, x)\mathcal{A}_{k+2} \in \mathcal{A}_{k+1}/\mathcal{A}_{k+2}$. From relation $\mathcal{A}_{k+1}/\mathcal{A}_{k+2} \subseteq \mathcal{Z}(Q/\mathcal{A}_{k+2})$ we get

$$\alpha(a, x, y)\mathcal{A}_{k+2}, \beta(a, x, y)\mathcal{A}_{k+2}, (a, x)\mathcal{A}_{k+2} \in \mathcal{Z}(Q/\mathcal{A}_{k+2}). \tag{6.3}$$

Now we consider the quotient loop $Q/\mathcal{Z}_{k-2} = F$ and let the image of series (6.1) under the homomorphism $\psi : Q \to Q/\mathcal{Z}_{k-2}$ have the form

$$1 = \mathcal{Z}_{k-2}/\mathcal{Z}_{k-2} \subset \mathcal{Z}_{k-1}/\mathcal{Z}_{k-2} \subset \mathcal{Z}_k/\mathcal{Z}_{k-2} \subseteq \ldots$$

From relation $\mathcal{Z}_k/\mathcal{Z}_{k-1} \subseteq \mathcal{Z}(Q/\mathcal{Z}_{k-1})$ it follows that

$$\alpha(\mathcal{Z}_k, Q, Q), \beta(\mathcal{Z}_k, Q, Q), (\mathcal{Z}_k, Q) \in \mathcal{Z}(Q/\mathcal{Z}_{k-1}). \tag{6.4}$$

According to Corollary 3.13 let $G \supset \mathcal{A}_1(G) \supset \mathcal{A}_2(G) = e$ be the lower central series of loop G. Let also $f \subset \mathcal{Z}_1(F) \subset \mathcal{Z}_2(F) \subseteq \ldots$ be the upper central series of loop F (e, f are the units of G, F respectively). If H is a normal subloop of a loop L then by (1.3) $\alpha(xH, yH, zH) = \alpha(x, y, z)H$, $\beta(xH, yH, zH) = \beta(x, y, z)H$, $(xH, yH) = (x, y)H$.

By Corollaries 3.13, 3.15 $\mathcal{A}_t(L) =< w_t(x_1, \dots, x_j)|\forall w_t \in W_t(\alpha, \beta, 1)\forall x_1, \dots, x_j \in L >$, $\mathcal{Z}_t(L) = \{a \in L | w_t(a, x_1, \dots, x_r) = 1,\ \forall w_t \in W_t(\alpha, \beta, 1),\ \forall x_1, \dots, x_r \in L\}$. From here it follows that

$$\mathcal{A}_{k+2}(Q) = \varphi^{-1}(e), \mathcal{A}_{k+1}(Q) \subseteq \varphi^{-1}(\mathcal{A}_1(G)), \mathcal{A}_k(Q) \subseteq G,$$

$$\mathcal{Z}_k(Q) = \psi^{-1}(f), \mathcal{Z}_{k-1}(Q) \subseteq \psi^{-1}(\mathcal{Z}_1(F)), \mathcal{Z}_{k-2}(Q) \subseteq \psi^{-1}(\mathcal{Z}_2(F)). \quad (6.5)$$

From (6.3), (6.4), (3.4) it follows.

Lemma 6.1. *Let Q be a loop with (transfinite) upper and lower central series $\{\mathcal{Z}_\alpha(Q)\}$ and $\{\mathcal{A}_\xi(Q)\}$ respectively and let k be a natural number. Then:*

(i) $a \in \mathcal{A}_k(Q) \Rightarrow$

$\Rightarrow \alpha(a, x, y)\mathcal{A}_{k+2}(Q), \beta(a, x, y)\mathcal{A}_{k+2}(Q), (a, x)\mathcal{A}_{k+2}(Q) \in Z(Q/\mathcal{A}_{k+2})(Q)), \forall x, y \in Q;$

(ii) $a \in \mathcal{Z}_k(Q) \Rightarrow$

$\Rightarrow \alpha(a, x, y)\mathcal{Z}_{k-2}(Q), \beta(a, x, y)\mathcal{Z}_{k-2}(Q), (a, x)\mathcal{Z}_{k-2}(Q), \in Z(Q/\mathcal{Z}_{k-2}(Q)), \quad \forall x, y \in Q.$

Lemma 6.2. *Let G be a loop with (transfinite) upper central series $\{\mathcal{Z}_\alpha(G)\}$ or a centrally nilpotent loop of class 2.*

Then $a \in \mathcal{Z}_2(G) \Rightarrow \alpha(a, x, y), \beta(a, x, y), (a, x) \in Z(G)\ \forall x, y \in G$, where $Z(G)$ mean the centre of loop G.

Lemma 6.2 follows from Lemma 6.1. It also easily follows from Corollary 3.14 and (3.5).

As any Moufang loop is an IP-loop then the Moufang loops can be defined with the help of operations $\cdot$, $^{-1}$, where $x^{-1}x = xx^{-1} = 1$. But A-loops can be defined by operations $1, \cdot, \backslash, /$ and identities

$1 \cdot x = x \cdot 1 = x,\ x/y \cdot y = y \cdot y\backslash x = (x \cdot y)/y = y\backslash(y \cdot x) = x.$

The crucial result used in the proof of Lemma 6.3 is Lemma 6.2. We use it without mention while for Moufang loops we will Lemma 6.2.

Lemma 6.3. *Let 1) G be a loop with (transfinite) upper central series $\{\mathcal{Z}_\alpha(G)\}$ and let $x, y, z \in G$, $a, b \in \mathcal{Z}_2(G)$ or 2) let Q be a centrally nilpotent loop of class 2 and let $a, b, x, y, z \in G$. If G is a Moufang loop then*

$$[a, x, y] = [x, y, a] = [y, a, x],$$

$$[a, x, y]^{-1} = [a^{-1}, x, y] = [a, x^{-1}, y] = [a, y, x], \quad (6.6)$$

$$[ab, x] = [a, x][b, x][a, b, x]^3, [a, xy] = [a, x][a, y][a, x, y]^3, \quad (6.7)$$

$$[a, xy, z] = [a, x, z][a, y, z], \quad [a, x, yz] = [a, x, y][b, x, z], \quad (6.8)$$

$$[ab, x, y] = [a, x, y][b, x, y]. \quad (6.9)$$

If G is an A-loop then

$$(ab, x) = (a, x)(b, x), (a, xy) = (a, x)(a, y), \quad (6.10)$$

$$\gamma(ab, x, y) = \gamma(a, x, y)\gamma(b, x, y),$$

$$\gamma(a, xy, z) = \gamma(a, x, z)\gamma(a, y, z), \gamma(a, x, yz) = \gamma(a, x, y)(b, x, z), \tag{6.11}$$

$$(a\backslash b, x) = (a, x)^{-1}(b, x), (a, x\backslash y) = (a, x)^{-1}(a, y), \tag{6.12}$$

$$\gamma(a\backslash b, x, y) = \gamma(a, x, y)^{-1}\gamma(b, x, y), \gamma(a, x\backslash y, z) =$$
$$\gamma(a, x, z)^{-1}\gamma(a, y, z), \gamma(a, x, y\backslash z) = \gamma(a, x, y)^{-1}(b, x, z). \tag{6.13}$$

$$(a/b, x) = (a, x)(b, x)^{-1}, (a, x/y) = (a, x)(a, y)^{-1}, \tag{6.14}$$

$$\gamma(a/b, x, y) = \gamma(a, x, y)\gamma(b, x, y)^{-1}, \gamma(a, x/y, z) =$$
$$\gamma(a, x, z)\gamma(a, y, z)^{-1}, \gamma(a, x, y/z) \equiv \gamma(a, x, y)(b, x, z)^{-1}, \tag{6.15}$$

where $\gamma = \alpha$ or $\gamma = \beta$.

Proof. Let G be a Moufang loop. The equalities (6.6) follow from items (iii), (iv), (1.8) of Lemma 1.4 with $x = a$ in item (i), as $[a, x] \in Z(G)$, $[[a, x], y] = 1$.

The first equality from (6.7) follows from (1.9) with $x = a$, $y = b$ in item (i) of Lemma 1.4. Any Moufang loop is an IP-loop, then by (1.7) $(xy)^{-1} = y^{-1}x^{-1}$, $[x, y]^{-1} = [y^{-1}, x^{-1}]$, $[x, y, z]^{-1} = [z^{-1}, y^{-1}, x^{-1}]$. The centre of any loop is an abelian group. Then the second equality from (6.7) follows from item (ii) of Lemma 1.4 and (1.10) if $(xy)^{-1} = y^{-1}x^{-1}$ is used and z^{-1} is replaced by a, y^{-1} by x, and x^{-1} by y.

By (1.10), (1.11), it follows

$$L(z, a)(xy) \cdot [a^{-1}, z^{-1}] = L(z, a)x(L(z, a)y \cdot [a^{-1}, z^{-1}]),$$

$$(xy)[xy, a, z]^{-1} \cdot [a^{-1}, z^{-1}] = ((x[x, a, z]^{-1})(y[y, a, z]^{-1})) \cdot [a^{-1}, z^{-1}].$$

As $[a^{-1}, z^{-1}] \in Z(G)$ then, by (1.10), (1.11), it results

$$(xy)[xy, a, z]^{-1} = L(z, a)(xy) = L(z, a)x \cdot L(z, a)y = (x[x, a, z]^{-1})(y[y, a, z]^{-1}).$$

By (6.6), $[a, z, x], [a, z, y] \in Z(G)$ implies $[x, a, z], [y, a, z] \in Z(G)$.

Then, from $(xy)[xy, a, z]^{-1}] = (x[x, a, z]^{-1})(y[y, a, z]^{-1})$ it follows that $[xy, a, z]^{-1} = [x, a, z]^{-1}[y, a, z]^{-1}$, which, by (6.6), implies (6.8).

The subloops $\mathcal{Z}_2$, $\mathcal{G}_k$ are normal in G and $a, b \in \mathcal{Z}_2$ or $a, b \in \mathcal{G}_k$. Then, by (1.4) it follows that $L(y, x)a, L(y, x)b \in \mathcal{Z}_2$ or $L(y, x)a, L(y, x)b \in \mathcal{G}_k$. By (1.5), diassociativity of Moufang loops and (6.6) we get $[x^{-1}, y^{-1}] = x^{-1}y^{-1}xy$. We use (6.7), (6.8), then

$$[L(y, x)a, L(y, x)b, [x^{-1}, y^{-1}]] = [L(y, x)a, L(y, x)b, x][L(y, x)a,$$
$$L(y, x)b, y][L(y, x)a, L(y, x)b, x]^{-1}[L(y, x)a, L(y, x)b, y]^{-1} = 1.$$

According to (1.5), it follows

$$L(y, x)a(L(y, x)b \cdot [x^{-1}, y^{-1}]) = (L(y, x)a \cdot (L(y, x)b)[x^{-1}, y^{-1}]).$$

By (1.11), $L(y, x)(ab)[x^{-1}, y^{-1}] = L(y, x)a(L(y, x)b \cdot [x^{-1}, y^{-1}])$. Then $L(y, x)(ab) = L(y, x)aL(y, x)b$. We use (1.10). Then $(ab)[ab, x, y] = a[a, x, y] \cdot b[b, x, y]$. Consequently, $[ab, x, y] = [a, x, y][b, x, y]$ and (6.9) is proved.

Now, let Q be an A-loop. According to (3.5) one may assume that the loop Q is centrally nilpotent of class 2. Let a, b, x, y, z be elements of G. Then by Corollary 3.14 $a, b \in \mathcal{Z}_2(Q)$ and from (3.4), Corollary 3.14 it follows that the commutator-associators $\alpha(a, x, y), \beta(a, x, y), (a, x)$ belong to center $Z(Q)$ of loop Q. By definition $ax \cdot y = a\alpha(a, x, y) \cdot xy$, $y \cdot xa = yx \cdot \beta(a, x, y)a$, $ax = x(a(a, x))$. We prove the identities (6.10) - (6.15) only for associators of type $\alpha(a, x, y)$ as for $\beta(a, x, y)$, (a, x) the corresponding identities are proved analogically.

As $\alpha(a, x, y) \in Z(Q)$ then from $ax{\cdot}y = a\alpha(a, x, y){\cdot}xy$ we get $ax{\cdot}y = a(\alpha(a, x, y)x{\cdot}y)$, $R(y)L(a)x = L(a)R(y)(\alpha(a, x, y)x)$, $S(a, y)x = \alpha(a, x, y)x$, where $S(a, y) = R(y)^{-1}L(a)^{-1}R(y)L(a)$. Obviously, $S(a, y)1 = 1$, i.e. $S(a, y)$ is an inner mapping. Then $S(a, y)$ is an automorphism of loop Q. Hence $\alpha(a, xz, y)(xz) = S(a, y)(xz) = S(a, y)x \cdot S(a, y)z = \alpha(a, x, y)x \cdot \alpha(a, z, y)z = (\alpha(a, x, y)\alpha(a, z, y))(xz)$, i.e. $\alpha(a, xz, y) = \alpha(a, x, y)\alpha(a, z, y))$. The identities $\alpha(ab, x, y) = \alpha(a, x, y)\alpha(b, x, y)$, $\alpha(a, x, yz) = \alpha(a, x, y)\alpha(a, x, z))$, $(ab, x) = (a, x)(b, x)$, $(a, xy) = (a, x)(a, y)$ are proved analogously. Consequently, the identities (6.10), (6.11) hold for the associators of type $\alpha(x, y, z)$ and the commutators (x, y).

Further, according to (6.10) $(a, x) = (a/b \cdot b, x) = (a/b, x)(b, x)$, $(a/b, x) = (a, x)(b, x)^{-1}$. The other identities (6.10) - (6.15) are proved in a similar manner. ∎

Lemma 6.4. *Let Q be a loop with (transfinite) upper central series $\{\mathcal{Z}_\alpha(G)\}$ and lower central series $\{\mathcal{A}_\xi\}$, let k be a natural number. Further, let $a, b \in \mathcal{Z}_k(G)$ and $\mathcal{D}_k(Q) = \mathcal{Z}_{k-2}(Q)$ or let $a, b \in \mathcal{A}_k(Q)$ and $\mathcal{D}_k(Q) = \mathcal{A}_{k+2}(Q)$.*

If Q is a Moufang loop then, for any x, y, $z \in Q$,

$$[a, x, y] \equiv [x, y, a] \equiv [y, a, x] \pmod{\mathcal{D}_k},$$

$$[a, x, y]^{-1} \equiv [a^{-1}, x, y] \equiv [a, x^{-1}, y] \equiv [a, y, x] \pmod{\mathcal{D}_k}, \tag{6.16}$$

$$[ab, x] \equiv [a, x][b, x][a, b, x]^3 \pmod{\mathcal{D}_k},$$

$$[a, xy] \equiv [a, x][a, y][a, x, y]^3 \pmod{\mathcal{D}_k}, \tag{6.17}$$

$$[a, xy, z] \equiv [a, x, z][a, y, z] \pmod{\mathcal{D}_k},$$

$$[a, x, yz] \equiv [a, x, y][b, x, z] \pmod{\mathcal{D}_k}, \tag{6.18}$$

$$[ab, x, y] \equiv [a, x, y][b, x, y] \pmod{\mathcal{D}_k}. \tag{6.19}$$

If G is an A-loop then

$$(ab, x) \equiv (a, x)(b, x) \pmod{\mathcal{D}_k}, (a, xy) \equiv (a, x)(a, y) \pmod{\mathcal{D}_k}, \tag{6.20}$$

$$\gamma(ab, x, y) \equiv \gamma(a, x, y)\gamma(b, x, y) \pmod{\mathcal{D}_k},$$

$$\gamma(a, xy, z) \equiv \gamma(a, x, z)\gamma(a, y, z) \pmod{\mathcal{D}_k},$$

$$\gamma(a, x, yz) \equiv \gamma(a, x, y)(b, x, z) \pmod{\mathcal{D}_k}, \tag{6.21}$$

$$(a\backslash b, x) \equiv (a, x)^{-1}(b, x) \pmod{\mathcal{D}_k},$$

$$(a, x\backslash y) \equiv (a, x)^{-1}(a, y) \pmod{\mathcal{D}_k}, \tag{6.22}$$

$$\gamma(a\backslash b, x, y) \equiv \gamma(a, x, y)^{-1}\gamma(b, x, y),$$
$$\gamma(a, x\backslash y, z) \equiv \gamma(a, x, z)^{-1}\gamma(a, y, z) \pmod{\mathcal{D}_k},$$
$$\gamma(a, x, y\backslash z) \equiv \gamma(a, x, y)^{-1}(b, x, z) \pmod{\mathcal{D}_k}. \tag{6.23}$$

$$(a/b, x) \equiv (a, x)(b, x)^{-1} \pmod{\mathcal{D}_k},$$
$$(a, x/y) \equiv (a, x)(a, y)^{-1} \pmod{\mathcal{D}_k}, \tag{6.24}$$

$$\gamma(a/b, x, y) \equiv \gamma(a, x, y)\gamma(b, x, y)^{-1} \pmod{\mathcal{D}_k},$$
$$\gamma(a, x/y, z) \equiv \gamma(a, x, z)\gamma(a, y, z)^{-1} \pmod{\mathcal{D}_k},$$
$$\gamma(a, x, y/z) \equiv \gamma(a, x, y)(b, x, z)^{-1} \pmod{\mathcal{D}_k}, \tag{6.25}$$

where $\gamma = \alpha$ or $\gamma = \beta$.

Lemma 6.4 follows from (6.5).

In [6] (see, also, [2]) it is proved that if n is a positive integer and if $H_1, H_2, \dots, H_n$ are non-trivial abelian groups such that H_n has at least three elements, then there exists a TS-loop Q (defined by identities $xy = yx$, $(xy)x = y$) with upper central series $1 = \mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \dots \subset \mathcal{Z}_n = Q$ such that $\mathcal{Z}_i/\mathcal{Z}_{i-1}$ is isomorphic to H_i for $i = 1, 2, \dots, n$. The following statements show that a similarly situation for Moufang loops and A-loops is false. This statements generalizes the analogical results for commutative Moufang loops from [15].

Now we rewrite in more detail the relations (6.17), (6.19) – (6.21).

$$[ab, x] \equiv [a, x][b, x][a, b, x]^3 \pmod{\mathcal{A}_{k+2}(Q)},$$
$$[ab, x, y] \equiv [a, x, y][b, x, y] \pmod{\mathcal{A}_{k+2}(Q)}, \tag{6.26}$$
$$(ab, x) \equiv (a, x)(b, x) \pmod{\mathcal{A}_{k+2}(Q)},$$
$$\alpha(ab, x, y) \equiv \alpha(a, x, y)\alpha(b, x, y) \pmod{\mathcal{A}_{k+2}(Q)},$$
$$\beta(ab, x, y) \equiv \beta(a, x, y)\beta(b, x, y) \pmod{\mathcal{A}_{k+2}(Q)}, \tag{6.27}$$

if $a, b \in \mathcal{A}_k(Q)$, $x, y, z \in Q$.

$$[ab, x] \equiv [a, x][b, x][a, b, x]^3 \pmod{\mathcal{Z}_{k-2}(Q)},$$
$$[ab, x, y] \equiv [a, x, y][b, x, y] \pmod{\mathcal{Z}_{k-2}(Q)}, \tag{6.28}$$
$$(ab, x) \equiv (a, x)(b, x) \pmod{Z_{k-2}(Q)},$$
$$\alpha(ab, x, y) \equiv \alpha(a, x, y)\alpha(b, x, y) \pmod{Z_{k-2}(Q)},$$
$$\beta(ab, x, y) \equiv \beta(a, x, y)\beta(b, x, y) \pmod{Z_{k-2}(Q)}, \tag{6.29}$$

if $a, b \in \mathcal{D}_k(Q)$, $x, y, z \in Q$.

Let Q be an A-loop and let $a \in \mathcal{A}_k(Q)$. By Lemma 6.1 $\alpha(a,x,y)\mathcal{A}_{k+2}(Q)$, $\beta(a,x,y)\mathcal{A}_{k+2}(Q), (a,x)\mathcal{A}_{k+2}(Q) \in Z(Q/\mathcal{A}_{k+2}(Q))$. Hence the factors of the right side of (6.27) form an abelian group. Then the factors of form a, b also form an abelian group. Hence, for such elements, it is clear the degree a^n, b^{-n}. Any Moufang loop is power associative. Hence the similar situation holds for Moufang loops.

From (6.26) – (6.29) it follows that the following relations hold for all $u \in \mathcal{A}_k(Q)$, all $v \in \mathcal{Z}_k(Q)$, all $x, y \in Q$ and all integers m, n:

$$[u^n, x] \equiv [u, x]^n \pmod{\mathcal{A}_{k+2}(Q)},$$

$$[u^n, x, y] \equiv [u, x, y]^n \pmod{\mathcal{A}_{k+2}(Q)}, \tag{6.30}$$

$$[v^m, x] \equiv [v, x]^m \pmod{\mathcal{Z}_{k-2}(Q)},$$

$$[v^m, x, y] \equiv [v, x, y]^m \pmod{\mathcal{Z}_{k-2}(Q)}, \tag{6.31}$$

$$(u^n, x) \equiv (u, x)^n \pmod{\mathcal{A}_{k+2}(Q)},$$

$$\alpha(u^n, x, y) \equiv \alpha(u, x, y)^n \pmod{\mathcal{A}_{k+2}(Q)},$$

$$\beta(u^n, x, y) \equiv \beta(u, x, y)^n \pmod{\mathcal{A}_{k+2}(Q)}, \tag{6.32}$$

$$(v^m, x) \equiv (v, x)^m \pmod{\mathcal{Z}_{k-2}(Q)},$$

$$\alpha(v^m, x, y) \equiv \alpha(v, x, y)^m \pmod{\mathcal{Z}_{k-2}(Q)},$$

$$\beta(v^m, x, y) \equiv \beta(v, x, y)^m \pmod{\mathcal{Z}_{k-2}(Q)}. \tag{6.33}$$

Theorem 6.5. *Let Q be a Moufang loop or an A-loop with (transfinite) upper and lower central series $\{\mathcal{Z}_\alpha(Q)\}$ and $\{\mathcal{A}_\xi(Q)\}$, respectively.*

If for an integer $k \geq 0$ the quotient loop $\mathcal{A}_k(Q)/\mathcal{A}_{k+1}(Q)$ has a finite exponent λ_k, then the quotient loop $\mathcal{A}_{k+1}(Q)/\mathcal{A}_{k+2}(Q)$ has the finite exponent λ_k and λ_{k+1} divide λ_k.

If for a natural number $k \geq 1$ the quotient loop $\mathcal{Z}_{k-1}(Q)/\mathcal{Z}_{k-2}(Q)$ has the finite exponent λ_{k-1}, then the quotient loop $\mathcal{Z}_k(Q)/\mathcal{Z}_{k-1}(Q)$ has the finite exponent λ_k and λ_k is the divider of λ_{k-1}. Moreover, for any $k \geq 2$ the quotient loops $\mathcal{Z}_k(Q)/\mathcal{Z}_{k-1}(Q)$ and $\mathcal{A}^{\mathcal{Z}_k(Q)}(\mathcal{Z}_{k-2}(Q))$ have the same exponent.

Proof. We suppose that $\mathcal{A}_k(Q)/\mathcal{A}_{k+1}(Q)$ has a finite exponent λ_k that is for any $a \in \mathcal{A}_k(Q)$ we have $a^{\lambda_k} \in \mathcal{A}_{k+1}$. Then $[a^{\lambda_k}, x] \in \mathcal{A}_{k+2}(Q)$, $[a^{\lambda_k}, x, y] \in \mathcal{A}_{k+2}(Q)$ or $(a^{\lambda_k}, x]) \in \mathcal{A}_{k+2}(Q)$, $\alpha(a^{\lambda_k}, x, y) \in \mathcal{A}_{k+2}(Q)$, $\beta(a^{\lambda_k}, x, y) \in \mathcal{A}_{k+2}(Q)$ for all $x, y \in Q$. Further, according to (6.30), (6.32)

$$[a, x]^{\lambda_k} \equiv [a^{\lambda_k}, x] \pmod{\mathcal{A}_{k+1}(Q)} \equiv 1 \pmod{\mathcal{A}_{k+2}(Q)},$$

$$[a, x, y]^{\lambda_k} \equiv [a^{\lambda_k}, x, y] \pmod{\mathcal{A}_{k+1}(Q)} \equiv 1 \pmod{\mathcal{A}_{k+2}(Q)},$$

$$(a, x)^{\lambda_k} \equiv (a^{\lambda_k}, x) \pmod{\mathcal{A}_{k+1}(Q)} \equiv 1 \pmod{\mathcal{A}_{k+2}(Q)},$$

$$\alpha(a^{\cdot}x, y)^{\lambda_k} \equiv \alpha(a^{\lambda_k}, x, y) \pmod{\mathcal{A}_{k+1}(Q)} \equiv 1 \pmod{\mathcal{A}_{k+2}(Q)},$$

$$\beta(a, x, y)^{\lambda_k} \equiv \beta(a, x, y)^{\lambda_k} \pmod{\mathcal{A}_{k+1}(Q)} \equiv 1 \pmod{\mathcal{A}_{k+2}(Q)}.$$

As $\mathcal{A}_{k+1}/\mathcal{A}_{k+2} \subseteq Z(Q/\mathcal{A}_{k+2}$, the quotient loop $\mathcal{A}_{k+1}/\mathcal{A}_{k+2}$ is an abelian group. Hence $\mathcal{A}_{k+1}/\mathcal{A}_{k+2}$ is generated by elements whose order divides λ_k. Therefore, the quotient loop $\mathcal{A}_{k+1}/\mathcal{A}_{k+2}$ has the exponent λ_{k+1} and λ_{k+1} divides λ_k.

The statement for upper central series is proved analogous to the first case, by using (6.31), (6.33).

Finally, we will prove the last statement only for Moufang loops, as for A-loops it is proved by analogy. Indeed, let $v \in \mathcal{Z}_k(Q)$, $x, y \in Q$, $k \geq 2$, and let m be a natural number. By (6.31) we have $[v^m, x] \equiv [v, x]^m$ (mod)$Z_{k-2}(Q)$, $[v^m, x, y] \equiv [v, x, y]^m$ (mod)$Z_{k-2}(Q)$.

This leads to the conclusion that both quotient loops $\mathcal{Z}_k(Q)/\mathcal{Z}_{k-1}(Q)$ and $[\mathcal{Z}_k(Q), L]/\mathcal{Z}_{k-2}(Q)$ have the same exponent. ∎

Corollary 6.6. *If the subloop $\mathcal{A}_1(Q)$ of a Moufang loop or A-loop Q has a finite exponent, then the exponent of the quotient loop $\mathcal{Z}_2(Q)/\mathcal{Z}_1(Q)$ is a divider of the exponent of $\mathcal{A}_1(Q)$.*

Proof. Let Q be a Moufang loop. Then $[\mathcal{Z}_2(Q), Q] \subseteq [Q.Q] = \mathcal{A}_1(Q)$, $[\mathcal{Z}_2(Q), Q, Q] \subseteq [Q.Q, Q] = \mathcal{A}_1(Q)$. Hence $[\mathcal{Z}_2(Q), Q]$, $[\mathcal{Z}_2(Q), Q]$ has a finite exponent, which divides the exponent of $\mathcal{A}_1(Q)$. By Theorem 6.5 the exponent of the quotient loop $\mathcal{Z}_2(Q)/\mathcal{Z}_1(Q)$ is a divider of $\mathcal{A}_1(Q)$.

The statement for A-loops is proved analogically to the first case. ∎

Corollary 6.7. *Let Q be a Moufang loop or an A-loop. If $Q/\mathcal{Z}_1(Q)$ has a finite exponent, then the abelian group $\mathcal{A}_1(Q)/\mathcal{A}_2(Q)$ has the finite exponent and it is a divider of the exponent $L/\mathcal{Z}_1(Q)$.*

Proof. Again we prove it only for Moufang loops. Let H be the inverse image of subloop $\mathcal{Z}(Q/\mathcal{A}_2(Q))$ under homomorphism $Q \to Q/\mathcal{A}_2(Q)$. Then $H \supseteq \mathcal{A}_2(Q)\mathcal{Z}_1(Q) \supseteq \mathcal{Z}_1(Q)$. Hence the exponent m of the loop L/H is finite and it is a divider of the exponent of the loop $L/\mathcal{Z}_1(Q)$. But in this case, for any $v \in Q$ we have $v^m = b \cdot c$, where $b \in \mathcal{Z}_2(Q), c \in \mathcal{Z}_1(Q)$. By using (6.31), (6.28), it follows:

$$[v, x, y]^m \equiv [v^m, x, y] \equiv [b \cdot c, x, y] \equiv [b, x, y] \equiv 1 \pmod{\mathcal{A}_2(Q)},$$

$$[v, x]^m \equiv [v^m, x] \equiv [b \cdot c, x] \equiv [b, x] \equiv 1 \pmod{\mathcal{A}_2(Q)}$$

Hence the quotient loop $\mathcal{A}_1(Q)/\mathcal{A}_2(Q)$ (which is an abelian group as $\mathcal{A}_1(Q)/\mathcal{A}_2(Q) \subseteq \mathcal{Z}(Q/\mathcal{A}_2(Q))$ has a finite exponent, a divider of m. ∎

Obviously, from Corollary 6.6 we have.

Corollary 6.8. *Let Q be a centrally nilpotent of class 2 Moufang loop or A-loop. Then $\mathcal{A}_1(Q)$ and $Q/\mathcal{Z}_1(Q)$ have the same exponent.*

Proposition 6.9. *Let the Moufang loop (respect. A-loop) Q with lower central series $\{\mathcal{A}_\alpha(Q)\}$ be generated by the set M. Then for any integer $n \geq 0$ the quotient loop $\mathcal{A}_n(Q)/\mathcal{A}_{n+1}(Q)$ is generated by those conjugate classes that contain commutator-associators of type $(\mu, 1)$ (respect. $(\alpha, \beta, 1)$) of weight n of the elements of M.*

Proof. We use the induction with respect to n. For $n = 0$ the statement is obvious. We assume that the quotient loop $\mathcal{A}_{n-1}(Q)/\mathcal{A}_n(Q)$ is generated by conjugate classes containing commutator-associators of type $(\mu, 1)$ (respect. $(\alpha, \beta, 1)$) of weight $n-1$ of elements of M. By Corollary 3.13 $\mathcal{A}_n(Q)$ is generated by the elements $[a, x, y]$, $[a, x]$ (respect. $\alpha(a, x, y)$, $\beta(a, x, y)$, (a, x)), where $a \in \mathcal{A}_{n-1}(Q)$, $x, y \in Q$. By induction hypothesis, $a = (a_1^{\pm 1} \dots a_r^{\pm 1}) \cdot z$, where $z \in \mathcal{A}_n$, and each a_i is a commutator-associator of type $(\mu, 1)$ (respect. $(\alpha, \beta, 1)$) of weight $n - 1$ of elements of M. The elements x, y are terms with respect to variables $b_i, c_i \in M$. After using several times the identities (6.16) – (619) (respect. (6.20) – (6.25)), we get that $[a, x, y]$, $[a, x]$ (respect. $\alpha(a, x, y)$, $\beta(a, x, y)$, (a, x)) are products of terms of the forms $[a_i, b_i, c_i]$, $[a_i, b_i]$, $[z, x, y]$, $[z, x]$ (respect. $\alpha(a_i, b_i, c_i)$, $\beta(a_i, b_i, c_i)$, (a_i, b_i), $\alpha(z, x, y)$, $\beta(z, x, y)$, (z, x), where $b_i, c_i \in M$. Since $[z, x, y], [z, x], \alpha(z, x, y), \beta(z, x, y), (z, x) \in \mathcal{A}_{n+1}$ the proof ends. ∎

Remark 6.10. The Proposition 6.9 for Moufang loops is presented in paper [16] (Lemma 2.2). It has a crucial importance for the proof of the main results of the paper. But its proof is mistaken. Namely, the member L_n of lower central series is defined as a normal subloop generated by the associator-commutators of weight n, but in proof of Lemma 2.2 it is used (again by definition) as a subloop generated by the associator-commutators of weight n. This ignores the Lemma 1.3.

The following statement follows from Theorem 5.6, but for fullness we prove it.

Corollary 6.11.. *Any finitely generated centrally nilpotent Moufang loop or A-loop Q satisfies the maximum condition for its subloops.*

Proof. Let $Q = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \dots \mathcal{A}_n = \{1\}$ be the lower central series of the loop Q and let H be a subloop of Q. We denote $H_i = H \cap \mathcal{A}_i$. Using (1.3) it is easy to see that the subloop H_i is normal in H. By homomorphism theorems we get $H_i/H_{i+1} = H_i/(H \cap \mathcal{A}_{i+1}) \cong H_i\mathcal{A}_{i+1}/\mathcal{A}_{i+1} \subseteq \mathcal{A}_i/\mathcal{A}_{i+1}$. By Proposition 6.9 $\mathcal{A}_i/\mathcal{A}_{i+1}$ is a finitely generated abelian group. Then H_i/H_{i+1} also is finitely generated. From here it follows that the subloop H is finitely generated. ∎

As follows from Section 3, the theory of centrally nilpotent loops differs from theory of nilpotent groups in that the notions of commutators, of commutators of weight n, of lower central series, of upper central series etc. for groups are replaced by the notions of commutator-associator, of commutator-associators of weight n, of lower central series, of upper central series, etc. for loops respectively. Making such a mechanical replacement in [17] or [18] (see, for example [19], [16], [14]) using the Proposition 6.9 , Corollary 6.11 or the Theorem 5.6 we get.

Theorem 6.12. *The variety generated by a centrally nilpotent Moufang loop is finitely based.*

Theorem 6.13. *The variety generated by a centrally nilpotent A-loop is finitely based.*

Given the importance of the last two results, we make the following remark.

Remark 6.14. The Theorems 6.12 and 6.13 are related to the main results of papers [16], [14], which contain similar statements for nilpotent Moufang loops and nilpotent A-loops with the Theorems 6.12 and 6.13. But, unfortunately, these mentioned results from [16], [14] we cannot consider as proved. They are corrected and described in detail in this paper.

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