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ON A CLASS OF SURFACES IN $\mathbb{H}^+ \times \mathbb{R}$

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Abstract In this work we mainly discuss a particular class of surfaces in the ambient space $\mathbb{H}^+ \times \mathbb{R}$, namely the surfaces with a canonical principal direction and we provide their complete classification. The relations between these type of surfaces involving both the Minkowski model $\mathcal{H}$ and the upper half-plane model $\mathbb{H}^+$ of the hyperbolic plane are described.

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1. INTRODUCTION

Classically, the differential geometry of submanifolds starts with the study of surfaces in the Euclidean 3−space. Then, one could extend the study, by one hand varying the dimension or the codimension of the submanifold, and on the other hand changing the ambient space. A wide class of ambient spaces that gained interest among geometers is represented by homogeneous spaces. Roughly speaking, we say about a homogeneous space that it *looks the same* in every point. An impulse for the research of these spaces was given by the full classification of Thurston in [17]:

$$\mathbb{E}^3,\ \mathbb{S}^3,\ \mathbb{H}^3,\ \mathbb{H}^2 \times \mathbb{R},\ \mathbb{S}^2 \times \mathbb{R},\ \widetilde{T_1(\mathbb{H}^2)},\ \mathrm{Nil}_3,\ \mathrm{Sol}_3.$$

In the above list, the first three ambient spaces are the well-known space forms, then we have the product type ambient spaces, $\widetilde{T_1(\mathbb{H}^2)}$ is the universal covering space of the unit tangent space of $\mathbb{H}^2$, Nil_3 denotes the Heisenberg group and finally Sol_3 is a Lie group whose isometry group has dimension 3.

A particular class of surfaces, namely *constant angle surfaces* in the most of these ambient spaces, were intensively studied recently. By definition, a surface for which its normal makes a constant angle θ with a fixed direction is called constant angle surface. This problem was first solved in [3] for the ambient space $\mathbb{S}^2 \times \mathbb{R}$, and then for $\mathbb{H}^2 \times \mathbb{R}$ in [5] and [6], using both the upper half-plane model and the Minkowski model of $\mathbb{H}^2$, respectively. A new approach on the study of constant angle surfaces in $\mathbb{E}^3$ is included in [13] and curves generating constant angle surfaces in $\mathbb{E}^3$ are discussed in [16]. Related on the results obtained in the Euclidean 3−space, in [9] the classification of space-like constant angle surfaces in Minkowski space is given. Finally, the constant angle surfaces in the Heisenberg group Nil_3 are investigated in

[8] and in the case of the ambient space Sol_3 the problem is solved in [10]. The full classification of constant angle surfaces in the eight geometries from Thurston's list is not yet completed. We recommend the survey [15] for a concise description of the classification results on constant angle surfaces obtained until now.

Let us focus our attention on ambient spaces of product type $\mathbb{M}^2\times\mathbb{R}$. Among all the results on surfaces in these ambient spaces, let us mention only few published in last five years, when $\mathbb{M}^2$ has constant Gaussian curvature; see for example [1, 2, 11, 12].

In order to study constant angle surfaces in this framework, the angle θ is taken between the normal to the surface and the tangent to the second factor, namely the real line $\mathbb{R}$. Denoting by t the global parameter on $\mathbb{R}$, we have the following decomposition of the canonical field $\frac{\partial}{\partial t} :\overset{not}{=} \partial_t$,

$$\partial_t = T + \cos\theta\xi, \tag{1}$$

where T is the projection of ∂_t on the tangent plane to the surface M and ξ is the unit normal to the surface. A distinct property of these surfaces consists in the fact that the projection T is a principal direction for the surface with the corresponding principal curvature zero.

A natural generalization of this topic is the study of those surfaces for which T remains a principal direction but with non-null principal curvature, case in which T is called *canonical principal direction*. First results were obtained in the ambient space $\mathbb{S}^2\times\mathbb{R}$ in [4].

The problem may be also investigated in the ambient space $\mathbb{H}^2\times\mathbb{R}$, considering different models of the hyperbolic plane. A $3-$dimensional model is the Minkowski model of $\mathbb{H}^2$ and it was introduced as the analog of the sphere $\mathbb{S}^2$ in Euclidean space. Next to it, there are also $2-$dimensional models used especially to *visualize* the hyperbolic plane with *Euclidean eyes*. The most known are the upper half-plane model $\mathbb{H}^+$, the Klein model and the Poincaré disk. Surfaces with a canonical principal direction in $\mathbb{H}^2\times\mathbb{R}$ were classified in [7], when the Minkowski model was considered for $\mathbb{H}^2$.

In this paper we consider the upper half-plane model $\mathbb{H}^+$ for the hyperbolic plane in order to give also graphical representations for the obtained surfaces. Having an *Euclidean visualization* of these surfaces, we could understand better their geometry. This fact could be subject to future remarks comparing the results in $\mathbb{H}^2\times\mathbb{R}$ and those obtained in [14] classifying surfaces with a canonical principal direction in the Euclidean space $\mathbb{E}^3$.

The main result of this paper consists in the full classification of surfaces with a canonical principal direction in $\mathbb{H}^+\times\mathbb{R}$. The partial differential equations we obtain here are completely different from those in [7] mainly because $\mathbb{H}^2$, as Minkowski model, is embedded in $\mathbb{R}^3_1$ with the Lorentzian scalar product and hence, the surface M is thought as a submanifold of codimension 2 in $\mathbb{R}^3_1\times\mathbb{R}$.

Finally, we pass from the upper half-plane model $\mathbb{H}^+$ to the Minkowski model (where the classification is already known) via Cayley transformations, in order to show the consistency of our results.

2. PRELIMINARIES

Let us consider the immersion $F : M \to \widetilde{M}$, where $\widetilde{M}$ denotes the ambient space $\mathbb{H}^+ \times \mathbb{R}$ endowed with the metric $\tilde{g}$ and $\widetilde{\nabla}$ its corresponding Levi Civita connection. The metric on the surface is $g = \widetilde{g}|_M$ with the corresponding Levi Civita connection ∇. In these notations the expression of the angle function can be interpreted as

$$\cos\theta = \widetilde{g}(\partial_t, \xi). \tag{2}$$

The fundamental Gauss and Weingarten formulas are

(**G**) $\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$,

(**W**) $\widetilde{\nabla}_X \xi = -AX$,

where X, Y are tangent to M and h denotes the second fundamental form with A the associated shape operator.

In [7] there are proved the following results which allow us to work with coordinates in the hyperbolic plane independent of the model we choose.

Proposition A. *Let M be isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$ with T a principal direction. Then, we can choose the local coordinates $(u,\ v)$ such that ∂_u is in the direction of T, the metric is expressed by*

$$g = du^2 + \beta^2(u,\ v)dv^2, \tag{3}$$

and the shape operator with respect to $\{\partial_u,\ \partial_v\}$ can be written as

$$A = \begin{pmatrix} \theta_u & 0 \\ 0 & \tan\theta\frac{\beta_u}{\beta} \end{pmatrix}. \tag{4}$$

Moreover, the functions θ and β are related by

$$\beta_{uu} + \tan\theta\theta_u\beta_u - \beta\cos^2\theta = 0, \tag{5}$$

and $\theta_v = 0$.

Proposition B. *The partial differential equation* (5) *has the following solutions, after a change of the v-coordinate:*

(a) $\beta = \sinh(\phi(u) + \psi(v))$,

(b) $\beta = \cosh(\phi(u) + \psi(v))$,

(c) $\beta = e^{\pm\phi(u)}$,

where, in each case, $\phi(u)$ is a primitive function of $\cos\theta$, $\phi(u) = \int^u \cos\theta(\tau)d\tau$, *and $\psi(v)$ is a integration function depending only on v.*

3. MAIN RESULTS

In this section we study surfaces with a canonical principal direction given by T in $\mathbb{H}^+ \times \mathbb{R}$ using the upper half-plane model

$$\mathbb{H}^+ = \{(X,\ Y) \in \mathbb{R}^2, Y > 0\} \text{ endowed with the metric } g_{\mathbb{H}} = \tfrac{dX^2+dY^2}{Y^2}.$$

Let us consider a Riemannian immersion of our surface in $\widetilde{M}$,

$$F : M \to \widetilde{M} = \mathbb{H}^+ \times \mathbb{R},\ \ (u, v) \mapsto (X(u, v), Y(u, v), t(u, v)),$$

when the metric on the ambient space is given by $\widetilde{g} = g_{\mathbb{H}} + dt^2$. Without loss of generality, let us fix a point on M such that $F(0, 0) = (0, 1, 0)$; otherwise M is determined up to translations in the ambient space. In order to describe locally the surface M, we must determine the functions $X(u, v)$, $Y(u, v)$ and $t(u, v)$.

The third component of the parametrization can be immediately determined. The partial derivatives of the immersion are $F_u = (X_u, Y_u, t_u)$ and $F_v = (X_v, Y_v, t_v)$. Since $t_u = \widetilde{g}(F_u, \partial_t)$ and using (1) with $T = \sin\theta\partial_u$, one gets $t_u = \sin\theta$. Moreover, $t_v = \widetilde{g}(F_v, \partial_t) = 0$ which implies that t is a primitive function of $\sin\theta$, namely

$$t(u, v) = \int^{u} \sin\theta(\tau)d\tau. \tag{6}$$

Concerning the unit normal to M in $\mathbb{H}^+ \times \mathbb{R}$, from (2) it follows that

$$\xi(u, v) = (\xi_1(u, v), \xi_2(u, v), \cos\theta),$$

such that $\xi_1^2 + \xi_2^2 = Y^2 \sin^2\theta$. Substituting the expression of ξ in (1) we get

$$\xi_1 = \tan\theta X_u,\ \ \xi_2 = -\tan\theta Y_u. \tag{7}$$

From Proposition A we find the Levi Civita connection associated to the metric on the surface,

$$\nabla_{\partial_u}\partial_u = 0,\ \ \nabla_{\partial_u}\partial_v = \nabla_{\partial_v}\partial_u = \frac{\beta_u}{\beta}\partial_v,\ \ \nabla_{\partial_v}\partial_v = -\beta\beta_u\partial_u + \frac{\beta_v}{\beta}\partial_v. \tag{8}$$

Denoting $\widetilde{\Gamma}_{ij}^k$ with $i, j, k = \overline{1, 3}$ the Christoffel symbols for $\widetilde{\nabla}$ on $\mathbb{H}^+ \times \mathbb{R}$, we find the non zero components given by $\widetilde{\Gamma}_{12}^1 = \widetilde{\Gamma}_{21}^1 = \widetilde{\Gamma}_{22}^2 = -\frac{1}{Y}$ and $\widetilde{\Gamma}_{11}^2 = \frac{1}{Y}$.

Combining **(G)**, (4), (7) and (8) we obtain the following system of partial differential equations that must be satisfied by the first two components of the parametrization, namely $X(u, v)$ and $Y(u, v)$:

$$X_{uu} = \frac{2}{Y}X_uY_u - \tan\theta\theta_uX_u, \tag{9}$$

$$Y_{uu} = \frac{1}{Y}(Y_u^2 - X_u^2) - \tan\theta\theta_u Y_u, \tag{10}$$

$$X_{uv} = \frac{1}{Y}(X_u Y_v + Y_u X_v) + \frac{\beta_u}{\beta} X_v, \tag{11}$$

$$Y_{uv} = \frac{1}{Y}(Y_u Y_v - X_u X_v) + \frac{\beta_u}{\beta} Y_v, \tag{12}$$

$$X_{vv} = \frac{2}{Y} X_v Y_v - \frac{1}{\cos^2\theta}\beta\beta_u X_u + \frac{\beta_v}{\beta} X_v \tag{13}$$

$$Y_{vv} = \frac{1}{Y}(Y_v^2 - X_v^2) - \frac{1}{\cos^2\theta}\beta\beta_u Y_u + \frac{\beta_v}{\beta} Y_v. \tag{14}$$

In order to solve this system of equations we proceed as follows. Integrating once (9) we get

$$X_u = r(v)\cos\theta Y^2, \tag{15}$$

with $r(v)$ an integration function. Replacing the expression of X_u in (10) we obtain $Y_{uu} = \frac{1}{Y}(Y_u^2 - r^2(v)\cos^2\theta Y^4) - \tan\theta\theta_u Y_u$, which can be rewritten as

$$\rho_{uu} + r^2(v)\cos^2\theta e^{2\rho} + \tan\theta\theta_u\rho_u = 0, \tag{16}$$

where we denoted $\rho := \ln Y$.

Case 1. $r(v) \neq 0$, for all v.

Equation (16) can be equivalently written as

$$\frac{\partial}{\partial u}\Big(\Big(\frac{\rho_u}{\cos\theta}\Big)^2\Big) = -r^2(v)\frac{\partial}{\partial u}\Big(e^{2\rho}\Big),$$

and integrating it once we get $\frac{\rho_u}{\sqrt{\eta^2(v) - r^2(v)e^{2\rho}}} = \pm\cos\theta$, where $\eta(v)$ is an integration function always positive.

After a second integration with respect to u we have

$$\frac{\rho}{\eta(v)} - \frac{1}{\eta(v)}\ln\Big(\eta(v) + \sqrt{\eta^2(v) - r^2(v)e^{2\rho}}\Big) = \pm\phi(u) + \frac{\sigma(v)}{\eta(v)},$$

where $\phi(u)$ is a primitive of $\cos\theta$ and we denoted $\sigma(v) = -\ln\eta(v)$.

Further, this expression can be rewritten in an equivalent way as

$$\ln\Big(\eta(v) + \sqrt{\eta^2(v) - r^2(v)e^{2\rho}}\Big) = \ln\left(r(v)e^{\rho}\right) \mp \eta(v)\phi(u) + \overline{\sigma}(v),$$

with $\overline{\sigma}(v) := -\sigma(v) - \ln r(v)$. Straightforward computations yield

$$r^2(v)e^{2\rho} = \eta^2(v)\Big(1 - \tanh^2\left(\mp\eta(v)\phi(u) + \overline{\sigma}(v)\right)\Big),$$

and recalling that $e^{\rho} = Y$, we find the second component of the immersion, namely

$$Y(u,v) = \frac{\eta(v)}{r(v)\cosh\Big(\eta(v)\phi(u) + p(v)\Big)}. \tag{17}$$

Replacing the above expression of Y in (15) and integrating with respect to u we find also the first component of the immersion,

$$X(u,v) = \frac{\eta(v)\tanh\Big(\eta(v)\phi(u) + p(v)\Big)}{r(v)} + q(v). \tag{18}$$

Note that in (17) and (18) we redenoted the integration function depending on v by $p(v)$.

In order to determine the integration functions, or at least to find some expressions that could characterize them, we check first the conditions that follow from the fact that $F(u,v)$ is an isometric immersion.
Consequently, $\widetilde{g}(F_*\partial_u, F_*\partial_u) = 1$, $\widetilde{g}(F_*\partial_u, F_*\partial_v) = 0$ and $\widetilde{g}(F_*\partial_v, F_*\partial_v) = \beta^2$ yield respectively

$$\frac{1}{Y^2}(X_u^2 + Y_u^2) = \cos^2\theta, \tag{19}$$

$$X_u X_v + Y_u Y_v = 0, \tag{20}$$

$$\frac{1}{Y^2}(X_v^2 + Y_v^2) = \beta^2. \tag{21}$$

Combining (17), (18) and (19) we get $\eta^2(v)\cos^2\theta = \cos^2\theta$.
Since $Y(u,v) > 0$, suppose that $r(v) > 0$ and take $\eta(v) := 1$. Firstly, substituting this in (17) and (18) we get

$$X(u,v) = \frac{\tanh\left(\phi(u) + p(v)\right)}{r(v)} + q(v), \quad Y(u,v) = \frac{1}{r(v)\cosh\left(\phi(u) + p(v)\right)}. \tag{22}$$

Secondly, (20) yields

$$\frac{p'(v)}{r(v)} + q'(v) = 0. \tag{23}$$

Finally, from (21) and (22) we obtain

$$-(\ln r)'\cosh(\phi(u) + p(v)) - p'(v)\sinh(\phi(u) + p(v)) = \pm\beta. \tag{24}$$

At this point we found $X(u,v)$ and $Y(u,v)$ having expressions (22) such that the integration functions p, q and r depending on v satisfy (23) and (24).

Going back to the system of equations, we ask what supplementary conditions could we get from the partial differential equations (11)-(14). First, computing

$X_{uv} = (X_v)_u$ we get that (11) is identically satisfied. Second, using the fact that $(X_v)_u = (X_u)_v$ one gets

$$\pm \cos\theta \sinh(\phi(u) + p(v))\beta \mp \cosh(\phi(u) + p(v))\beta_u - p' \cos\theta = 0. \tag{25}$$

Following a similar approach, from (12) we get the same condition (25). Moreover, taking into account this condition, the last equations (13) and (14) are identically satisfied.

Comparing (24) and (25) we easily observe that (25) is just a consequence of (24). Hence, taking into account (24) and combining it with the expressions of β corresponding to each case from Proposition B, we conclude as follows.

If $\beta = \sinh(\phi(u) + \psi(v))$, then the integration functions $p,\ r$ satisfy

$$\begin{cases} p'(v) &= \mp\cosh(p(v) - \psi(v)) \\ (\ln r(v))' &= \pm\sinh(p(v) - \psi(v)). \end{cases} \tag{26}$$

If $\beta = \cosh(\phi(u) + \psi(v))$, then the integration functions $p,\ r$ satisfy

$$\begin{cases} p'(v) &= \pm\sinh(p(v) - \psi(v)) \\ (\ln r(v))' &= \mp\cosh(p(v) - \psi(v)). \end{cases} \tag{27}$$

If $beta = e^{\pm\phi(u)}$, then the integration functions $p,\ r$ satisfy

$$p'(v) = \pm(\ln r(v))'. \tag{28}$$

Case 2. $r(v) = 0$, for all v. In this case, we get from (15) that

$$X(u, v) = X(v). \tag{29}$$

Substituting $r(v) = 0$ in (16) and integrating the obtained equation we find

$$Y(u, v) = q(v)e^{p(v)\phi(u)}. \tag{30}$$

We can rewrite the equations (11) - (13) as follows

$$0 = \left(\frac{Y_u}{Y} + \frac{\beta_u}{\beta}\right)X_v, \tag{31}$$

$$Y_{uv} = \left(\frac{Y_u}{Y} + \frac{\beta_u}{\beta}\right)Y_v, \tag{32}$$

$$X_{vv} = \left(2\frac{Y_v}{Y} + \frac{\beta_v}{\beta}\right)X_v, \tag{33}$$

while equation (14) remains unchanged.

Replacing the expressions of X and Y given by (29) respectively (30) in (19) - (21) we get $p^2(v) = 1$, $q(v)q'(v) = 0$ and respectively $X_v^2 = (\beta Y)^2$. Taking into account

now expression (31), one has $Y(u,v) = qe^{\mp\phi(u)}$, $q \in \mathbb{R}_+^*$.
Since $Y(u,v) > 0$ satisfies the initial condition $F(0,0) = 1$ one easily remarks that q cannot vanish and it is exactly $q = e^{\pm\phi(0)}$. Moreover, $X_v \neq 0$. Now (32) is automatically satisfied and from (33) it follows that X_v is constant. More precisely, the equation (14) gives us extra information, $X_v = q$, namely $X(u,v) = qv$, since the initial condition $X(0,0) = 0$ must be satisfied. Hence, the first two components of the parametrization are

$$X(u,v) = qv, \qquad Y(u,v) = qe^{\mp\phi(u)}, \text{ where } q = e^{\pm\phi(0)}. \tag{34}$$

Recalling that $\phi(u)$ is a primitive function of $\cos\theta$, and assuming $\phi(0) = 0$, we get $q = 1$ and from (34) we have

$$X(u,v) = v, \qquad Y(u,v) = e^{\mp\phi(u)}.$$

Particular case: $p(v) = p$, $q(v) = q$, where $p, q \in \mathbb{R}$, namely the integration functions are constant. If q is a real constant, we can assume without loss of the generality that $q = 0$ because the translations of X along x-axis are isometries. Thus,

$$X(u,v) = \frac{\tanh(\phi(u) + p)}{r(v)}, \quad Y(u,v) = \frac{1}{r(v)\cosh(\phi(u) + p)}.$$

From the initial condition $F(0,0) = (0,1,0)$, we get $p = -\phi(0)$, $r(0) = 1$. As $r(v) > 0$ for all v, we can write it as $r(v) = e^{-\zeta(v)}$ with $\zeta(0) = 0$. Taking into account that $\phi(u)$ is a primitive of $\cos\theta$, we can state the following result:

Proposition 3.1. *In particular, as the integration functions p and q are constant, then the isometric immersion $F : M \to \mathbb{H}^+ \times \mathbb{R}$ is given locally, up to isometries of the ambient space, by*

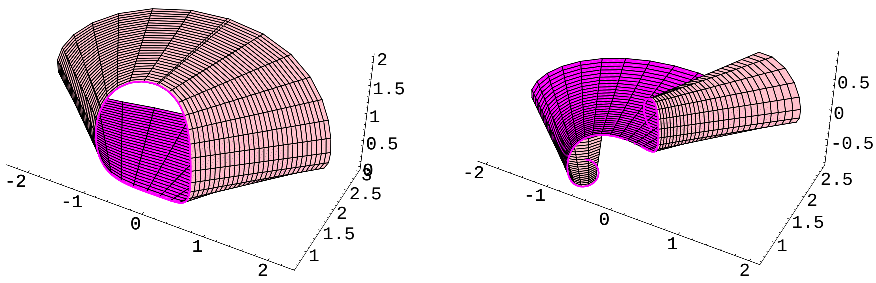
$$F(u,v) = \left(e^{\zeta(v)}\gamma(u), \int^u \sin\theta(\tau)d\tau \right), \tag{35}$$

where $\gamma(u) = \left(\tanh(\phi(u)), \frac{1}{\cosh(\phi(u))}\right)$ is a geodesic line in $\mathbb{H}^+$ given by a semicircle centered in origin and passing through $(0,1)$.

Note that these are ruled surfaces with rulings given by Euclidean straight lines. To illustrate this, we depict two examples of surfaces parameterized by (35), with the corresponding parameters mentioned in Figure 1.

We conclude this section classifying the obtained results and giving some graphical representations of surfaces corresponding to certain parameters.

Theorem 3.1 (**Classification Theorem).** *Let $F : M \to \mathbb{H}^+ \times \mathbb{R}$ be an isometric immersion with angle function $\theta \neq 0, \frac{\pi}{2}$. Then M has a principal direction T if and only if F is given locally, up to isometries of the ambient space, by :*



$\zeta(v) = \log v$, $\theta(u) = u$ $\qquad\qquad$ $\zeta(v) = v$, $\theta(u) = u^2$

Fig. 1. Two examples in Proposition 3.1.

1. $F(u,v) = \left(\dfrac{\tanh(\phi(u) + p(v))}{r(v)} + q(v),\ \dfrac{1}{r(v)\cosh(\phi(u) + p(v))},\ \displaystyle\int^{u} \sin\theta(\tau)d\tau \right)$

where $p,\ q,\ r$ *are integration functions satisfying* (23), (26) - (28) *and* $r(v) \neq 0$ *for all* v.

2. $F(u,v) = \left(v,\ e^{\mp\phi(u)},\ \displaystyle\int^{u} \sin\theta(\tau)d\tau \right)$. *In both cases* $\phi(u) = \displaystyle\int^{u} \cos\theta(\tau)d\tau$.

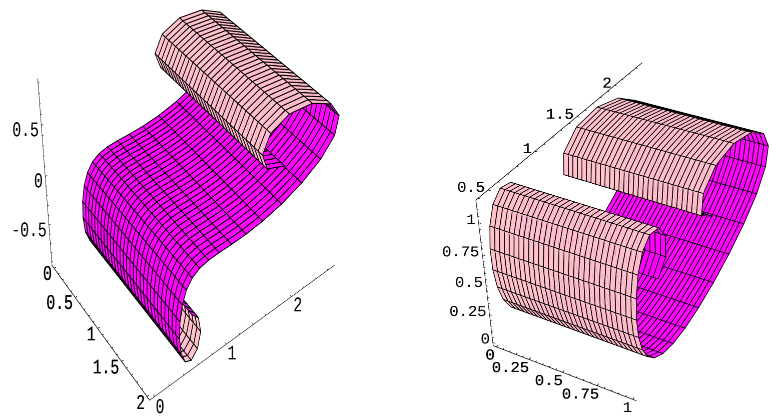
Proof. The direct implication is just the conclusion of the above reasoning.

Conversely, it can be shown by straightforward computations that surfaces parameterized in the two cases have T as a principal direction. ∎

Remark 3.1. *The surfaces from Case* 2 *of the classification Theorem* 3.1 *are right cylinders over the curves parameterized by*

$$t \mapsto \left(e^{\int^{t} \cos\theta(\tau)d\tau},\ \int^{t} \sin\theta(\tau)d\tau \right).$$

Hence, they are ruled surfaces with vanishing Gaussian curvature. (This is suggested also in the above two pictures in Figure 2 for different values of the angle function θ which determines the generating curve).



$\theta(u) = u^2$ $\theta(u) = \sinh u$

Fig. 2. Case 2 of Theorem 3.1.

4. UPPER HALF-PLANE MODEL VERSUS MINKOWSKI MODEL. EXAMPLES

In the previous section we classified all surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with a canonical principal direction using the upper half-plane model $\mathbb{H}^+$ in order to facilitate their visualization in $\mathbb{E}^3$. In the sequel, we show that our results are consistent with those obtained in [7], where $\mathbb{H}^2$ was modeled with the Minkowski space.

Let us denote $\mathcal{H} = \{x \in \mathbb{R}^3_1 \,|\langle x, x\rangle = -1\}$ the Minkowski model of $\mathbb{H}^2$. If we consider the coordinate functions of $x \in \mathbb{R}^3_1$, $x = (x_1, x_2, x_3)$, by symmetry reasons, $\mathbb{H}^2$ can be modeled only after the upper sheet of the hyperboloid, when $x_3 > 0$. Moreover, recall that $\langle\ ,\ \rangle$ denotes the Lorentzian metric $\langle\ ,\ \rangle = dx_1^2 + dx_2^2 - dx_3^2$ and the Lorentzian cross-product for two vectors $x, y \in \mathbb{R}^3_1$ is defined as $\boxtimes : \mathbb{R}^3_1 \times \mathbb{R}^3_1 \to \mathbb{R}^3_1$, $((x_1, x_2, x_3),\ (y_1, y_2, y_3)) \mapsto (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_2y_1 - x_1y_2)$.

If (X, Y) denote the coordinates of a point in the upper half-plane model $\mathbb{H}^+$ and (x_1, x_2, x_3) are the coordinates in the Minkowski model $\mathcal{H}$, they are related via Cayley transformations:

$$X = \frac{x_1}{x_3 - x_2}, \qquad x_1 = \frac{X}{Y},$$
$$Y = \frac{1}{x_3 - x_2}. \qquad x_2 = \frac{X^2 + Y^2 - 1}{2Y},$$
$$x_3 = \frac{X^2 + Y^2 + 1}{2Y}.$$

Starting from the classification theorem in $\mathbb{H}^2 \times \mathbb{R}$ using the upper half-plane model $\mathbb{H}^+$, we retrieve the following theorems from [7] in (u, v)-coordinates on M, where the Minkowski model was used.

Theorem A. *If $F : M \to \mathcal{H} \times \mathbb{R}$ is an isometric immersion with angle function $\theta \neq 0, \frac{\pi}{2}$, then T is a principal direction if and only if F is given locally, up to isometries of the ambient space by*

$$F(u, v) = \Big(A(v) \sinh \phi(u) + B(v) \cosh \phi(u), \chi(u)\Big), \tag{36}$$

where $A(v)$ is a regular curve in $\mathbb{S}^2_1$, $B(v)$ is a regular curve in $\mathcal{H}$, such that $\langle A, B\rangle = 0$, $A' \| B'$ and where $(\phi(u), \chi(u))$ is a regular curve in $\mathbb{R}^2$ parameterized by arc length. The angle function θ on M depends only on u and coincides with the angle function of the curve (ϕ, χ), i.e. $\theta'(u) = \kappa(u)$.

Theorem B. *Let $F : M \to \mathcal{H} \times \mathbb{R}$ be an isometrically immersed surface M in $\mathcal{H} \times \mathbb{R}$. Then M has T as a principal direction if and only if F is given, up to rigid motions of the ambient space, by*

$$F(u, v) = \Big(f(v) \cosh \phi(u) + N_f(v) \sinh \phi(u), \chi(u)\Big), \tag{37}$$

where $f(v)$ is a regular curve in $\mathcal{H}$ and $N_f(v) = \frac{f(v) \boxtimes f'(v)}{\sqrt{\langle f'(v), f'(v)\rangle}}$ represents the normal of f in $\mathcal{H}$. Moreover, (ϕ, χ) is a regular curve in $\mathbb{R}^2$ and the angle function θ of this curve is the same as the angle function of the surface parameterized by F.

Hence, from the first case of our classification Theorem 3.1, we get that

$$X(u, v) = \frac{\tanh(\phi(u) + p(v))}{r(v)} + q(v), \quad Y(u, v) = \frac{1}{r(v) \cosh(\phi(u) + p(v))},$$

which yield

$$\begin{aligned}
x_1(u, v) &= \sinh(\phi(u) + p(v)) + q(v) r(v) \cosh(\phi(u) + p(v)),\\
x_2(u, v) &= q(v) \sinh(\phi(u) + p(v)) + \frac{r(v)}{2}\Big(q^2(v) + \frac{1}{r^2(v)} - 1\Big) \cosh(\phi(u) + p(v)),\\
x_3(u, v) &= q(v) \sinh(\phi(u) + p(v)) + \frac{r(v)}{2}\Big(q^2(v) + \frac{1}{r^2(v)} + 1\Big) \cosh(\phi(u) + p(v)).
\end{aligned}$$

Hence, the isometric immersion is given by

$$F : M \to \mathbb{H}^2 \times \mathbb{R},\ F(u,v) = \big(x_1(u,v), x_2(u,v), x_3(u,v), F_4(u)\big),$$

where the last component is the same in both considered models for $\mathbb{H}^2$. According to Theorem A, we rewrite F as in (36) with

$$\begin{aligned} A(v) &= \Big(\cosh p(v) + q(v)r(v)\sinh p(v), \\ &\quad q(v)\cosh p(v) + \frac{r(v)}{2}\left(q^2(v) + \frac{1}{r^2(v)} - 1\right)\sinh p(v), \\ &\quad q(v)\cosh p(v) + \frac{r(v)}{2}\left(q^2(v) + \frac{1}{r^2(v)} + 1\right)\sinh p(v)\Big), \end{aligned}$$

$$\begin{aligned} B(v) &= \Big(\sinh p(v) + q(v)r(v)\cosh p(v), \\ &\quad q(v)\sinh p(v) + \frac{r(v)}{2}\left(q^2(v) + \frac{1}{r^2(v)} - 1\right)\cosh p(v), \\ &\quad q(v)\sinh p(v) + \frac{r(v)}{2}\left(q^2(v) + \frac{1}{r^2(v)} + 1\right)\cosh p(v)\Big). \end{aligned}$$

By straightforward computations one easily sees that A is a curve in $\mathbb{S}^2_1$, B is a curve in $\mathbb{H}^2$ such that they are orthogonal with parallel speeds.

Moreover, $\langle B'(v), B'(v)\rangle = (\frac{r'}{r}\cosh p(v) + p'(v)\sinh p(v))^2$. Supposing that B' is not constant, denote it with f such that $\langle f', f'\rangle > 0$. But in this case $A(v) = \frac{f(v)\boxtimes f'(v)}{\sqrt{\langle f'(v), f'(v)\rangle}}$, which is exactly N_f in (37) from Theorem B. Hence, this result obtained when the upper half-plane model is used for $\mathbb{H}^2$ is consistent with Theorem B formulated when the Minkowski model is chosen.

Let us follow same steps also in the second case of Theorem 3.1 when the coordinates for a point in the upper half-plane are given by $X(u,v) = v$, $Y(u,v) = e^{\mp\phi(u)}$. Hence, using the Cayley transformations one gets that the parametrization F can be written in form (36) with

$$A(v) = \Big(v, \frac{v^2}{2} - 1, \frac{v^2}{2}\Big) \quad \text{and} \quad B(v) = \Big(v, \frac{v^2}{2}, \frac{v^2}{2} + 1\Big),$$

such that A lies on $\mathbb{S}^2_1$, B lies on $\mathcal{H}$ with $\langle A, B\rangle = 0$ and $A' \| B'$. Moreover, the function f from Theorem B is given by B.

Finally, the particular case treated in Proposition 3.1 yielding the following coordinates in Minkowski model:

$$\begin{aligned} x_1(u,v) &= \sinh(\phi(u) - \phi(0)), \\ x_2(u,v) &= \sinh\zeta\cosh(\phi(u) - \phi(0)), \\ x_3(u,v) &= \cosh\zeta\cosh(\phi(u) - \phi(0)), \end{aligned}$$

is formulated as follows. The parametrization F may be written in form (36) with

$$A(v) = \Big(\cosh\phi(0),\ -\sinh\zeta\sinh\phi(0),\ -\cosh\zeta\sinh\phi(0)\Big),$$

$$B(v) = \Big(-\sinh\phi(0),\ \sinh\zeta\cosh\phi(0),\ \cosh\zeta\cosh\phi(0)\Big),$$

such that the hypothesis from Theorem A and Theorem B are satisfied.

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