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NORMED COERCIVITY FOR MONOTONE FUNCTIONALS

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Abstract A normed coercivity result is established for order nonsmooth functionals fulfilling Palais-Smale conditions. The core of this approach is an asymptotic type statement obtained via local versions of the monotone variational principle in Turinici [An. Şt. UAIC Iaşi, 36 (1990), 329-352].

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1. INTRODUCTION

Let $(X, ||.||)$ be a (real) Banach space. Given a (proper) functional $F : X \to R \cup \{\infty\}$, we say that it is *coercive*, provided

(a01) $F(u) \to \infty$, whenever $u \to \infty$ (in the sense: $||u|| \to \infty$).

Sufficient conditions for such a property are stated in a differential setting, by means of the celebrated 1964 Palais-Smale condition [15]. A typical result in this direction is due to Caklovic, Li and Willem [3]; it states that, whenever

(a02) F is Gâteaux differentiable and lower semicontinuous (lsc),

the relation (a01) is deductible under a Palais-Smale requirement like

(a03) each sequence (v_n) in X with $(F(v_n))$ bounded and
$F'(v_n) \to 0$ (in X^*) has a convergent (in X) subsequence.

(Here, $(X^*, ||.||)$ is the *topological dual* of X). Note that (a02) holds when $F \in C^1(X)$; hence, their statement includes Brezis-Nirenberg's [2]. An extension of it (under (a02)) was obtained by Goeleven [7], for $F = F_1 + F_2$, where

(a04) F_1 is Gâteaux differentiable lsc and F_2 is (proper) convex lsc;

and the Palais-Smale condition (a03) is adapted to this decomposition. Further enlargements of (a04) were given by Motreanu and Motreanu [11]; under

(a05) F_1 is locally Lipschitz (hence continuous) on X, and a Palais-Smale condition as in Motreanu and Panagiotopoulos [13, Ch 3].

The basic tool of all these is Ekeland's variational principle [6] (in short: EVP); and as such, F must be lsc (on X). So, we may ask whether this is removable; an appropriate answer is available in a (linear) quasi-order context. Precisely, let K stand for a (closed) *convex cone* in X; and $(\leq)$, its associated *quasi-order*. Then, (a01) is still retainable under

(a06) F is $(\geq)$-lsc over (the whole of) X

(cf. Section 2) and a specific (modulo K) Palais-Smale condition involving the directional derivatives of F; see Motreanu, Motreanu and Turinici [12] for details. A basic particular case of (a06) is

(a07) F is $(\leq)$-increasing ($x \leq y \Longrightarrow F(x) \leq F(y)$).

Unfortunately, for such functionals, the quoted result is not (directly) applicable; because the Palais-Smale condition does not work. This comes from the fact that (a01) is not retainable in such a context. (Just take $X = R$ and F=identity). A way of correcting it is to restrict F to the cone K; and to rephrase (a01) as

(a08) $F(u) \to \infty$, whenever $u \in K$, $u \to \infty$.

For details, we refer to Motreanu and Turinici [14]; where a conical type coercivity result was formulated via Palais-Smale conditions involving directional (modulo K) derivatives of F. It is our aim in the following to show that this last procedure may be ultimately viewed as a "global" one with respect to the "natural" prolongation

(a09) $G(x) = F(x)$, $x \in K$, $G(x) = \infty$, otherwise;

details will be given in Section 4 below. The basic tool of these investigations is an asymptotic type result involving such functionals (described in Section 3), obtained by means of **i)** a "local" version of the monotone EVP obtained by Turinici [16] (cf. Section 2) and **ii)** a conical variant of the *slope* concept introduced by DeGiorgi, Marino and Tosques [5]. Further aspects will be delineated elsewhere.

2. LOCAL MAXIMALITY PRINCIPLES

Let X be a nonempty set; and $(\leq)$ be a *quasi-order* (i.e.: reflexive and transitive relation) over it. Also, fix a map $d : X \times X \to R_+$; supposed to be *reflexive* [$d(x, x) = 0, \forall x \in X$]; it will be referred to as a *pseudometric* over X. Given the (nonempty) subset $M \in 2^X$, call $z \in X$, *$(\leq, d)$-maximal* over M, if: $u, v \in M$, $z \leq u \leq v \Longrightarrow d(u, v) = 0$. It is our aim in the following to give sufficient conditions for such a property. Then, an application is given to "local" Ekeland principles.

(A) Call the $(\leq)$-ascending sequence $(x_n) \subseteq X$, *d-Cauchy* when [$\forall \varepsilon > 0, \exists n$: $n \leq p \leq q \Longrightarrow d(x_p, x_q) \leq \varepsilon$]; and *d-asymptotic*, if [$d(x_n, x_{n+1}) \to 0$, as $n \to \infty$]. Clearly, each $(\preceq)$-ascending d-Cauchy sequence is d-asymptotic too. The reverse implication is also true in a global sense; i.e., the conditions below are equivalent:

(b01) each $(\preceq)$-ascending sequence is d-Cauchy

(b02) each $(\preceq)$-ascending sequence is d-asymptotic.

In fact, suppose that (b02) holds; but some $(\preceq)$-ascending (x_n) is not entitled with the d-Cauchy property; i.e. (for some $\varepsilon > 0$)

for each n, there exists (p, q) with $n \leq p \leq q, d(x_p, x_q) > \varepsilon$ (hence $p < q$).

Fix some rank $i(0)$. By this assumption, there exists $(i(1), i(2))$ with $i(0) \leq i(1) < i(2)$, $d(x_{i(1)}, x_{i(2)}) > \varepsilon$. Further, given the rank $i(2)$, there exists $(i(3), i(4))$ with $i(2) \leq i(3) < i(4)$, $d(x_{i(3)}, x_{i(4)}) > \varepsilon$. By induction, we get a subsequence $(y_n = x_{i(n)})$ of (x_n) with $d(y_{2n+1}, y_{2n+2}) > \varepsilon$, for all n. This contradicts (b02); hence the claim. By definition, either of these conditions will be referred to as: d is $(\preceq)$-*regular* on X. Note that, as a consequence of this property,

(b03) $(\forall M \in 2^X)$ d is weakly $(\preceq)$-regular on M: $\forall x \in M, \forall \varepsilon > 0$,
$\exists y = y(x, \varepsilon) \in M(x, \preceq)$: $u, v \in M, y \preceq u \preceq v \Longrightarrow d(u, v) \leq \varepsilon$.

[Here, $M(x, \preceq) = \{u \in M; x \preceq u\}$, $\forall x \in X$]. Indeed, assume this would be false; that is (for some $M \in 2^X$, $x \in M, \varepsilon > 0$)

for each $y \in M(x, \preceq)$ there exist $u, v \in M$ with $y \preceq u \preceq v$, $d(u, v) > \varepsilon$.

By the same reasoning as before, there exists (from the starting $y_0 = x$) a $(\preceq)$-ascending sequence (y_n) in M with $d(y_{2n+1}, y_{2n+2}) > \varepsilon$, for all n. But then, (b02) would be contradicted; hence the claim. Further, define a d-convergence structure on X by the convention: $x_n \to x$ iff $d(x_n, x) \to 0$ as $n \to \infty$; referred to as: x is a d-limit of (x_n). The set of all these will be denoted $\lim_n(x_n)$; when it is nonempty, we call (x_n), *d-convergent*. Note that, by the lack of symmetry, a relationship like [(for each sequence) d-convergent $\Longrightarrow$ d-Cauchy] is not in general true. However, it will make sense for us to consider the condition

(b04) d is $(\preceq)$-complete on X:
each $(\preceq)$-ascending d-Cauchy sequence is d-convergent.

For the last requirement, we need some conventions, Given the (nonempty) subset $Z \in 2^X$, let us say that $w \in X$ is a $(\preceq)$-*adherence* point of it when $w = \lim_n z_n$ for some $(\preceq)$-ascending sequence (z_n) of Z; the class of all these will be denoted as $\mathcal{L}_{(\preceq)}(Z)$. Note that, $Y \mapsto \mathcal{L}_{(\preceq)}(Y)$ is not in general an involution over 2^X; i.e., it is not a closure operator, as in Kuratowski [10, Ch I, Sect 4]. However, we shall say that Z is $(\preceq)$-*closed* when $Z = \mathcal{L}_{(\preceq)}(Z)$; i.e.: the limit of each $(\preceq)$-ascending sequence in Z belongs to Z. Now, the condition to be considered is

(b05) $(\preceq)$ is *self-closed* (on X): $X(x, \preceq)$ is $(\preceq)$-closed, for each $x \in X$.

Proposition 2.1. *Assume that (b03)-(b05) hold; and fix some nonempty subset $M \in 2^X$. Then, for each $u \in M$ there exists $v \in \mathcal{L}_{(\leq)}(M)$ with*
i) *$u \leq v$,* **ii)** *v is $(\leq, d)$-maximal over M.*

Proof. From (b03), we may construct a $(\leq)$-ascending sequence (u_n) in M with
j) $u \leq u_0$; **jj)** $[(\forall n), (\forall y, z \in M)]$: $u_n \leq y \leq z \Longrightarrow d(y, z) \leq 2^{-n}$.
This sequence is $(\leq)$-ascending d-Cauchy; so, by (b04), $u_n \to v$ as $n \to \infty$, for some $v \in \mathcal{L}_{(\leq)}(M)$. Moreover, from (b05), $u_n \leq v$, $\forall n$ (hence $u \leq v$). It is not hard to see that v is our desired element; and the conclusion follows. ∎

In particular, when $M = X$, Proposition 2.1 is just the ordering principle in Kang and Park [9]. Moreover, as remarked, a sufficient condition for (b03) is (b01)/(b02). Hence, Proposition 2.1 includes as well the related statement in Turinici [17]. Note that all these are ultimately equivalent with the Brezis-Browder's ordering principle [1]; we do not give details.

(B) A basic application of these facts is to "local" monotone variational principles. Let X be a nonempty set; and $(\leq)$ be a quasi-order on it. Further, let $d(.,.)$ be a pseudometric over X; supposed to be *triangular* $[d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in X]$ and *sufficient* $[d(x, y) = 0$ implies $x = y]$; it will be referred to as an *almost metric* over X. (This comes the fact that d has all the properties of a metric, excepting *symmetry*). All quasi-order notions to be used refer to the *dual* $(\geq)$; but these may be also formulated in terms of our initial relation. Precisely, take the couple $(\leq, d)$ according to

(b06) $(\geq)$ is self-closed and d is $(\geq)$-complete (on X).

In addition, let the function $\varphi : X \to R \cup \{\infty\}$ be such that

(b07) φ is inf-proper $(\mathrm{Dom}(\varphi) \neq \emptyset$ and $\inf[\varphi(X)] > -\infty)$

(b08) φ is $(\geq)$-lsc on X: $[\varphi \leq t] := \{x \in X; \varphi(x) \leq t\}$ is $(\geq)$-closed, $\forall t \in R$.

Call the (nonempty) subset $M \in 2^X$, *φ-admissible* when $M \cap \mathrm{Dom}(\varphi) \neq \emptyset$. The following "local" type variational statement is available.

Proposition 2.2. *Let $M \in 2^X$ be φ-admissible; and fix $u \in M \cap \mathrm{Dom}(\varphi)$. There exists then $v \in \mathcal{L}_{(\geq)}(M) \cap \mathrm{Dom}(\varphi)$ with*

$$u \geq v, d(u, v) \leq \varphi(u) - \varphi(v) \text{ (hence } \varphi(u) \geq \varphi(v)) \tag{1}$$

$$d(v, x) > \varphi(v) - \varphi(x), \text{ for each } x \in M(v, \geq) \setminus \{v\}. \tag{2}$$

Proof. Denote $X[u] = \{x \in X; u \geq x, \varphi(u) \geq \varphi(x)\}$, $M[u] = M \cap X[u]$. Clearly, $X[u] \subseteq \mathrm{Dom}(\varphi)$; moreover, from (b06) and (b08),

$$X[u] \text{ is } (\geq)\text{-closed; hence } d \text{ is } (\geq)\text{-complete on } X[u]. \tag{3}$$

Let $(\preceq)$ stand for the relation (over X):

$$x \preceq y \text{ iff } x \geq y,\ d(x,y) + \varphi(y) \leq \varphi(x).$$

Clearly, $(\preceq)$ is an *order* (antisymmetric quasi-order) on $\mathrm{Dom}(\varphi)$; so, it remains as such on $X[u]$. We claim that conditions of Proposition 2.1 are fulfilled on $(X[u]; \preceq; d)$ and $M[u]$. In fact, let (x_n) be an ascending (modulo $(\preceq)$) sequence in $X[u]$:

(b09) $x_n \geq x_m$ and $d(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m)$, if $n \leq m$.

The sequence $(\varphi(x_n))$ is descending and (via (b07)) bounded from below; hence a Cauchy one. This, along with the preceding relation shows that (x_n) is a $(\preceq)$-ascending d-Cauchy sequence; wherefrom d is $(\preceq)$-regular on $X[u]$. Moreover, the obtained properties give us (by (3)) some $y \in X[u]$ with $x_n \to y$; whence, d is $(\preceq)$-complete on $X[u]$. Finally, fix some rank n. From (b06), $x_n \geq y$; in addition, the triangular property of d gives (via (b08))

$$d(x_n, y) \leq d(x_n, x_m) + d(x_m, y) \leq \varphi(x_n) - \varphi(y) + d(x_m, y), \quad \forall m \geq n;$$

wherefrom (passing to limit as $m \to \infty$) $d(x_n, y) \leq \varphi(x_n) - \varphi(y)$. Summing up, $y \in X[u]$ is an upper bound (modulo $(\preceq)$) of (x_n); and this shows that $(\preceq)$ is self-closed on $X[u]$; hence the claim. By Proposition 2.1 it then follows that, for the starting $u \in M[u]$ there exists $v \in \mathcal{L}_{(\preceq)}(M[u]) \subseteq \mathcal{L}_{(\geq)}(M) \cap \mathrm{Dom}(\varphi)$, with the properties **i)** and **ii)** described there. The former of these is just (1). And the latter one gives at once (2); for (as d=almost metric), it reads: $v \preceq x \in M[u]$ implies $v = x$. ∎

A basic particular case corresponds to the choice $(\leq) = (\geq) = X \times X$ (=the trivial quasi-order on X). The regularity condition (b08) may then be written as

(b10) φ is lsc over X: $\liminf_n \varphi(x_n) \geq \varphi(x)$, whenever $x_n \to x$;

and Proposition 2.2 is nothing but a "local" version of Ekeland's variational principle [6]. On the other hand, the same requirement holds under (a07) (modulo φ) and the self-closeness of $(\geq)$. In this case, Proposition 2.2 is a "local" version of the monotone variational principle obtained by Turinici [16]. Further aspects may be found in Hyers, Isac and Rassias [8, Ch 5].

3. MAIN RESULTS

Let $(X, ||,||)$ be a (real) normed space; and K, some *convex cone* of X [$\alpha K + \beta K \subseteq K$, for all $\alpha, \beta \geq 0$]; supposed to be *non-degenerate* ($K \neq \{0\}$). Denote by $(\leq)$ the associated quasi-order [$x \leq y$ if and only if $y - x \in K$]; this, by the choice of K is *compatible* with the linear structure of X. On the other hand, $(-K)$ is also a (non-degenerate) convex cone of X; and its associated quasi-order is just $(\geq)$ (the dual of $(\leq)$). Let d stand for the (standard) metric induced by $||.||$. As before, it is compatible with the linear structure of X. Finally, note that, $\forall u \in X$,

$$X[u, \delta; \geq] := X[u, \delta] \cap X(u, \geq) \neq \{u\}, \quad \forall \delta > 0; \tag{4}$$

where $X[u,\delta] = \{x \in X; d(u,x) \leq \delta\}$, $X(u,\geq) = \{x \in X; u \geq x\}$, $\delta > 0$. In fact, given $h \in K \setminus \{0\}$, the open half-line $\{u - th; t > 0\} \subseteq X \setminus \{u\}$ has a nonempty intersection with both such objects; we do not give details. Assume that

(c01) K is $(\leq)$-closed and d is $(\leq)$-complete.

Clearly, by the linear-compatible character of $(\geq)$ and d, (b06) holds for these data. Further, take some map $\Gamma : X \to R_+$ with the properties

(c02) $(\exists(\lambda,\mu), 0 < \lambda < 1 < \mu)$: $(d(x,y) \leq \lambda, x \geq y) \Longrightarrow |\Gamma(x) - \Gamma(y)| \leq \mu$

(c03) $\sup[\Gamma(X)] = \infty$ (hence $[\Gamma \geq \sigma] := \{x \in X; \Gamma(x) \geq \sigma\} \neq \emptyset$, $\forall \sigma \geq 0$).

Note that a useful consequence of these facts is

$$\mathcal{L}_{(\geq)}[\Gamma \geq \rho] \subseteq [\Gamma \geq \rho - \mu], \ \forall \rho \geq \mu. \tag{5}$$

For, let $v \in \mathcal{L}_{(\geq)}[\Gamma \geq \rho]$ be arbitrary fixed. By definition, there exists a $(\geq)$-ascending sequence (u_n) in $[\Gamma \geq \rho]$ with $u_n \to v$ (hence, by (c01), $u_n \geq v$, for all n). In particular, there exists some rank $n = n(\lambda)$ with $d(u_n, v) < \lambda$ (and $u_n \geq v$); hence (by (c02)) $|\Gamma(u_n) - \Gamma(v)| \leq \mu$. But then, $\Gamma(v) \geq \Gamma(u_n) - \mu \geq \rho - \mu$; and the claim follows. Finally, pick some functional $F : X \to R \cup \{\infty\}$ with (cf. Section 2)

(c04) F is inf-proper and $(\geq)$-lsc over all of X.

The quantity $m(\Gamma, F)(\sigma) := \inf[F([\Gamma \geq \sigma])]$ is well defined for each $\sigma \geq 0$, via (c03). Moreover, $m(\Gamma, F)(.)$ is increasing from R_+ to $R\cup$; wherefrom

$$\liminf_{\Gamma(u)\to\infty} F(u) := \sup_{\sigma\geq 0} m(\Gamma, F)(\sigma) \ [= \lim_{\sigma\to\infty} m(\Gamma, F)(\sigma)]$$

exists, as an element of $R \cup \{\infty\}$, in view of

$$F_* \leq m(\Gamma, F)(\sigma) \leq \alpha(\Gamma, F) := \liminf_{\Gamma(u)\to\infty} F(u) \leq \infty, \ \ \forall \sigma \geq 0. \tag{6}$$

Here, as usually, $F_* := \inf[F(X)]$. When $\alpha(\Gamma, F) = \infty$, the functional F will be referred to as Γ-*coercive*. It is our aim in the following to get sufficient conditions in order that such a property be attained. These, as a rule, require a *differential* setting. Denote, for each $u \in \mathrm{Dom}(F)$, $|\nabla_{(K)}|F(u) = \max\{0, \nabla_{(K)}F(u)\}$; where

(c05) $\nabla_{(K)}F(u) = \limsup\{\dfrac{F(u) - F(x)}{d(u,x)};\ x \to u, x \leq u\}$.

Note that, the quantity in the right member means

$$\nabla_{(K)}F(u) = \inf_{\delta>0} \sup\{\frac{F(u) - F(x)}{d(u,x)}; x \in X[u,\delta;\geq] \setminus \{u\}\};$$

hence, by (4), it is meaningful; and this tells us that (c05) is well defined. The object in question is a conical version of the one introduced by DeGiorgi, Marino and

Tosques [5]; we shall term it the $(d, \geq)$-*slope* of F at u. The usefulness of its amorphous version ($K = X$) for the critical point theory was underlined by Corvellec, DeGiovanni and Marzocchi [4]. Here, we shall establish that the concept (c05) is appropriate for our "conical" coercivity. The following asymptotic type statement is a basic step to the answer we are looking for. Let $\mathrm{fin}(X)$ stand for the class of all finite parts of X; and (x_n) be a sequence in X. Given $Q \in \mathrm{fin}(X)$, we say that (x_n) *avoids* Q when $[\exists m = m(Q), \forall n \geq m\colon x_n \notin Q]$. If this holds for all $Q \in \mathrm{fin}(X)$, the obtained property will be referred to as: (x_n) avoids $\mathrm{fin}(X)$.

Theorem 3.1. *Suppose that*

(c06) $\alpha(\Gamma, F) < \infty$ *(hence (cf. (6)) $\alpha(\Gamma, F)$ is finite).*

There exists then a sequence $(v_n) \subseteq \mathrm{Dom}(F)$ *with*

$$\Gamma(v_n) \to \infty \text{ as } n \to \infty \text{ (hence } (v_n) \text{ avoids } \mathrm{fin}(X)) \tag{7}$$

$$F(v_n) \to \alpha(\Gamma, F) \text{ and } |\nabla_{(K)}|F(v_n) \to 0 \text{ as } n \to \infty. \tag{8}$$

Proof. **(I)** Let $\eta > 0$ be taken according to

(c07) $\eta < \dfrac{\lambda}{2\mu}$; hence (cf. (c02)) $\dfrac{1}{\eta} > \mu > \dfrac{\lambda}{2} > \eta.$

By (c06), there exists $r(\eta)$ with

$$r(\eta) \geq 1/\eta; \quad m(\Gamma, F)(r) > \alpha(\Gamma, F) - \eta^2, \ \forall r \geq r(\eta). \tag{9}$$

Having this precise, we claim that there exists $v_\eta \in \mathrm{Dom}(F)$ so that

$$\Gamma(v_\eta) \geq r(\eta), \ |F(v_\eta) - \alpha(\Gamma, F)| < \eta^2, \ |\nabla_{(K)}|F(v_\eta) \leq \eta. \tag{10}$$

In fact, by (9) (and the definition of these quantities) we have an evaluation like $\alpha(\Gamma, F) - \eta^2 < m(\Gamma, F)(4r(\eta)) < \alpha(\Gamma, F) + \eta^2$; wherefrom

$$F(u_\eta) < \alpha(\Gamma, F) + \eta^2, \quad \text{for some } u_\eta \in [\Gamma \geq 4r(\eta)]. \tag{11}$$

[Hence, in particular, $u_\eta \in \mathrm{Dom}(F)$]. Taking (c01) and (c04) into account, Proposition 2.2 applies to ($(X, \leq; d)$=as before; $\varphi = (1/\eta)F$; $M = [\Gamma \geq 2r(\eta)]$). So, given $u_\eta \in M \cap \mathrm{Dom}(F)$, there must be some $v_\eta \in \mathcal{L}_{(\geq)}(M) \cap \mathrm{Dom}(F)$ with

$$u_\eta \geq v_\eta, \eta d(u_\eta, v_\eta) \leq F(u_\eta) - F(v_\eta) \text{ (hence } F(u_\eta) \geq F(v_\eta)) \tag{12}$$

$$\eta d(v_\eta, x) > F(v_\eta) - F(x), \text{ for all } x \in M(v_\eta, \geq) \setminus \{v_\eta\}. \tag{13}$$

We claim that v_η fulfills (10). In fact, (5) gives (by (c07)) $v_\eta \in [\Gamma \geq 2r(\eta) - \mu] \subseteq [\Gamma \geq r(\eta)]$; so that, v_η is an element of $\mathrm{Dom}(F)$ fulfilling the first part of (10). Combining with (9)+(11)+(12)

$$\alpha(\Gamma, F) - \eta^2 < F(v_\eta) \leq F(u_\eta) < \alpha(\Gamma, F) + \eta^2; \tag{14}$$

which tells us that the second part of (10) holds too. This, again coupled with (12) yields (via (c07)) $d(u_\eta, v_\eta) \leq (1/\eta)2\eta^2 < \lambda$; so, by (c02), $v_\eta \in [\Gamma \geq 4r(\eta) - \mu]$; hence, in particular, $v_\eta \in M$. Finally, again by (c03) (and (c07)),

$$X(v_\eta, \delta; \geq) \subseteq X(v_\eta, \lambda; \geq) \subseteq [\Gamma \geq 4r(\eta) - 2\mu] \subseteq [\Gamma \geq 2r(\eta)] = M, \quad \forall \delta \in]0, \lambda].$$

This, along with (13), gives the third part of (10); and the claim follows.

(II) Let (η_n) be a descending to zero sequence with $0 < \eta_n < \lambda/2\mu$, $\forall n$; and put $r_n = r(\eta_n)$[=the quantity of (9)], $n \geq 0$. Note that, by this choice, $r_n \geq 1/\eta_n$, for all n; hence $r_n \to \infty$ as $n \to \infty$. Moreover, the developments in **(I)** give us a sequence $(v_n = v_{\eta_n})$ in Dom(F) fulfilling (for each n)

$$\Gamma(v_n) \geq r_n, \ |F(v_n) - \alpha(\Gamma, F)| < \eta_n^2, \ |\nabla_{(K)}|F(v_n) \leq \eta_n. \tag{15}$$

But, from this, (7)+(8) are clear. The proof is thereby complete. ∎

We are now in position to give the promised answer to our coercivity question. For the arbitrary fixed $Q \in \text{fin}(X)$, let us consider the "hybrid" condition:

(c08) each avoiding Q sequence $(x_n) \subseteq$ Dom(F) with $(F(x_n))$=convergent and $|\nabla_{(K)}|F(x_n) \to 0$ has a subsequence (y_n) with $(\Gamma(y_n))$=bounded.

This will be referred to as a Palais-Smale condition (modulo (Q, K)) upon F. When $Q \in \text{fin}(X)$ is generic in such a convention, we shall term the resulting property as: a Palais-Smale condition (modulo $(\text{fin}(X), K)$) upon F.

Theorem 3.2. *Suppose that (in addition) F satisfies a Palais-Smale condition (modulo* $(\text{fin}(X), K)$*). Then, F is* Γ*-coercive.*

Proof. By definition, there exists $Q \in \text{fin}(X)$ such that (c08) holds. If, by absurd, F is not Γ-coercive, the relation (c06) must be true. By Theorem 3.1, we have promised a sequence (v_n) in Dom(F) with the properties (7)+(8); note that, as a consequence of this, (v_n) avoids Q, $(F(v_n))$ is convergent and $|\nabla_{(K)}|F(v_n) \to 0$. Combining with (c08) one deduces that (v_n) must have a subsequence (y_n) with $(\Gamma(y_n))$=bounded. On the other hand, $\Gamma(y_n) \to \infty$, by (7); contradiction. ∎

Now, a "dual" version of this result (with $(-K)$ in place of K) is immediate; we do not give details. On the other hand, (c02) trivially holds under

(c09) $(\exists(\lambda, \mu), 0 < \lambda < 1 < \mu)$: $d(x, y) \leq \lambda \Longrightarrow |\Gamma(x) - \Gamma(y)| \leq \mu$.

Note that, in such a case, Theorem 3.2 includes a statement due to Motreanu, Motreanu and Turinici [12].

4. RELATIVE ASPECTS

The obtained results are global (=absolute) ones; so, it would be useful having local (=relative) forms of them (expressed in a standard differential setting).

Let $(X, \|.\|)$ be a normed space; and K, some (non-degenerate) convex cone of X. The general condition (c01) is still accepted; here, the couple $(\leq, d)$ is introduced as in Section 3. Further, take some map $\Gamma : X \to R_+$ as in (c02); and consider a (non-degenerate) sub-cone Y of K with

(d01) Y is $(\geq)$-closed and $\sup[\Gamma(Y)] = \infty$ (hence $\sup[\Gamma(X)] = \infty$).

Finally, pick some functional $G : X \to R$ with

(d02) (G is bounded below on Y) $G_*^Y := \inf[G(Y)] > -\infty$

(d03) (G is $(\geq)$-lsc on Y) $[Y; G \leq t] := Y \cap [G \leq t]$ is $(\geq)$-closed, $\forall t \in R$.

For each $\sigma \geq 0$, $[Y;\Gamma \geq \sigma] := Y \cap [\Gamma \geq \sigma] \neq \emptyset$; so, $m_Y(\Gamma, G)(\sigma) := \inf[G([Y;\Gamma \geq \sigma])]$ is well defined. Moreover, $m_Y(\Gamma, G)(.)$ is increasing from R_+ to R; wherefrom,

$$\liminf_{\Gamma(u)\to\infty,\, u\in Y} G(u) := \sup_{\sigma\geq 0} m_Y(\Gamma, G)(\sigma) \; [= \lim_{\sigma\to\infty} m_Y(\Gamma, G)(\sigma)]$$

exists, as an element of $R \cup \{\infty\}$, in view of

$$G_*^Y \leq m(\Gamma, G)(\sigma) \leq \alpha_Y(\Gamma, G) := \liminf_{\Gamma(u)\to\infty,\, u\in Y} G(u) \leq \infty, \quad \forall \sigma > 0. \tag{16}$$

When $\alpha_Y(\Gamma, G) = \infty$, the functional G will be referred to as Γ-*coercive* on Y. We are now interested to get sufficient conditions for such a property in a standard differential setting. For each u in $Y_0 := Y \setminus \{0\}$ denote

(d04) $Y[u; -K] = \{h \in -K; \exists \sigma > 0$ such that $u + \tau h \in Y, \forall \tau \in [0, \sigma]\}$.

This is a sub-cone of $(-K)$; which, in addition, is non-degenerate (because $-u \in Y[u; -K]$). For each $h \in Y[u; -K]$ put

$$\Theta G(u)(h) = \limsup_{t\to 0+} \frac{1}{t}[G(u) - G(u + th)].$$

This object [referred to as the *h-directional* derivative of G at u] always exists, as an element of $R \cup \{-\infty, \infty\}$; in addition, the map $h \mapsto \Theta G(u)(h)$ is positively homogeneous. As a consequence, $|\lambda_{(K,Y)}|G(u) = \max\{0, \lambda_{(K,Y)}G(u)\}$ is well defined (for each $u \in Y_0$); where

(d05) $\lambda_{(K,Y)}G(u) = \sup\{\Theta G(u)(h); h \in Y[u; -K] \cap X_{(1)}\}$.

(Here, $X_{(1)} = \{x \in X; \|x\| = 1\}$). Now, for the arbitrary fixed $Q \in \mathrm{fin}(X)$, let us consider the "hybrid" condition:

(d06) each avoiding Q sequence $(x_n) \subseteq Y_0$ with $(G(x_n))$=convergent
and $|\lambda_{(K,Y)}|G(x_n) \to 0$, has a subsequence (y_n) with $(\Gamma(y_n))$=bounded.

This is referred to as a Palais-Smale differential condition (modulo (Q, K, Y)) upon G. When $Q \in \mathrm{fin}(X)$ is generic in such a convention, we shall term the resulting property as: a Palais-Smale differential condition (modulo $(\mathrm{fin}(X), K, Y)$) upon F.

Theorem 4.1. *Suppose that (in addition) G satisfies a Palais-Smale differential condition (modulo* $(\mathrm{fin}(X), K, Y)$*). Then, G is Γ-coercive.*

To establish this, let $F = F(Y, G)$ stand for the function

(d07) $F(x) = G(x)$, if $x \in Y$; $F(x) = \infty$, otherwise.

For the moment, $\mathrm{Dom}(F) = Y(\neq \emptyset)$ and $F_* = G^Y_* > -\infty$ (by (d02)); so, F is inf-proper. Moreover, as $[F \leq t] = [Y; G \leq t]$, $\forall t \in R$, it follows via (d03) that F is $(\geq)$-lsc; so, (c04) holds. Finally, we have (by definition) $m_Y(\Gamma, G)(\sigma) = m(\Gamma, F)(\sigma), \forall \sigma \geq 0$; hence $\alpha_Y(\Gamma, G) = \alpha(\Gamma, F)$. In other words: G is Γ-coercive on Y iff F is Γ-coercive. Consequently, the conclusion of our statement is retainable as soon as conditions in Theorem 2 are verified for F; precisely, when the Palais-Smale condition in Section 3 holds. The following auxiliary statement will verify this.

Lemma 4.1. *Let $u \in Y_0$ be arbitrary fixed. Then,*

$$\lambda_{(K,Y)}G(u) \leq \nabla_{(K)}F(u); \quad \textit{hence} \quad |\lambda_{(K,Y)}|G(u) \leq |\nabla_{(K)}|F(u). \tag{17}$$

Proof. Let $h \in Y[u; -K] \cap X_{(1)}$ be arbitrary fixed; and $\sigma = \sigma(h)$ stand for the number given by (d04). For each δ in $]0, \sigma[$, one has

$$\sup\{\frac{G(u) - G(u + th)}{t}; 0 < t \leq \delta\} \leq \sup\{\frac{F(u) - F(x)}{d(u, x)}; 0 < d(u, x) \leq \delta, u \geq x\}.$$

Passing to infimum upon δ yields

$$\Theta G(u)(h) \leq \nabla_{(K)}F(u), \text{ for each } h \in Y[u; -K] \cap X_{(1)};$$

wherefrom, taking the supremum over all such h, we are done. ∎

Proof. (of Theorem 4.1) By definition, there exists $Q \in \mathrm{fin}(X)$ such that (d06) holds. From the statement above, the Palais-Smale condition (modulo (Q, K, Y)) upon G implies the Palais-Smale condition (modulo (P, K)) upon F, where $P = Q \cup \{0\}$. In other words, conditions of Theorem 3.2 are fulfilled by such data; and this concludes the argument. ∎

Now, a basic choice for G so as to satisfy (d02)+(d03) is (a07) (relative to G). In such a case, Theorem 4.1 reduces to the result in Motreanu and Turinici [14] proved under direct methods (involving the monotone variational principle in Turinici [16]). An extension of these facts under the lines in Motreanu and Motreanu [11] is also possible; we do not give details.

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