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FUZZY INCLUSION AND DESIGN OF MEASURE OF FUZZY INCLUSION

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Abstract Fuzzy inclusion between fuzzy subsets of a crisp universe is defined as a binary operation on the set of all fuzzy subsets of a universe of discourse *X*. The fuzzy set defined as a fuzzy set of inclusion is then converted into a degree of inclusion with the help of a suitable measure. It is shown that the pointwise character of fuzzy inclusion allows many interesting properties to hold. Furthermore, the technique of applying a fuzzy measure to a fuzzy set of inclusion is used to construct mappings which provide degrees of inclusion.

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1. INTRODUCTION

Fuzzy subsethood or fuzzy inclusion is an important concept in the field of fuzzy set theory and it provides a basis for fuzzy similarity and measures of similarity (see [18] and [25]). First attempt to define fuzzy subsethood was made by Zadeh [26]; the graphical interpretation of his definition is: he defines the fuzzy set *A* as a subset of the fuzzy set *B*, if the graph of *A* never goes above the graph of *B*. Later on it was realized that defining fuzzy subsethood in this way is though highly appreciable and useful but is still against the spirit of fuzzy set theory, in the sense that it represents a crisp decision about being a subset or not (for details see [1] and [14]). Researchers working in the area of fuzzy inclusion remained interested in assigning a degree of inclusion of one fuzzy set into another (for details see [2], [4], [5], [6], [7], [8], [10], [13], [14], [19] and [24]). Many measures of inclusion have been proposed in literature; most of these take values in $[0, 1]$ or sometimes in the Boolean lattices *L*. Many researchers formulated axioms of inclusion i.e., they provide a list of properties which a reasonable inclusion measure must satisfy. A significant contribution in this direction is by Sinha and Dougherty [22]; they list properties that a reasonable inclusion measure should possess. Cornelis [10] later proved that the scalar defined

by Bandler and Kohout [1] as $\inf_{x\in X}\{I(A(x),B(x))\}$ is a fuzzy inclusion satisfying all the axioms of fuzzy inclusion constructed by Sinha and Dougherty.
In this paper, a different approach towards fuzzy inclusion is adopted. Basic intuition is that the inclusion defined should capture both the features of amount of overlap and orientation of one fuzzy set with respect to the other fuzzy set. This is in accordance with the spirit of fuzzy set theory. The properties possessed by such an inclusion are observed to be similar to the ones stated as axioms of inclusion by Sinha and Dougherty [10]. In our work, we use the Lukasiewicz's implicator I_W along with its corresponding t-norm and t-conorm. The only exception to this setting is Proposition 3.7 where max and min are used along with Lukasiewicz's implicator as per requirement of Sinha and Dougerty axioms of inclusion. The implicator I_W selects between two values depending upon the region of its domain. If a fuzzy set A is subset of another fuzzy set B in Zadeh's sense, the Lukasiewicz's implicator assigns to it a degree of inclusion equal to 1 at each point of its domain. When measure is applied to such a fuzzy set of inclusion it assigns a degree of inclusion of the fuzzy set A into B to be equal to 1. Intuitively speaking, it partially agrees with the idea of Zadeh. It differs from Zadeh's inclusion in the fact that Zadeh allocates a degree zero to $m_{Inc}(A,B)$ (see Definition 4.2) if A takes a value greater than B even at a single point of the domain, while $m_{Inc}(A,B)$ calculated through Lukasiewicz's implicator would take into account all the other $1s$ and would not assign a value equal to zero abruptly at the point where $A(x) \leq B(x)$ is violated. Measure m_{Inc} is much closer to reality than all the previously defined measures of inclusion. The important part of the theory is that the fuzzy set of inclusion provides with a permanent basis for inclusion theories and measures of inclusion. Every time one selects an appropriate fuzzy measure by it's application to the fuzzy set of inclusion, one gets a new example of measure of fuzzy inclusion.

2. PRELIMINARIES

Definition 2.1. [26, Sec II, p.339] *Let $F(X)$ be the set of all fuzzy subsets of a universe X. For all $A,B \in F(X)$, A is said to be a subset of B if for all $x \in X$, $A(x) \leq B(x)$, where $A(x)$ and $B(x)$ represent the membership grades of x in A and B respectively. In this case, we write $A \subseteq B$ and call it the Zadeh's inclusion. Two fuzzy sets A and B are said to be equal if and only if $A(x) = B(x)$ for all $x \in X$.*

Fuzzy sets X and $\varnothing$ will be denoted by $\underline{1}$ and $\underline{0}$ respectively. This notation has been borrowed from Boolean lattices in order to remain specific about the maximal and minimal elements of the range of the newly introduced mappings in this paper. The t-norms and t-conorms will be used for the pointwise conjunction and disjunction of fuzzy sets respectively.

Definition 2.2. [20, Definition 1.13 p. 11] *The triangular norm (t-norm) T and triangular conorm (t-conorm) δ are increasing, associative, commutative and* $[0,1]^2 \to [0,1]$ *mappings satisfying: $T(1,a) = a$ and $\delta(a,0) = a$ for all $a \in [0,1]$.*

In the following there is a list [17, Sec 3 p 219] of some choices for t-norms and their dual t-conorms:
1. The *min* and *max* operators $M(x,y) = min(x,y)$, and $M^*(x,y) = max(x,y)$.
2. The product and probabilistic sum $P(x,y) = xy$, and $P^*(x,y) = x + y - xy$.
3. The Lukasiewicz's pair
$W(x,y) = max(x + y - 1, 0)$, and $W^*(x,y) = min(x + y, 1)$.
In this paper the notation $T(A,B)$ shall represent a fuzzy set defined as: $T(A,B)(x) = T(A(x), B(x))$ for any t-norm T and for all $x \in X$.

Definition 2.3. [14, Sec2.2, Definition 1 p. 3] *A negator N is an order-reversing* $[0,1] \to [0,1]$ *mapping such that $N(0) = 1$ and $N(1) = 0$. A negator is called strict if it is continuous and strictly decreasing. A strict negator is said to be strong if it is involutive too i.e., $N(N(x)) = x$ for all $x \in [0,1]$.*

The negators are used to model pointwise complements in the literature of fuzzy sets. Throughout this paper A^c, the complement of a fuzzy set A will be calculated by the standard negator i.e., $A^c(x) = N(x) = 1 - x$.

Next we give an introduction to the fuzzy implicator which is the back bone of the theory on fuzzy set of similarity.

Definition 2.4. [11, Sec III (C) (2), p.277] *A fuzzy implicator is a binary operation on* $[0,1]$ *with order reversing first partial mappings and order preserving second partial mappings such that:*

$$I(0,1) = I(0,0) = I(1,1) = 1, I(1,0) = 0.$$

Remark 2.1. [9, Section2, p.213, Definition 1] From an axiomatic point of view the following properties are very important for fuzzy implicators I:
A1. Contraposition: $(\forall x, y \in [0,1]), I(x,y) = I(N(y), N(x))$;
A2. Exchange Principle: $(\forall x, y, z \in [0,1]), I(x, I(y,z)) = I(y, I(x,z))$;
A3. Hybrid Monotonicity: stated in definition 2.4;
A4. Ordering Principle: $x \leq y \Longleftrightarrow I(x,y) = 1$ for all $x, y \in [0,1]$;
A5. Neutrality Principle: $I(1,x) = x$ for all $x \in [0,1]$;
A6. Continuity: I is continuous.
Where N is a strong negator.

Definition 2.5. [14, Sec 2.4.1, Definition 7] *For a t-norm T, the R-implicator (or residuated implicator) I_T is defined as:*

$$I_T(a,b) = \sup\{x \in [0,1] \mid T(a,x) \leq b\} \ \forall a, b \in [0,1].$$

Definition 2.6. [1, *Definition* 2.1 (5)] *The Lukasiewicz implicator* (I_W) *is defined as:*

$$I_W(x, y) = min(1 - x + y, 1) \textit{ for all } x, y \in [0, 1].$$

The Lukasiewicz implicator possesses all the properties from A1 to A6, Moreover it is an R-implicator with respect to the t-norm W and satisfies following property of R-implicators [21, Sec 2, p 3]:

$$W(I_W(x, y), I_W(y, z)) \leq I_W(x, z). \tag{1}$$

Definition 2.7. *[12, Sec 1.3.2] The scalar cardinality of a fuzzy subset A of a finite universe X, is defined as:*

$$|A| = \sum_{x \in X} A(x).$$

Definition 2.8. *[16, Definition 2.2] Given a t-norm T, a T-equivalence relation on a set X is a fuzzy relation E on X that satisfies:*
(i) $E(x, x) = 1$ *for all* $x \in X$. *(Reflexivity)*
(ii) $E(x, y) = E(y, x)$ *for all* $x, y \in X$. *(Symmetry)*
(iii) $T(E(x, y), E(y, z)) \leq E(x, z)$ *for all* $x, y, z \in X$. *($T-$transitivity)*

Definition 2.9. *[10, Definition 2.1]Let Inc be a* $F(X) \times F(X) \to [0, 1]$ *mapping, and* $A, B, C \in F(X)$ *be fuzzy sets in a given universe X. Sinha-Dougherty axioms imposed on Inc for it to be an inclusion operator are as follows:*
Axiom 1. $Inc(A, B) = 1$ *if and only if* $A \subseteq B$ *(in Zadeh's sense).*
Axiom 2. $Inc(A, B) = 0$ *if and only if* $Ker(A) \cap (supp(B)^c) \neq \emptyset$, *where* $ker(A = \{x \in X \mid A(x) = 1\}$, $supp(B) = \{x \in X \mid B(x) > 0\}$.
Axiom 3. $B \subseteq C$ *implies* $Inc(A, B) \leq Inc(A, C)$, *i.e., Inc has increasing second partial mappings.*
Axiom 4. $B \subseteq C$ *implies* $Inc(B, A) \geq Inc(C, A)$, *i.e., Inc has decreasing first partial mappings.*
Axiom 5. $Inc(A, B) = Inc(S(A), S(B))$ *where* S *is a* $F(X) \to F(X)$ *mapping defined by, for every* $x \in X$, $S(A)(x) = A(s(x))$, *s denoting an* $X \to X$ *mapping.*
Axiom 6. $Inc(A, B) = Inc(B^c, A^c)$.
Axiom 7. $Inc(B \cup C, A) = \min(Inc(B, A), Inc(C, A))$.
Axiom 8. $Inc(A, B \cap C) = \min(Inc(A, B), (A, C))$.

Definition 2.10. *[3] Let R be a fuzzy relation on X, I any fuzzy implicator and T a t-norm, then the fuzzy transitivity relation* $tr_R^{I,T}$ *is a fuzzy relation on X defined as:*

$$tr_R^{I,T}(x, z) = \inf_{y \in X} I(T(R(x, y), R(y, z)), R(x, z)).$$

The transitivity function so defined, assigns a degree of transitivity to the given fuzzy relation at each point of $X \times X$.

Definition 2.11. *[3] For a given fuzzy relation R, the measure of transitivity of R is given by:*

$$mTr(R) = m(tr_R^{I,T});$$

where, m is Sugeno's measure (see [23]). In this paper plinth of a fuzzy set will be taken as the measure which is defined as: $Plinth(A) = \inf_{x\in X} A(x)$, while only I_W and W will be used in place of I and T respectively, consequently

$$mTr(R) = \inf_{x,y,z\in X}(I_W(W(R(x,y), R(y,z)), R(x,z))).$$

For this specific choice we shall write tr instead of $tr_R^{I,T}$ and Tr instead of mTr. If $Tr(R) = \epsilon$, then R is called an ϵ-fuzzy transitive relation. A fuzzy relation R is called fuzzy transitive if $Tr(R) > 0$, strong fuzzy transitive if $\epsilon \geq 0.5$ and R is nontransitive if $Tr(R) = 0$.

Definition 2.12. *[3] A binary operation O on $F(X)$ is an $\epsilon-$local fuzzy order on $F(X)$, if for all $x \in X$ and $A, B, C \in F(X)$, following conditions hold :*
E1. fuzzy Reflexivity at x: if and only if $O(A, A)(x) = 1$.
E2. antisymmetry at x: $O(A, B)(x) = O(B, A)(x) > 0 \Rightarrow A(x) = B(x)$.
E3. $\epsilon-$fuzzy transitivity at x:
$I_M(min(O(A, B)(x), O(B, C)(x)), O(A, C)(x)) = \epsilon$.
If $\epsilon > 0$, then O is called fuzzy transitive at x. If $\epsilon \in [0.5, 1]$, then O will be called strong fuzzy transitive at x and it is called weak fuzzy transitive at x, otherwise.

Definition 2.13. *[3] A fuzzy relation R on $F(X)$ is called a fuzzy order relation on $F(X)$ if for all $A, B, C \in F(X)$:*
(i) $R(A, A) = 1$; (fuzzy reflexivity)
(ii) $R(A, B) = R(B, A) \Rightarrow A = B$; (fuzzy antisymmetry)
(iii) $I_W(W(R(A, B), R(B, C)), R(A, C)) = \epsilon > 0$. (fuzzy transitivity)
If $\epsilon \in [0.5, 1]$, then R will be called Strong fuzzy transitive order relation. A fuzzy relation satisfying (i) and (iii) only is called a fuzzy quasi order.

3. FUZZY INCLUSION UNDER I_W

Definition 3.1. *The fuzzy inclusion is a mapping $Inc : F(X) \times F(X) \to F(X)$, which assigns to every $A, B \in F(X)$ a fuzzy set $Inc(A, B) \in F(X)$ defined as:*

$$Inc(A, B)(x) = I(A(x), B(x)) \text{ for all } x \in X. \tag{2}$$

We will only use I_W in this paper for the study of properties of this inclusion.

Proposition 3.1. *For any $A, B \in F(X)$, for any $x \in X$,*

$$Inc(A, B)(x) = Inc(B, A)(x) \text{ if and only if } A(x) = B(x).$$

Proof. For any $A, B \in F(X)$ and $x \in X$,

$Inc(A,B)(x) = Inc(B,A)(x)$.
$\Longleftrightarrow I_W(A(x),B(x)) = I_W(B(x),A(x))$.
$\Longleftrightarrow 1 = 1 - B(x) + A(x)$ *(without loss of generality, let* $A(x) \leq B(x)$*).*
$\Longleftrightarrow A(x) = B(x)$.

Theorem 3.1. *The fuzzy inclusion defined in Definition 3.1 is a local fuzzy order on* $F(X)$ *which is* $1-$*fuzzy transitive.*

Proof. . For all $A, B, C \in F(X)$.
E1. *local fuzzy reflexivity*: For any $x \in X$, $Inc(A,A)(x) = I_W(A(x),A(x)) = 1$ by definition of I_W. So, $Inc(A,A) = X$.
E2. *local fuzzy antisymmetry*: Follows from Proposition 3.1.
E3. *Strong local fuzzy transitivity*: Let $x \in X$,

$$\begin{aligned} & I_W(W(Inc(A,B)(x), Inc(B,C)(x)), Inc(A,C)(x))) \\ = \quad & I_W(W(I_W(A(x),B(x)), I_W(B(x),C(x)), I_W(A(x),C(x))). \end{aligned} \tag{3}$$

By the property stated in (1)

$$W(I_W(A(x),B(x)), I_W(B(x),C(x))) \leq I_W(A(x),C(x)). \tag{4}$$

using (4) in (3), we get for all $x \in X$,

$$I_W(W(Inc(A,B)(x), Inc(B,C)(x)), Inc(A,C)(x)) = 1.$$

Hence Inc is $1-$fuzzy transitive fuzzy local order. ∎

Proposition 3.2. *For all* $A, B \in F(X)$ *following hold:*
1. $Inc(A,B) = X \Longleftrightarrow A(x) \leq B(x)$ *for all* $x \in X$.
2. $Inc(A,B) = \emptyset \Longleftrightarrow A = X$ *and* $B = \emptyset$.
3. $Inc(A,A^c) = \emptyset \Longleftrightarrow A = X$.

Proof. For all $A, B \in F(X)$,
1. Let $Inc(A,B) = X \Longleftrightarrow$ for all $x \in X$, $I_W(A(x),B(x)) = 1 \Longleftrightarrow A(x) \leq B(x)$ for all $x \in X$ by definition of I_W.
2. Let $Inc(A,B) = \emptyset$
$\Longleftrightarrow$for all $x \in X$, $I_W(A(x),B(x)) = 0$
$\Longleftrightarrow$for all $x \in X$, $1 - A(x) + B(x) = 0$
$\Longleftrightarrow$for all $x \in X$, $A(x) = 1$ and $B(x) = 0$.
3. 2 implies 3. ∎

Proposition 3.3. *For all* $A, B \in F(X)$, *we have:*

$$Inc(A,B) = Inc(B^c, A^c).$$

Proof. For any $x \in X$, there arise two cases:

1. $A(x) \le B(x)$: It implies that $B^c(x) \le A^c(x)$. Therefore

$$\begin{aligned} Inc(B^c, A^c)(x) &= I_W(B^c(x), A^c(x)) = 1 \\ &= I_W(A(x), B(x)) = Inc(A, B)(x). \end{aligned}$$

2. $A(x) > B(x)$: It implies that $B^c(x) > A^c(x)$. Thus

$$\begin{aligned} Inc(B^c, A^c)(x) &= I_W(B^c(x), A^c(x)) = \min(1, 1 - B^c(x) + A^c(x)) \\ &= B(x) + 1 - A(x) = 1 - A(x) + B(x) \\ &= I_W(A(x), B(x)) = Inc(A, B)(x). \end{aligned}$$

∎

Proposition 3.4. *For all $A, B \in F(X)$,*

$$W^*(Inc(A, B), Inc(B, A)) = X.$$

Proof. If $A, B \in F(X)$, then we have for any $x \in X$, without loss of generality assume that $A(x) \le B(x)$,

$$\begin{aligned} W^*(Inc(A, B), Inc(B, A))(x) &= W^*(Inc(A, B)(x), Inc(B, A)(x)) \\ &= W^*(I_W(A(x), B(x)), I_W(B(x), A(x))) \\ &= W^*((1), (1 - B(x) + A(x)) \\ &= 1 \text{ by definition of } W^*. \end{aligned}$$

Hence

$$W^*(Inc(A, B), Inc(B, A)) = X.$$

∎

Proposition 3.5. *For all $A, B, C \in F(X)$,*
1. $B \subseteq C \implies Inc(A, B) \subseteq Inc(A, C)$ i.e., the fuzzy inclusion is increasing in second variable.
2. $B \subseteq C \implies Inc(C, A) \subseteq Inc(B, A)$ i.e., the fuzzy inclusion is decreasing in first variable.

Proof. For all $A, B, C \in F(X)$, there arise following cases:
1. If $B \subseteq C$, then $B(x) \le C(x)$ for all $x \in X$. Using definition of I_W for any $x \in X$, so, $Inc(A, B)(x) = I_W(A(x), B(x)) \le I_W(A(x), C(x))$ due to the fact that I_W order preserving second partial mappings.
2. Again $B \subseteq C$ implies that $B(x) \le C(x)$ for all $x \in X$. Using definition of I_W for any $x \in X$, $Inc(C, A)(x) = I_W(C(x), A(x)) \le I_W(B(x), A(x))$ due to the fact that I_W has order reversing first partial mappings. ∎

Proposition 3.6. *For all* $A, B, C, D \in F(X)$, *we have*

$$\begin{array}{rl} & W(Inc(A,B), Inc(C,D)) \\ \subseteq & W[Inc(W(A,C), W(B,D)), Inc(W^*(A,C), W^*(B,D))] \end{array} \tag{5}$$

$$\subseteq W^*[Inc(W(A,C), W(B,D)), Inc(W^*(A,C), W^*(B,D))] \tag{6}$$

$$\subseteq W^*[Inc(A,B), Inc(C,D)]. \tag{7}$$

Proof. . For any $x \in X$, there arise following possibilities:
Case I. $A(x) \leq B(x)$ and $C(x) \leq D(x)$,

$$\begin{array}{rcl} W(Inc(A,B)(x), Inc(C,D)(x)) & = & W(I_W(A(x), B(x)), I_W(C(x), D(x))) \\ & = & 1 \wedge 1 = 1 \text{ by assumption.} \end{array} \tag{8}$$

Similarly, the assumption also implies that for this particular x,
$W(A(x), C(x)) \leq W(B(x), D(x))$ and $W^*(A(x), C(x)) \leq W^*(B(x), D(x))$.

$$\begin{array}{rc} & W(Inc(W(A,C), W(B,D)), Inc(W^*(A,C), W^*(B,D)))(x) \\ = & W(I_W(W(A(x), C(x)), W(B(x), D(x))), \\ & I_W(W^*(A(x), C(x)), W^*(B(x), D(x)))) \\ = & W(1,1) = 1 \text{ by assumption and definition of } I_W. \end{array} \tag{9}$$

Similar argument implies that for the x selected in case 1,

$$W^*(Inc(W(A,C)(x), W(B,D)(x)), Inc(W^*(A,C), W^*(B,D)(x)) = 1, \tag{10}$$

and

$$W^*(Inc(A,B), Inc(C,D))(x) = 1. \tag{11}$$

From equations (8), (9), (10) and (11), we obtain the result.
Case II. $x \in X$, is such that $A(x) > B(x)$ and $C(x) > D(x)$

$$\begin{array}{rcl} W(Inc(A,B)(x), Inc(C,D)(x)) & = & W(I_W(A(x), B(x)), I_W(C(x), D(x))) \\ & = & W(1 - A(x) + B(x), 1 - C(x) + D(x)) \\ & = & \max(0,\ 1 - (A(x) + C(x)) + B(x) + D(x)), \end{array} \tag{12}$$

and

$$\begin{array}{rl} & W(Inc(W(A,C), W(B,D)), Inc(W^*(A,C), W^*(B,D)))(x) \\ = & W(I_W(W(A(x), C(x)), W(B(x), D(x))), \\ & I_W(W^*(A(x), C(x)), W^*(B(x), D(x)))) \\ = & W(1 - W(A(x), C(x)) + W(B(x), D(x)), 1 - W^*(A(x), C(x)) \\ & +W^*(B(x), D(x))) \\ = & \max(0,\ 1 - (W(A(x), C(x)) + W^*(A(x), C(x))) + \\ & W(B(x), D(x)) + W^*(B(x), D(x))) \\ = & \max(0,\ 1 - (A(x) + C(x)) + B(x) + D(x)). \end{array} \tag{13}$$

We used the fact $W(x,y) + W^*(x,y) = x + y$ in reaching at the last expression in (13). Lastly, Comparing (12), (13) the equality is again obtained.
Case III. For any $x \in X$, $A(x) \le B(x)$ and $C(x) > D(x)$. So,

$$W(Inc(A,B), Inc(C,D))(x) = W(I_W(A(x), B(x)), I_W(C(x), D(x)))$$
$$= W(1, 1 - C(x) + D(x))$$
$$= 1 - C(x) + D(x) \tag{14}$$

There arise many different cases for the right side of (5), without loss of generality we state the proof only for the situation when $W(B(x), D(x)) < W(A(x), C(x))$ and $W^*(B(x), D(x)) < W^*(A(x), C(x))$, all the other situations can be handled similarly.

$$\begin{array}{cc} & W(Inc(W(A,C), W(B,D)), Inc(W^*(A,C), W^*(B,D)))(x) \\ = & W(I_W(W(A(x),C(x)), W(B(x),D(x))), I_W(W^*(A(x),C(x)), W^*(B(x),D(x)))) \\ = & 1 - (A(x) + C(x)) + B(x) + D(x) \\ = & 1 - C(x) + D(x) + (B(x) - A(x)) \\ \ge & 1 - C(x) + D(x). \end{array} \tag{15}$$

The proof of (5) is complete.
Case IV can be proved on more or less the same lines as Case III so we omit the proof.
The inequality (6) is straightforward outcome of results about conjunction and disjunction respectively, the last inequality (7) can be proved on very same lines as (5). ∎

Proposition 3.7. *For all $A, B, C \in F(X)$ we have:*

1. $Inc(\max(A,B), C) = \min(Inc(A,C), Inc(B,C))$.
2. $Inc(\min(A,B), C) = \max(Inc(A,C), Inc(B,C))$.
3. $Inc(A, \min(B,C)) = \min(Inc(A,B), Inc(A,C))$.
4. $Inc(A, \max(B,C)) = \max(Inc(A,B), Inc(A,C))$.

Proof. 1. For all $A, B, C \in F(X)$ and $x \in X$,

$$\min(Inc(A,C)(x), Inc(B,C)(x))$$
$$= \min(I_W(A(x), C(x)), I_W(B(x), C(x)))$$
$$= \min(\min(1 - A(x) + C(x), 1),\ \min(1 - B(x) + C(x), 1)). \tag{16}$$

Similarly,

$$Inc(\max(A,B), C)(x) = I_W(\max(A(x), B(x)), C(x))$$
$$= \min(1 - \max(A(x), B(x)) + C(x), 1)$$
$$= \min(\min(1 - A(x), 1 - B(x)) + C(x), 1)$$

$$= \min(\min(1 - A(x) + C(x), 1 - B(x) + C(x), 1) \tag{17}$$

Comparing equations (16) and (17) we get the result.
2. It can proved on the similar lines as 1.
3. For all $A, B, C \in F(X)$ and for any $x \in X$, without loss of generality, suppose $B(x) \leq C(x)$,

$$\begin{aligned} \min(Inc(A, B), Inc(A, C)(x)) &= \min(I_W(A(x), B(x)), I_W(A(x), C(x))) \\ &= \min(1 - A(x) + B(x), 1 - A(x) + C(x)) \\ &= 1 - A(x) + B(x). \end{aligned} \tag{18}$$

Next we consider the other side

$$\begin{aligned} Inc(A, \min(B, C))(x) &= I_W(A(x) + \min(B(x), C(x))) \\ &= \min(1 - A(x) + \min(B(x), C(x), 1) \\ &= \min(1 - A(x) + B(x), 1) \\ &\quad \text{according to the selection made in eq (18).} \end{aligned}$$

4. Proof is similar to part 3. ∎

Proposition 3.8. *For all $A, B, C \in F(X)$ we have*

$$Inc(A, B) \leq W(Inc(W(A, C), W(B, C)), Inc(W^*(A, C), W^*(B, C))).$$

Proof. This inequality can be proved only by putting $D = C$ in the first part of inequalities of Proposition 3.6. ∎

4. MEASURE OF FUZZY INCLUSION

Fuzzy inclusion as defined in Section 3 is a fuzzy set and defines inclusion in a very general setting. The advantage of this approach is that it provides a basis for all scalar valued inclusions. Several researchers have tried to give a degree of inclusion of one fuzzy set into another. In this section we aim at refining this approach with the help of fuzzy measures to the fuzzy set of inclusions. We observe that all the previously defined measures of inclusion are examples of this newly defined concept. Two types of observations are highly important in dealing with measures of inclusion instead of fuzzy inclusion. The measure of fuzzy inclusion of a set loses some properties satisfied by the fuzzy set of inclusion itself particularly antisymmetry. If we define the fuzzy set of inclusion and local fuzzy order in the above fashion then we successfully order the given fuzzy sets by fuzzy inclusion. But if we attempt to order the given fuzzy sets with the help of measure of fuzzy inclusion then we can at the most obtain a quasi order. Moreover the scalar associated with fuzzy inclusion has advantages. It has been proved about certain measures of inclusion that they satisfy all the axioms of Sinha and Dougherty [22].

Definition 4.1. [15, Definition 2.7] *Let* (X, ρ) *be a measurable space. A function* $m : \rho \to [0, \infty[$ *is a fuzzy measure if it satisfies the following properties:*
m1*:* $m(\emptyset) = 0$*, and* $m(X) = 1$*;* **m2***:* $A \subseteq B$ *implies that* $m(A) \leq m(B)$*.*

The concept of measure considers that $\rho \subseteq \{0, 1\}^X$, but this consideration can be extended to a set of fuzzy subsets $\mathfrak{I}$ of X, $\mathfrak{I} \subseteq F(X)$, satisfying the properties of measurable space $(F(X), \mathfrak{I})$.
Remark 4.1. Some times it is beneficial to require following property from the measure m: For any $A \in F(X)$, $m(A) = 1$ implies that $A = X$ and $m(A) = 0$ implies that $A = \emptyset$.

Definition 4.2. *A fuzzy measure of inclusion is a mapping* m_{Inc}*:*$F(X) \times F(X) \longrightarrow [0, 1]$, *which allocates to all A,B* $\in F(X)$ *a value in the interval* $[0, 1]$ *defined as:*

$$m_{Inc}(A, B) = m(Inc(A, B)),$$

where $Inc(A, B)$ *is the fuzzy set of inclusion of A into B defined in (2), and m is a fuzzy measure defined in Definition 4.1.*

Examples 4.1. Here are some examples of the measure m: for all $A \in F(X)$,
1a. $m_1(A) = Plinth(A) = \inf_{x \in X} A(x)$;
2a. $m_2(A) = \frac{1}{2}[Plinth(A) + Height(A)] = \frac{1}{2}[\inf_{x \in X} A(x) + \sup_{x \in X} A(x)]$;
3a. In case of finite universes,

$$m_3(A) = \frac{|A|}{|X|};$$

4a. In case of bounded universes equipped with a measure m,

$$m_4(A) = \frac{\int A(x) dm}{m(X)}.$$

Applying these measures on the fuzzy set of inclusion defined in (2), we get:
1b. $m_{1Inc}(A, B) = \inf_{x \in X} I_W(A(x), B(x))$.
2b.

$$m_{2Inc}(A, B) = \frac{1}{2}[\inf_{x \in X} I_W(A(x), B(x)) + \sup_{x \in X} I_W(A(x), B(x))].$$

3b. In case of finite universes:

$$m_{3Inc}(A, B) = \frac{|Inc(A, B)|}{|X \times X|}.$$

4b. In case of bounded universes equipped with a measure m the measure $m_{4Inc}(A, B)$ is defined as:

$$m_{4Inc}(A, B) = \frac{\int Inc(A, B) dm}{m(X \times X)}.$$

Fuzzy inclusion measure defined in 1b is the one defined in [1]. Next we give an example to demonstrate the role of $Inc(A, B)$ and comparison of different measures. For simplicity we take crisp sets A and B.
Example 4.2. Let $X = \{1, 2, 3, ...10\}$, $A = \{1, 2, 3, 8, 9\}$ and $B = \{2, 3, 4, 5, 6, 7\}$ then:

$$\begin{aligned} Inc(A, B) &= \{(1, 0), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (7, 1), (8, 0), (9, 0), (10, 1)\}, \\ Inc(B, A) &= \{(1, 1), (2, 1), (3, 1), (4, 0), (5, 0), (6, 0), (7, 0), (8, 1), (9, 1), (10, 1)\} \end{aligned}$$

where, The first coordinate of every ordered pair represents the element and the second coordinate represents its membership in the fuzzy set of inclusion.
1. $m_1 Inc(A, B) = 0$ and $m_1 Inc(B, A) = 0$;
2. $m_2 Inc(A, B) = 0.5$ and $m_2 Inc(B, A) = 0.5$;
3. $m_3 Inc(A, B) = 0.7$ and $m_3 Inc(B, A) = 0.6$.

Theorem 4.1. *If the fuzzy measure m defined in Definition 4.1 also possesses the additional property stated in Remark 4.1, then it is a fuzzy quasi order on $F(X)$.*

Proof. We will prove the two conditions one by one
O1. *Reflexivity*: For all $A \in F(X)$,
Using property m_1 and Proposition 3.2 we get $m_{Inc}(A, A) = m(X) = 1$.
O3. *Fuzzy transitivity*: Let on contrary suppose that m_{Inc} is not fuzzy transitive so,

$$I_W(W(\ m_{Inc}(A, B), m_{Inc}(B, C)), m_{Inc}(A, C)) = 0.$$

It implies that

$$m_{Inc}(A, B) = 1,\ m_{Inc}(B, C) = 1 \text{ and } m_{Inc}(A, C) = 0.$$

It further implies that

$$Inc(A, B) = X \text{ and } Inc(B, C) = X \text{ and } m_{Inc}(A, C) = 0.$$

Therefore

$$A \subseteq B \text{ and } B \subseteq C \text{ and } m_{Inc}(A, C) = 0.$$

a contradiction. So m_{Inc} is fuzzy transitive. ∎

Proposition 4.1. *For all $A, B \in F(X)$ and the measure m satisfying the additional property stated in Remark 4.1, then we have:*
1. $m_{Inc}(A, B) = 1$ *if and only if* $A(x) \leq B(x)$ *for all* $x \in X$;
2. $m_{Inc}(A, B) = 0$ *if and only if* $A = X$ *and* $B = \emptyset$;
3. $Inc(A, A^c) = 0$ *if and only if* $A = X$.

Proof. For all $A, B \in F(X)$,
1. Let $A(x) \leq B(x)$. It implies that $Inc(A, B) = X$, applying measure to both sides it further implies that $m_{Inc}(A, B) = 1$.

Conversely let $m_{Inc}(A, B) = 1$ which implies that $Inc(A, B) = X$, which further implies that $A(x) \leq B(x)$.

2. $A = X$ and $B = \emptyset$ implies that $Inc(A, B) = \emptyset$ by Proposition 3.2, applying measure to both sides we get $m_{Inc}(A, B) = 0$.

Conversely let $m_{Inc}(A, B) = 0$ which implies that $Inc(A, B) = \emptyset$ due to property of the measure stated in Remark 4.1, and which in turn implies that $A = X$ and $B = \emptyset$ due to Proposition 3.2.

3. Using the Proposition 4.1 (2) with $A^c = B$. ∎

Proposition 4.2. *For all $A, B \in F(X)$ we have:*

1. $m_{Inc}(A, B) = m_{Inc}(B^c, A^c)$.

2. $m(W^*(Inc(A, B), Inc(B, A))) = 1$.

Proof. These identities can be obtained by applying fuzzy measure to both sides of the statements proved in Propositions 3.5 and 3.6. ∎

Proposition 4.3. *For all $A, B, C \in F(X)$,*

1. $B \subseteq C \Longrightarrow m_{Inc}(A, B) \leq m_{Inc}(A, C)$.

2. $B \subseteq C \Longrightarrow m_{Inc}(C, A) \leq m_{Inc}(B, A)$.

Proof. 1. Since $B \subseteq C \Longrightarrow Inc(A, B) \subseteq Inc(A, C)$ by Proposition 3.3. Using properties of fuzzy measure we get

$$m_{Inc}(A, B) \leq m_{Inc}(A, C).$$

2. Proof is on the same lines as of part 1. ∎

Proposition 4.4. *For all $A, B, C \in F(X)$,*

1. $m_{Inc}(\max(A, B), C) = m(\min(Inc(A, C), Inc(B, C)))$;

2. $m_{Inc}(\min(A, B), C) = m(\max(Inc(A, C), Inc(B, C)))$;

3. $m_{Inc}(A, \min(B, C)) = m(\min(Inc(A, C), Inc(B, C)))$;

4. $m_{Inc}(A, \max(B, C)) = m(\max(Inc(A, C), Inc(B, C)))$.

Proof. All these equalities can be obtained by applying measures to both sides of equalities in Proposition 3.7. ∎

Theorem 4.2. *Fuzzy inclusion measure defined in :*

$$m_{Inc}(A, B) = \inf_{x \in X} I_W(A(x), B(x))$$

is a 1-transitive quasi order.

Proof. For all $A, B, C \in X$, we have

1. m_{Inc} is reflexive since $m_{Inc}(A, A) = \inf_{x \in X} I_W(A(x), A(x)) = 1$.

2. *Strong Fuzzy transitivity*: The use of Theorem 3.1 yields

$$\begin{aligned} W(m_{Inc}(A,B), m_{Inc}(B,C)) &= W(\inf_{x\in X}(Inc(A,B)(x)), \inf_{x\in X}(Inc(B,C)(x))) \\ &\le \inf_{x\in X}(Inc(A,C)(x)) = m_{Inc}(A,C). \end{aligned}$$

It implies that m_{Inc} is a W-transitive fuzzy equivalence relation on $F(X)$ and hence 1-fuzzy transitive. ∎

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