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ABOUT FINITENESS CONDITIONS FOR COMMUTATIVE MOUFANG LOOPS

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Abstract It is proved that a commutative Moufang loop satisfies one of the following properties: is finite; is finitely generated; has a finite (special) rank; maximum condition for its subloops; minimum condition for its subloops if and only if this property is satisfied by the centralizer of one of its finitely generated subloops.

Keywords: commutative Moufang loop, centralizer, finitely generated, special rank, minimum condition for subloops, maximum condition for subloops.

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1. INTRODUCTION

It is known that, in some classes of groups or loops, the different finiteness conditions on their centers are transferred to these groups, loops, respectively. For example, in [1] it was proved that if the centre of finitely generated nilpotent group is finite then the group itself is finite. Further, let Ω denote one of the following classes of loops: the class of finite loops; the class of finitely generated loops; the class of loops of finite rank; the class of loops with maximum conditions for its subloops; the class of loops with minimum conditions for its subloops. In [2 – 4] it is proved that a commutative Moufang ZA-loop belongs to a class Ω if and only if its centre belongs to the same class Ω.

There exists a commutative Moufang loop (CML) with trivial centre [5]. Then for the described CML with finiteness conditions it is reasonable to use the notion of centralizer, more general that the notion of centre. This paper generalizes the aforementioned result for ZA-loops. It is proved that a CML belongs to a class Ω if and only if the centralizer of one of its finitely generated subloops belongs to Ω.

2. THEORETICAL BACKGROUND

Let us recall some notions and results of the theory of the *commutative Moufang loops* (abbreviated *CMLs*) from [5], which are the commutative loops characterized by the identity $x^2 \cdot yz = xy \cdot xz$.

The *associator* (a, b, c) of the elements a, b, c of the CML L is defined by the equality $ab \cdot c = (a \cdot bc)(a, b, c)$. The identities:

$$(xy, u, v) = (x, u, v)((x, u, v), x, y)(y, u, v)((y, u, v), y, x), \tag{1}$$

$$(x, y, z) = (y^{-1}, x, z) = (y, x, z)^{-1} = (y, z, x) \tag{2}$$

hold in the CML L.

The *centre* $Z(L)$ of the CML L is a normal subloop

$$Z(L) = \{x \in L|(x, y, z) = 1, \ \forall y, z \in L\}$$

. The *upper central series* of the CML L is the series

$$1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots \subseteq Z_\alpha \subseteq \ldots$$

of the normal subloops of the CML L, satisfying the conditions:

1) $Z_\alpha = \sum_{\beta<\alpha} Z_\beta$ for the limit ordinal;

2) $Z_{\alpha+1}/Z\alpha = Z(L/Z_\alpha)$ for any α.

If the CML possesses a central series, then this loop is called *ZA-loop*.

If the upper central series of the *ZA*-loop have a finite length, then the loop is called *centrally nilpotent*.

The least of such a length is called the *class* of the central nilpotentcy.

Lemma 2.1. *(Bruck-Slaby Theorem). Let n be a positive integer, $n \geq 3$. Then every commutative Moufang loop L, which can be generated by n elements, is centrally nilpotent of class at most $n - 1$.*

Lemma 2.2. *For any CML, L, with centre $Z(L)$, the quotient loop $L/Z(L)$ is locally finite 3-loop of exponent 3 and it is finite if L is finitely generated [5].*

3. MAIN RESULTS

The following concept is the natural generalization of the concept of centre.

Let M be a subset and H be a subloop of the CML L. The set $Z_H(M) = \{x \in H|x \cdot yz = xy \cdot z, \ \ \forall y, z \in M\}$ is called *centralizer* of the subset M into subloop H. $Z_H(M)$ is a subloop of L [6].

The (*special*) *rank* of loop L is called the least positive number rL with the following feature: any finitely generated subloop of loop L can be generated by rL elements; if there are not such numbers, then we suppose that $rL = \infty$.

Lemma 3.1. *A centrally nilpotent CML L belongs to class Ω if and only if the centralizer of some of its finitely generated subloop H also belongs to class Ω.*

Proof. The necessity of lemma is obvious. To prove the sufficiency, it is enough to proceed by induction on the class of central nilpotence of CML L. The sufficiency may be proved by using the well known facts from [2-4]: if the center of commutative Moufang ZA-loop belongs to class Ω, then the CML itself belongs to Ω. In [2-4] this result is proved in different way for various classes. We prove the sufficiency only for the case of rank finiteness as the proof of other cases of class Ω are identical to this case.

Let the subloop H be generated by the set $A = \{a_1, a_2, \dots, a_n\}$ and let the centralizer $Z_L(H)$ have a finite rank. We will suppose that the CML L is non-associative, as for abelian groups (centrally nilpotent CML of class $k = 1$) the statement holds.

Let Z be the centre of the CML L and let k be the class of the central nilpotence. Obviously, $Z \subseteq Z_L(H)$, hence the rank of Z is finite. As the quotient loop L/Z is centrally nilpotent of class $k-1$, then by inductive supposition, the rank of L/Z will be finite if the centralizer D/Z of image HZ/Z of subloop H into the quotient loop L/Z has a finite rank. We will prove this below.

Indeed, let $x, y \in D$, and let $A_i = \{a_{i_1}, a_{i_2}\},\ i = 1, 2, \dots, t$, be an arbitrary fixed pair of elements $a_{i_1}, a_{i_2} \in A$. We have $(x, a_{i_1}, a_{i_2}) \in Z$, then by (1), it follows $(xy, a_{i_1}, a_{i_2}) = (x, a_{i_1}, a_{i_2})(y, a_{i_1}, a_{i_2})$. This equality shows that the mapping $x \to (x, a_{i_1}, a_{i_2})$ is a homomorphism of D into Z.

For each A_i we consider the homomorphisms $\varphi_i(x) = (x, a_{i_1}, a_{i_2}),\ x \in D$. Obviously, $\ker\varphi_i = Z_D(A_i)$. We mentioned above that the centre Z has a finite rank. Hence the quotient loops $D/Z_D(A_i)$ are abelian groups of finite ranks. In particular they are finitely generated. Since t is a finite integer, then the direct product $\prod_{i=1}^{t} D/Z_D(A_i)$ is a finitely generated abelian group. It is known that: *if an arbitrary Abelian group has m generators then any of its subgroup have at most m generators*. Then from definition of special rank it follows that the direct product $\prod_{i=1}^{t} D/Z_D(A_i)$ has a finite rank. Further, by (1) and (2), it is easy to see that $\bigcap_{i=1}^{t} Z_D(A_i) = Z_D(H)$.

Analogously to Remak Theorem for groups [7], it may be proved that the quotient loop $D/\bigcap_{i=1}^{t} Z_D(A_i) = D/Z_D(H)$ is isomorphic to a subloop of the direct product $\prod_{i=1}^{t} D/Z_D(A_i)$. Then $D/Z_D(H)$ has a finite rank $r(D/Z_D(H)$. As $Z_D(H) \subseteq Z_L(H)$, then $Z_D(H)$ also has a finite rank $r(Z_D(H)$. Thus, from the definition of special rank it follows that and CML D have a finite rank $\leq r(D/Z_D(H))r(Z_D(H))$.

Consequently, L/Z is a CML of finite rank $r(L/Z)$. The centre Z has a finite rank $r(Z)$. Then the CML L also has a finite rank $r(L/Z)r(Z)$. ∎

Lemma 3.2. *Let H be a finitely generated subloop of the CML L. If the centralizer $Z_L(H)$ belongs to class Ω then the centralizer $Z_{L/Z(L)}(Z(L)H/Z(L))$ belongs to class Ω.*

Proof. In [9, 10] it is proved that for a CML the condition of finite generation and maximum condition for subloops are equivalent, and, in [3], it was proved that this conditions are equivalent with the maximum condition for associative subloops. Further, in [2], it was proved that, for a CML, the minimum condition for subloops and the minimum condition for associative subloops are equivalent, and for p-loops, these conditions are equivalent with the condition of finiteness of rank [4].

Now, let $Z(L)$ be the centre of the CML L, $\overline{L} = L/Z(L)$ and $\overline{H} = Z(L)H/Z(L)$. Let us suppose that the centralizer $Z_{\overline{L}}(\overline{H})$ does not belong to class S. By Lemma 2.2, the quotient loop $L/Z(L)$ satisfies the identity $x^3 = 1$. Then by the aforementioned, $Z_{\overline{L}}(\overline{H})$ contains an infinite elementary abelian 3-group $\overline{B}$, which decomposes into a direct product of cyclic groups of order 3. Let $A/Z(L) = \overline{A} = \overline{A}_1 \times \overline{A}_2 \times \ldots \times \overline{A}_i \times \ldots$ be the maximal subgroup of $\overline{B}$ regarding to property $\overline{A}_i \not\subseteq \overline{H}$, and let $\overline{A}_i =< \overline{a}_i >$. We denote by $M(\overline{a}_i)$ a maximal subloop of the CML $\overline{R} =< \overline{A}, \overline{H} >$ such that $\overline{a}_i \notin M(\overline{a}_i)$. As the element $\overline{a}_i$ has order 3 then $\overline{A}_i \cap M(\overline{a}_i) = \overline{1}$. Every maximal subloop of a CML is normal in this CML [9]. Let $I(\overline{R})$ be the inner mapping group of the CML $\overline{R}$. Every inner mapping of the CML is an automorphism of this CML [5]. Then $I(\overline{R})\overline{A}_i = \overline{A}_i$. Hence the subloop $\overline{A}_i$ is normal in $\overline{R}$. In [10] it was proved that, *if an element of order* 3 *of CML generates a normal subloop then this element belongs to the centre of this CML*. Hence $\overline{A}_i \subseteq Z(\overline{R})$. $\overline{A} \subseteq Z(\overline{R})$. From here it follows that $\overline{R} = \overline{AH}$. But $\overline{A} \cap \overline{H} = \overline{1}$. Then from $I(\overline{R})\overline{A} = \overline{A}$ it follows that $I(\overline{R})\overline{H} = \overline{H}$, i.e. the subloop $\overline{H}$ is normal in $\overline{R}$. Consequently, $\overline{R} = \overline{A} \times \overline{H}$.

The subloop H is finitely generated. Then, by Lemma 2.1, it is centrally nilpotent. The subloop $\overline{H}$ is also centrally nilpotent. Then the subloop $\overline{R}$ is also centrally nilpotent. As $\overline{B} \subseteq \overline{R}$ and $\overline{B}$ do not belong to the class S then $\overline{R}$ does not belong to class S, too. The inverse image of $\overline{R}$, under the homomorphism $L \to L/Z(L)$, is AH. This CML is centrally nilpotent and does not belong to class S. Then, by Lemma 3.1, the centralizer $Z_{AH}(H)$ does not belong to class S. We get a contradiction, as $Z_{AH}(H) \subseteq Z_L(H)$ and $Z_L(H)$ belong to class S. Consequently, the centralizer $Z_{\overline{L}}(\overline{H})$ belongs to class S. ∎

Lemma 3.3. *Let H be a finitely generated subloop of a CML L. If the centralizer $Z_L(H)$ belongs to a class Ω, then the centre $Z(L)$ belongs to Ω if the CML L is different from unity loop.*

Proof. If $a \in L$ is an element of infinite order, then by Lemma 2.2, $1 \neq a^3 \in Z(L)$. Let us suppose that L is a periodic CML. In these cases, L decomposes into a direct product of its maximal p-subloops L_p, and, in addition, L_p belongs to the centre $Z(L)$ under $p \neq 3$. Hence, in order to prove Lemma 3.1, it is sufficient to suppose that L is a 3-loop. By Lemma 2.1, every finitely generated CML is centrally nilpotent, and consequently we will suppose that CML L is infinite.

By analogy with the group theory [7], we say that the system $\{G_\alpha\}$ $(\alpha \in I)$ of subloops of loop G is a *local system* if the union $\bigcup_{\alpha \in I} G_\alpha$ coincides with G and every two members of this system are contained in a certain third member of this system. Using the definition of the local system, it is easy to prove the statement: if $\{G_\alpha\}$ $(\alpha \in I)$ is some local system of loop G and $I = I_1 \bigcup I_2 \bigcup \ldots \bigcup I_k$ is a certain partition of the set of indices I into a finite number of subsets, I_j, $j = 1, 2, \ldots, k$, then at least one subset I_j corresponds to the set of subloops $\{G_\beta\}$, $\beta \in I_j$, which will also be a local system for loop G.

Let now $\{L_\alpha\}$, $\alpha \in I$, be the local system of all finitely generated subloops of CML L, which contain the subloop H. By Lemma 2.1, $Z(L_\alpha) \neq \{1\}$. For each $\alpha \in I$, we fixed an arbitrary non-unitary element a_α in the centre $Z(L_\alpha)$ and let K be the subloop of L generated by all chosen a_α, $\alpha \in I$. From $K \subseteq Z_L(H)$ it follows that K belongs to class Ω. We suppose that the 3-subloop K has a finite rank. Then by [4], K satisfies the minimum condition for its subloops and, by [2], $K = R \times T$, where $R \subseteq Z(L)$ and T is a finite subloop. Further, any periodic CML is locally finite [5]. Hence, to prove Lemma 3.1, it is sufficient to consider that K is a finite subloop.

Further, let us decompose the set of indices I into a finite number of subsets $I = I_1 \bigcup I_2 \bigcup \ldots \bigcup I_k$ by rule: $\beta, \gamma \in I_j$ if and only if $a_\beta = a_\gamma$. According to the aforementioned statement we have received that at least one of the subsets I_j (e.g. I_1) corresponds to the subset of subloops L_α, $\alpha \in I_1$, which will be a local system for the CML L. Next, let us fix index $\alpha \in I_1$, and consider the set of indices $S \subseteq I_1$, such that $L_\alpha \subseteq L_\beta, \beta \in S$. We notice that the set S corresponds to the set of subloops $\{L_\beta\}, \beta \in S$, which will give a local system for CML L. Let us denote the value of the corresponding members by b, $b = a_\alpha = a_\beta = \ldots$. Then $b \in Z(L_\beta)$ for all $\beta \in S$ and, consequently, $b \in Z(L)$. ∎

Theorem 3.1. *A CML L belongs to class Ω if and only if the centralizer of some of its finitely generated subloop H also belongs to class Ω.*

Proof. The "necessity" is obvious. Conversely, let the centralizer $Z_L(H)$ belong to class Ω. We denote $\overline{L} = L/Z(L)$, $\overline{H} = HZ(L)/Z(L)$. Any periodic CML is locally finite [5]. Then from Lemma 2.2 it follows that the subloop $\overline{H}$ is finite. From Lemmas 3.1, 3.2, it follows that the upper central series of CML $\overline{L}$ has the form $\overline{1} \subset Z_1(\overline{L}) \subset \ldots \subset Z_k(\overline{L}) \subset \ldots$, and, for a natural number n, $Z_n(\overline{L}) \neq Z_{n+1}(\overline{L})$ if $Z_n(\overline{L}) \neq \overline{L}$. As $\overline{H}$ is finite, then, for some k, $\overline{H} \not\subseteq Z_{k-1}(\overline{L})$ but $\overline{H} \subseteq Z_k(\overline{L})$. Hence $Z_{\tilde{L}}(\tilde{H}) = \tilde{L}$, where $\tilde{L} = \overline{L}/\overline{Z_k}$, $\tilde{H} = \overline{HZ_k}/\overline{Z_k}$.

By Lemma 2.2, $\tilde{L}$ is a 3-loop and, by Lemma 3.2, $\tilde{L}$ belongs to the class Ω. In [4] it was proved that the minimum condition for subloops and the condition of finiteness rank are equivalent for the CML $\tilde{L}$. In this case $\tilde{L} = \tilde{R} \times \tilde{T}$, where $\tilde{R} \subseteq Z(\tilde{L})$, and $\tilde{T}$ is a finite CML which, by Lemma 2.1, is centrally nilpotent. Then CML $\tilde{L} = \overline{L}/\overline{Z_k} = (L/Z)/(Z_k/Z) \cong L/Z_k$ is also centrally nilpotent. From central nilpotence of L/Z_k it follows the central nilpotence of L, and by Lemma 3.1, L belongs to class Ω. ∎

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