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MONOTONE ITERATIVE TECHNIQUE FOR INTEGRAL BOUNDARY VALUE PROBLEMS OF SINGULAR DIFFERENTIAL EQUATIONS ON THE WHOLE LINE

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Abstract This paper deals with the existence of positive solutions for some boundary value problems of the singular differential equations on the whole line. Our approach is based on the fixed point theorem and the monotone iterative technique. Without the assumption of the existence of lower and upper solutions, we obtain not only the existence of nonnegative solutions for the problems, but also establish iterative schemes for approximating the solutions.

Keywords: second order differential equation with $p-$Laplacian on the whole line, integral type boundary value problem, solution, fixed point theorem.

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1. INTRODUCTION

Differential equations governed by nonlinear differential operators have been widely studied. In this setting the most investigated operator is the classical $p-$Laplacian, that is $\Phi_p(x) = |x|^{p-2}x$ with $p > 1$, which, in recent years, has been generalized to other types of differential operators, that preserve the monotonicity of the $p-$Laplacian, but are not homogeneous. These more general operators, which are usually referred to as $\Phi-$Laplacian (or quasi-Laplacian), are involved in some models, e.g. in non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces. The related nonlinear differential equation has the form

$$[\Phi(x')]' = f(t, x, x'), \quad t \in (-\infty, +\infty),$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\Phi(0) = 0$. More recently, equations involving other types of differential operators have been studied from a different point of view arising from other types of models, e.g. reaction diffusion equations with non-constant diffusivity and porous media equations. This leads to consider nonlinear differential operators of the type $[a(t, x, x')\Phi(x')]'$, where a is a

positive continuous function. For a comprehensive bibliography on this subject, see e.g. [1-10].

Philos and Purnaras [1] study a class of the boundary value problem (BVP for short) for the second order nonlinear ordinary differential equations on the whole line. Two existence results are established in [1]. The first theorem is established by the use of the Schauder theorem and concerns the existence of solutions, while the second theorem is concerned with the existence and uniqueness of solutions and is derived by the Banach contraction principle.

In [2], Bianconi and Papalini investigate the existence of solutions of the following boundary value problem

$$\begin{cases} [\Phi(x'(t))]' + a(t, x(t))b(x(t), x'(t)) = 0, \quad t \in \mathbb{R}, \\ \lim_{t\to-\infty} x(t) =: x(-\infty) = 0, \\ \lim_{t\to+\infty} x(t) =: x(+\infty) = 1, \end{cases} \tag{1}$$

where Π is a monotone function which generalizes the one-dimensional $p-$Laplacan operator. The criteria for the existence and non-existence of solutions of BVP(1.1) is established.

In [3,4], Avramescu and Vladimirescu study the following boundary value problem

$$\begin{cases} x''(t) + 2f(t)x'(t) + x(t) + g(t, x(t)) = 0, \quad t \in \mathbb{R}, \\ \lim_{t\to\pm\infty} x(t) =: x(\pm\infty) = 0, \\ \lim_{t\to\pm\infty} x'(t) =: x(\pm\infty) = 0, \end{cases} \tag{2}$$

where f and g are given functions. The existence of solutions of BVP(1.2) is obtained.

In [5], Avramescu and Vladimirescu study the following boundary value problem

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in \mathbb{R}, \\ \lim_{t\to-\infty} x(t) = \lim_{t\to+\infty} x(t), \\ \lim_{t\to-\infty} x'(t) = \lim_{t\to+\infty} x'(t), \end{cases} \tag{3}$$

under adequate hypothesis and using the Bohnenblust-Karlin fixed point theorem, the existence of solutions of BVP(1.3) is established.

Cabada and Cid [6] prove the solvability of the boundary value problem on the whole line

$$\begin{cases} [\Phi(x'(t)]' + f(t, x(t), x'(t)) = 0, \quad t \in \mathbb{R}, \\ \lim_{t\to-\infty} x(t) = -1, \\ \lim_{t\to+\infty} x(t) = 1, \end{cases} \tag{4}$$

where f is a continuous function, $\Phi : (-a, a) \to \mathbb{R}$ is a homeomorphism with $a \in (0, +\infty)$, i.e., Φ is singular.

Calamai [7] and Cristina Marcelli and Papalini [8] discuss the solvability of the following strongly nonlinear BVP:

$$\begin{cases} [a(x(t))\Phi(x'(t)]' + f(t, x(t), x'(t)) = 0, \quad t \in \mathbb{R}, \\ \lim_{t\to-\infty} x(t) = \alpha, \\ \lim_{t\to+\infty} x(t) = \beta, \end{cases} \tag{5}$$

where $\alpha < \beta$, Φ is a general increasing homeomorphism with bounded domain (singular Φ-Laplacian), a is a positive, continuous function and f is a Caratheodory nonlinear function. conditions for the existence and non-existence of heteroclinic solutions in terms of the behavior of $y \to f(t, x, y)$ and $y \to \Phi(y)$ as $y \to 0$, and of $t \to f(t, x, y)$ as $|t| \to +\infty$. The approach is based on fixed point techniques suitably combined to the method of upper and lower solutions.

Most of the known papers only consider the existence and uniqueness of positive solutions of various boundary value problems. A natural question which arises is "How can we find the solutions when they are known to exist?"

To fill this gap, we consider the following more generalized boundary value problem for the second order differential equation on the whole line with $p-$Laplacian

$$\begin{cases} [\Phi(\rho(t)a(t, x(t), x'(t))x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in \mathbb{R}, \\ \lim_{t\to-\infty} \rho(t)x'(t) = -\int_{-\infty}^{+\infty} g(s, x(s), x'(s))ds, \\ \lim_{t\to+\infty} x(t) = \int_{-\infty}^{+\infty} h(s, x(s), x'(s))ds, \end{cases} \tag{6}$$

where

- f, g, h defined on $\mathbb{R}^3$ are nonnegative Caratheodory functions, $f(t, 0, 0) \not\equiv 0$ on each subinterval of $\mathbb{R}$.
- $\rho \in C(\mathbb{R}, (0, \infty))$ satisfies

$$\int_0^{+\infty} \frac{1}{\rho(s)} ds < +\infty, \quad \int_{-\infty}^{+\infty} \frac{1}{\rho(s)} ds = +\infty.$$

- $a : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to (0, +\infty)$ is continuous and satisfies that there exist constants $m, M > 0$ such that

$$M \geq a\left(t, \left(1 + \int_t^{+\infty} \frac{1}{\rho(s)} ds\right) x, \frac{y}{\rho(t)}\right) \geq m, \ t \in \mathbb{R}, x \in \mathbb{R}, y \in \mathbb{R}$$

and for each $r > 0$, $|x|, |y| \leq r$ imply that $a\left(t, \left(1 + \int_t^{+\infty} \frac{1}{\rho(s)} ds\right) x, \frac{y}{\rho(t)}\right) \to a_{\pm\infty}$ uniformly as $t \to \pm\infty$ and

$(x, y) \to a\left(t, \left(1 + \int_t^{+\infty} \frac{1}{\rho(s)} ds\right) x, \frac{y}{\rho(t)}\right)$ is uniformly continuous on $[-r, r] \times [-r, r]$.

- $\Phi : \mathbb{R} \to \mathbb{R}$ is a sup-multiplicative-like function and its inverse function is denoted by Φ^{-1}.

The purpose of this paper is to establish the existence of positive solutions for BVP(1.6). Our approach is based on the fixed point theorem and the monotone iterative technique. Without the assumption of the existence of lower and upper solutions, we obtain not only the existence of positive solutions for the problems, but also establish iterative schemes for approximating the solutions.

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the main results are presented in Section 3.

2. PRELIMINARY RESULTS

In this section, we first present some background definitions in Banach spaces and state an important three fixed point theorem. Then the main results are given and proved.

Definition 2.1. *F defined on $\mathbb{R}^3$ is called a quasi-Carathédory function if*

(i) $t \to f\left(t, \left(1 + \int_t^{+\infty} \frac{1}{\rho(s)} ds\right) x, \frac{1}{\rho(t)} y\right)$ is measurable for any $(x, y) \in \mathbb{R}^2$,

(ii) $(x, y) \to f\left(t, \left(1 + \int_t^{+\infty} \frac{1}{\rho(s)} ds\right) x, \frac{1}{\rho(t)} y\right)$ is continuous for a.e. $t \in \mathbb{R}$,

(iii) for each $r > 0$, there exists nonnegative function $\phi_r \in L^1(\mathbb{R})$ such that $|x|, |y| \leq r$ implies

$$\left| f\left(t, \left(1 + \int_t^{+\infty} \frac{1}{\rho(s)} ds\right) x, \frac{1}{\rho(t)} y\right) \right| \leq \phi_r(t), a.e. t \in \mathbb{R}.$$

Definition 2.2. *[11]. An odd homeomorphism Φ of the real line $\mathbb{R}$ onto itself is called a sup-multiplicative-like function if there exists a homeomorphism ω of $[0, +\infty)$ onto itself which supports Φ in the sense that for all $v_1, v_2 \geq 0$ it holds*

$$\Phi(v_1 v_2) \geq \omega(v_1)\Phi(v_2). \tag{7}$$

ω is called the supporting function of Φ.

Remark 2.1. *Note that any sup-multiplicative function is sup-multiplicative-like function. Also any function of the form*

$$\Phi(u) := \sum_{j=0}^{k} c_j |u|^j u, \quad u \in \mathbb{R}$$

is sup-multiplicative-like, provided that $c_j \geq 0$. Here a supporting function is defined by $\omega(u) := \min\{u^{k+1}, \quad u\}$, $u \geq 0$.

Remark 2.2. *It is clear that a sup-multiplicative-like function Φ and any corresponding supporting function ω are increasing functions vanishing at zero and moreover their inverses Φ^{-1} and ν respectively are increasing and such that*

$$\Phi^{-1}(w_1 w_2) \leq \nu(w_1)\Phi^{-1}(w_2), \tag{8}$$

for all $w_1, w_2 \geq 0$. ν *s also called the supporting function of* Φ^{-1}.

In the sequel, we suppose that Φ is a sup-multiplicative-like function with its supporting function ω. The inverse function of Φ is denoted by Φ^{-1}, whose supporting function is denoted by ν.

Choose

$$X = \left\{ x \in C^1(\mathbb{R}) : \begin{array}{c} \frac{x(t)}{1+\int_t^{+\infty} \frac{1}{\rho(s)} ds} \text{ is bounded on } \mathbb{R} \text{ and the limits} \\ \lim_{t\to+\infty} x(t),\ \lim_{t\to-\infty} \rho(t)x'(t), \\ \text{and } \lim_{t\to+\infty} \rho(t)x'(t) \text{ exist} \end{array} \right\}.$$

For $x \in X$, define the norm of x by

$$\|x\| = \max\left\{ \sup_{t\in\mathbb{R}} \frac{|x(t)|}{1 + \int_t^{+\infty} \frac{1}{\rho(s)} ds},\ \sup_{t\in\mathbb{R}} \rho(t)|x'(t)| \right\}.$$

One can prove that X is a Banach space with the norm $\|x\|$ for $x \in X$.

For $x, y \in X$, we call $x \leq y$ if $x(t) \leq y(t)$ and $|x'(t)| \leq |y'(t)|$ hold for all $t \in \mathbb{R}$. Then $(X, \|\cdot\|, \leq,)$ is a partially ordered Banach space.

Denote $\tau(t) = \int_t^{+\infty} \frac{ds}{\rho(s)}$. Define the cone P in X by

$$P = \{x \in X : x(t) \geq 0,\ t \in \mathbb{R}\}.$$

For $x \in X$, define Tx by

$$\begin{aligned} (Tx)(t) &= \int_{-\infty}^{+\infty} h(u, x(u), x'(u))du \\ &+ \int_t^{+\infty} \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{s} f(u, x(u), x'(u))du\right)}{a(s, x(s), x'(s))\rho(s)} ds, \\ & t \in (-\infty, \infty). \end{aligned}$$

Lemma 2.1. *Suppose that* f, g, h *are Caratheodory functions. Then*

(i) $T : X \to X$ *is well defined and* Tx *satisfies*

$$\begin{cases} [\Phi(\rho(t)a(t, x(t), x'(t))(Tx)'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (-\infty, +\infty), \\ \lim_{t\to-\infty} \rho(t)(Tx)'(t) = -\int_{-\infty}^{+\infty} g(s, x(s), x'(s))ds, \\ \lim_{t\to+\infty} (Tx)(t) = \int_{-\infty}^{+\infty} h(s, x(s), x'(s))ds, \end{cases} \tag{9}$$

(ii) $Tx \in P$ *for each* $x \in P$;

(iii) $x \in X$ *is a positive solution of BVP(1.6) if and only if* $x \in P$ *is a solution of the operator equation* $x = Tx$.

Proof. (i) For $x \in X$, we know that

$$r = \max\left\{\sup_{t\in\mathbb{R}} \frac{|x(t)|}{1+\int_t^{+\infty}\frac{1}{\rho(s)}ds},\ \sup_{t\in\mathbb{R}}\rho(t)|x'(t)|\right\} < +\infty.$$

Since f, g, h are Caratheodory functions, there exist $\phi_r \in L^1(\mathbb{R})$ such that

$$|f(t, x(t), x'(t))| = \left|f\left(t, \left(1+\int_t^{+\infty}\frac{ds}{\rho(s)}\right)\frac{x(s)}{1+\int_t^{+\infty}\frac{ds}{\rho(s)}}, \frac{1}{\rho(t)}\rho(s)x'(s)\right)\right| \le \phi_r(t)$$

and

$$|f(t, x(t), x'(t))| \le \phi_r(t), \quad |g(t, x(t), x'(t))| \le \phi_r(t), \quad t \in \mathbb{R}.$$

Then

$$\int_{-\infty}^{+\infty}|f(u, x(u), x'(u))|du, \quad \int_{-\infty}^{+\infty}|g(u, x(u), x'(u))|du, \quad \int_{-\infty}^{+\infty}|h(u, x(u), x'(u))|du$$

are convergent. So $Tx \in C^0(\mathbb{R})$ and

$$(Tx)'(t) = -\frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{t} f(u, x(u), x'(u))du\right)}{a(t, x(t,), x'(t))\rho(t)}.$$

It is easy to see that $(Tx)' \in C^0(\mathbb{R})$ and

$$\frac{(Tx)(t)}{1+\int_t^{+\infty}\frac{1}{\rho(s)}ds}$$

is bounded on $\mathbb{R}$ and there exist the limits

$$\lim_{t\to+\infty}(Tx)(t) \text{ and } \lim_{t\to-\infty}\rho(t)(Tx)'(t), \lim_{t\to+\infty}\rho(t)(Tx)'(t).$$

Hence $Tx \in X$. So $T : X \to X$ is well defined. It is easy to show that (2.9) holds.

(ii) Since f, g, h are nonnegative Caratheodory functions, we get that $T : P \to P$.

(iii) It is easy to show that $x \in X$ is a positive solution of BVP(1.6) if and only if $x \in P$ is a solution of the operator equation $x = Tx$. ∎

Lemma 2.2. *$T : X \to X$ is completely continuous.*

Proof. From Lemma 2.1, $T : X \to X$ is well defined. Now we prove that T is continuous and maps bounded subsets into relatively compact sets.

First, we show that T is continuous. Let $x_n \to x_0$ as $n \to +\infty$ in X, then we get

$$r = \sup_{n=0,1,2,\cdots} \max\left\{\sup_{t\in\mathbb{R}} \frac{|x_n(t)|}{1+\int_t^{+\infty}\frac{1}{\rho(s)}ds}, \sup_{t\in\mathbb{R}} \rho(t)|x_n'(t)|\right\} < +\infty.$$

Because f, g, h are Caratheodory functions, then there exists $\phi_r \in L^1(-\infty, +\infty)$ such that

$$\begin{aligned}
&\left|f\left(t, x_n(t), x_n'(t)\right)\right| = \left|f\left(t, \left(1+\int_t^{+\infty}\frac{ds}{\rho(s)}\right)\frac{x_n(s)}{1+\int_t^{+\infty}\frac{ds}{\rho(s)}}, \frac{1}{\rho(t)}\rho(s)x_n'(s)\right)\right| \\
\leq\ & \phi_r(t),\ t\in\mathbb{R}, \\
&\left|g\left(t, x_n(t), x_n'(t)\right)\right| \leq \phi_r(t), t\in\mathbb{R}, \\
&\left|h\left(t, x_n(t), x_n'(t)\right)\right| \leq \phi_r(t),\ t\in\mathbb{R}.
\end{aligned}$$

Then

$$\int_{-\infty}^{+\infty} f(s, x(s), x'(s))ds, \int_{-\infty}^{+\infty} g(s, x(s), x'(s))ds \text{ and } \int_{-\infty}^{+\infty} h(s, x(s), x'(s))ds$$

are convergent.

We need to prove that $Tx_n \to Tx_0$ as $n \to \infty$.

From the definition of T, we find that

$$\begin{aligned}
(Tx_n)(t) &= \int_{-\infty}^{+\infty} h(u, x_n(u), x_n'(u))du \\
&+ \int_t^{+\infty} \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_n(u), x_n'(u))ds\right) + \int_{-\infty}^{s} f(u, x_n(u), x_n'(u))du\right)}{a(s, x_n(s,), x_n'(s))\rho(s)}ds,
\end{aligned}$$

and

$$\begin{aligned}
&\rho(t)(Tx_n)'(t) \\
=\ & -\frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_n(u), x_n'(u))ds\right) + \int_{-\infty}^{t} f(u, x_n(u), x_n'(u))du\right)}{a(t, x_n(t,), x_n'(t))}
\end{aligned}$$

for $n = 0, 1, 2, \cdots$. By the definition of G, we get

$$
\begin{aligned}
& \frac{|(Tx_n)(t) - (Tx_0)(t)|}{1 + \int_t^{+\infty} \frac{ds}{\rho(s)}} \\
\leq\ & \frac{1}{1 + \int_t^{+\infty} \frac{ds}{\rho(s)}} \int_{-\infty}^{+\infty} |h(u, x_n(u), x_n'(u)) - h(u, x_0(u), x_0'(u))| du \\
& + \frac{1}{1 + \int_t^{+\infty} \frac{ds}{\rho(s)}} \times \\
& \int_t^{+\infty} \left| \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_n(u), x_n'(u)) du\right) + \int_{-\infty}^{s} f(u, x_n(u), x_n'(u)) du\right)}{a(s, x_n(s), x_n'(s))\rho(s)} \right. \\
& \left. - \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_0(u), x_0'(u)) du\right) + \int_{-\infty}^{s} f(u, x_0(u), x_0'(u)) du\right)}{a(s, x_0(s), x_0'(s))\rho(s)} \right| ds \\
\leq\ & \int_{-\infty}^{+\infty} |h(u, x_n(u), x_n'(u)) - h(u, x_0(u), x_0'(u))| du \\
& + \frac{1}{1 + \int_t^{+\infty} \frac{ds}{\rho(s)}} \int_t^{+\infty} \frac{1}{\rho(s) a(s, x_n(s,), x_n'(s))} \\
& \times \left| \Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_n(u), x_n'(u)) du\right) + \int_{-\infty}^{s} f(u, x_n(u), x_n'(u)) du\right) \right. \\
& \left. - \Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_0(u), x_0'(u)) du\right) + \int_{-\infty}^{s} f(u, x_0(u), x_0'(u)) du\right) \right| ds \\
& + \frac{1}{1 + \int_t^{+\infty} \frac{ds}{\rho(s)}} \times \\
& \int_t^{+\infty} \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_0(u), x_0'(u)) du\right) + \int_{-\infty}^{s} f(u, x_0(u), x_0'(u)) du\right)}{\rho(s)} \\
& \times \frac{|a(s, x_n(s,), x_n'(s)) - a(s, x_0(s,), x_0'(s))|}{a(s, x_n(s), x_n'(s)) a(s, x_0(s,), x_0'(s))} ds \\
\leq\ & \int_{-\infty}^{+\infty} |h(u, x_n(u), x_n'(u)) - h(u, x_0(u), x_0'(u))| du \\
& + \frac{1}{m} \frac{1}{1 + \int_t^{+\infty} \frac{ds}{\rho(s)}} \int_t^{+\infty} \frac{1}{\rho(s)} \\
& \times \left| \Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_n(u), x_n'(u)) du\right) + \int_{-\infty}^{s} f(u, x_n(u), x_n'(u)) du\right) \right. \\
& \left. - \Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_0(u), x_0'(u)) ds\right) + \int_{-\infty}^{s} f(u, x_0(u), x_0'(u)) du\right) \right| ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{1+\int_t^{+\infty}\frac{ds}{\rho(s)}}\int_t^{+\infty}\frac{\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\phi_r(u)du\right)+\int_{-\infty}^{+\infty}\phi_r(u)du\right)}{\rho(s)}\\
&\times\frac{|a(s,x_n(s,),x_n'(s))-a(s,x_0(s),x_0'(s))|}{m^2}ds\\
\leq\ &\int_{-\infty}^{+\infty}|h(u,x_n(u),x_n'(u))-h(u,x_0(u),x_0'(u))|du\\
&+\frac{1}{m}\frac{1}{1+\int_t^{+\infty}\frac{ds}{\rho(s)}}\int_t^{+\infty}\frac{1}{\rho(s)}\\
&\times\left|\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}g(u,x_n(u),x_n'(u))du\right)+\int_{-\infty}^{s}f(u,x_n(u),x_n'(u))du\right)\right.\\
&\left.-\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}g(u,x_0(u),x_0'(u))ds\right)+\int_{-\infty}^{s}f(u,x_0(u),x_0'(u))du\right)\right|ds\\
&+\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\phi_r(u)du\right)+\int_{-\infty}^{+\infty}\phi_r(u)du\right)\times\\
&\frac{|a(s,x_n(s),x_n'(s))-a(s,x_0(s,),x_0'(s))|}{m^2}ds.
\end{aligned}
$$

For any $\epsilon>0$, since

$$\int_{-\infty}^{+\infty}|h(u,x_n(u),x_n'(u))-h(u,x_0(u),x_0'(u))|du\leq 2\int_{-\infty}^{+\infty}\phi_r(s)ds,$$

by the Lebesgue dominated convergence theorem, there exists $N_1>0$ such that

$$\int_{-\infty}^{+\infty}|h(u,x_n(u),x_n'(u))-h(u,x_0(u),x_0'(u))|du<\epsilon,\ n>N_1. \tag{10}$$

Similarly, we get

$$a_-\int_{-\infty}^{+\infty}g(u,x_n(u),x_n'(u))du\to a_-\int_{-\infty}^{+\infty}g(u,x_0(u),x_0'(u))du \text{ as } n\to\infty,$$

and

$$\int_{-\infty}^{s}f(u,x_n(u),x_n'(u))du\to\int_{-\infty}^{s}f(u,x_0(u),x_0'(u))du \text{ uniformly as } n\to\infty.$$

Since

$$\left|a_-\int_{-\infty}^{+\infty}g(u,x_n(u),x_n'(u))du\right|\leq|a_-|\int_{-\infty}^{+\infty}\phi_r(s)ds=:r_1,$$

and

$$\left|\int_{-\infty}^{s} f(u, x_n(u), x_n'(u))du\right| \leq \int_{-\infty}^{+\infty} \phi_r(s)ds =: r_2,$$

and Φ is uniformly continuous on $[-r_1, r_1]$, Φ^{-1} is uniformly continuous on $[-\Phi(r_1)-r_2, \Phi(r_1)+r_2]$, we can easily know that there exists $N_2 > 0$ such that

$$\begin{aligned}&\left|\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_n(u), x_n'(u))du\right) + \int_{-\infty}^{s} f(u, x_n(u), x_n'(u))du\right)\right.\\ &\left.-\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_0(u), x_0'(u))ds\right) + \int_{-\infty}^{s} f(u, x_0(u), x_0'(u))du\right)\right| \\ &< m\epsilon, n > N_2.\end{aligned} \tag{11}$$

By

$$\begin{aligned}&|a(s, x_n(s,), x_n'(s)) - a(s, x_0(s,), x_0'(s))|\\ &= \left|a\left(s, \left(1 + \int_t^{+\infty} \tfrac{ds}{\rho(s)}\right) \tfrac{x_n(s)}{1+\int_t^{+\infty} \frac{ds}{\rho(s)}}, \tfrac{1}{\rho(t)}\rho(t)x_n'(s)\right)\right.\\ &\left.-a\left(t, \left(1 + \int_t^{+\infty} \tfrac{ds}{\rho(s)}\right) \tfrac{x_0(t)}{1+\int_t^{+\infty} \frac{ds}{\rho(s)}}, \tfrac{1}{\rho(t)}\rho(t)x_0'(t)\right)\right|,\end{aligned}$$

and

$$a\left(t, \left(1 + \int_t^{+\infty} \frac{ds}{\rho(s)}\right)x, \frac{1}{\rho(t)}y\right) \text{ is uniformly continuous on } [-r, r] \times [-r, r]$$

we get the there exists $\delta > 0$ such that

$$\begin{aligned}&\left|a\left(t, \left(1 + \int_t^{+\infty} \tfrac{ds}{\rho(s)}\right)x_1, \tfrac{1}{\rho(t)}y_1\right) - a\left(t, \left(1 + \int_t^{+\infty} \tfrac{ds}{\rho(s)}\right)x_2, \tfrac{1}{\rho(t)}y_2\right)\right|\\ &< \frac{m^2\epsilon}{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} \phi_r(u)du\right) + \int_{-\infty}^{+\infty} \phi_r(u)du\right)}\end{aligned}$$

holds for $|x_1 - x_2| < \delta, |y_1 - y_2| < \delta$. From $x_n \to x_0$, we get that there exists $N_3 > 0$ such that

$$\left|\frac{x_n(s)}{1 + \int_t^{+\infty} \frac{ds}{\rho(s)}} - \frac{x_0(s)}{1 + \int_t^{+\infty} \frac{ds}{\rho(s)}}\right| < \delta,$$

and

$$|\rho(t)x_n'(s) - \rho(t)x_0'(s)| < \delta$$

hold for $n > N_3$ and all $t \in \mathbb{R}$. Hence $n > N_3$ implies that

$$\begin{aligned}&\left|a\left(t, x_n(t), x_n'(t)\right) - a\left(t, x_0(t), x_0'(t)\right)\right|\\ &< \frac{m^2\epsilon}{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} \phi_r(u)du\right) + \int_{-\infty}^{+\infty} \phi_r(u)du\right)}.\end{aligned} \tag{12}$$

It follows from (2.10)-(2.12) that $n > \max\{N_1, N_2, N_3\}$ implies that

$$\frac{|(Tx_n)(t) - (Tx_0)(t)|}{1 + \int_t^{+\infty} \frac{ds}{\rho(s)}} < \epsilon + \epsilon + \epsilon = 3\epsilon. \tag{13}$$

Furthermore, we have

$$\begin{aligned}
&\rho(t)|(Tx_n)'(t) - (Tx_0)'(t)| \\
\leq\ & \left| \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_n(u), x_n'(u))ds\right) + \int_{-\infty}^{t} f(u, x_n(u), x_n'(u))du\right)}{a(t, x_n(t,), x_n'(t))} \right. \\
& \left. - \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_0(u), x_0'(u))ds\right) + \int_{-\infty}^{t} f(u, x_0(u), x_0'(u))du\right)}{a(t, x_0(t,), x_0'(t))} \right| \\
& \frac{2\int_{-\infty}^{+\infty} M_r(s)ds}{\int_{-\infty}^{+\infty} \frac{dr}{\rho(r)}} + \int_{-\infty}^{+\infty} M_r(s)ds.
\end{aligned}$$

By the same methods used above, we get

$$|\rho(t)(Tx_n)'(t) - \rho(t)(Tx_0)'(t)| \to 0 \text{ uniformly as } n \to +\infty, \tag{14}$$

So, (2.13) and (2.14) imply that $Tx_n \to Tx_0$ as $n \to \infty$. So T is continuous.

Second, we show that T maps bounded sets into relatively compact sets.

Recall $W \subset X$ is relatively compact if

(i) it is bounded,

(ii) both $\frac{1}{1+\tau(t)}W$ and $\rho(t)W$ are equi-continuous on any closed subinterval $[e, f]$ of $(-\infty, +\infty)$,

(iii) both $\frac{1}{1+\tau(t)}W$ and $\rho(t)W$ are equi-convergent at $t = -\infty$,

(iv) both $\frac{1}{1+\tau(t)}W$ and $\rho(t)W$ are equi-convergent at $t = +\infty$.

Let Ω be any bounded subset of X. Then there exists $r > 0$ such that $\|x\| \leq r$ for all $x \in \Omega$. Because f, g, h are Caratheodory functions, then there exists $\phi_r \in L^1(-\infty, +\infty)$ such that

$$\begin{aligned}
&\left|f\left(t, x(t), x'(t)\right)\right| \leq \phi_r(t), t \in \mathbb{R}, \\
&\left|g\left(t, x(t), x'(t)\right)\right| \leq \phi_r(t), t \in \mathbb{R}, \\
&\left|h\left(t, x(t), x'(t)\right)\right| \leq \phi_r(t), t \in \mathbb{R}, \\
&a(t, x(t), x'(t)) \geq m, t \in \mathbb{R}.
\end{aligned}$$

(i) We show that $T\Omega$ is bounded.

Obviously, for $x \in \Omega$, one sees from the definition of G that

$$\begin{aligned}
\tfrac{|(Tx)(t)|}{1+\tau(t)} &= \tfrac{1}{1+\tau(t)} \Big| \int_{-\infty}^{+\infty} h(u, x(u), x'(u))du \\
&+ \int_t^{+\infty} \frac{\Phi^{-1}\Big(\Phi\Big(a_- \int_{-\infty}^{+\infty} g(u,x(u),x'(u))du\Big) + \int_{-\infty}^{s} f(u,x(u),x'(u))du\Big)}{a(s,x(s,),x'(s))\rho(s)} ds \Big| \\
&\le \int_{-\infty}^{+\infty} \phi_r(s)ds + \tfrac{1}{1+\tau(t)} \int_t^{+\infty} \frac{\Phi^{-1}\Big(\Phi\Big(a_- \int_{-\infty}^{+\infty} \phi_r(s)ds\Big) + \int_{-\infty}^{+\infty} \phi_r(s)\Big)}{m\rho(s)} ds \\
&\le \int_{-\infty}^{+\infty} \phi_r(s)ds + \tfrac{1}{m}\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} \phi_r(s)ds\right) + \int_{-\infty}^{+\infty} \phi_r(s)\right).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\rho(t)|(Tx)'(t)| &= \frac{\Phi^{-1}\Big(\Phi\Big(a_- \int_{-\infty}^{+\infty} g(u,x(u),x'(u))ds\Big) + \int_{-\infty}^{s} f(u,x(u),x'(u))du\Big)}{a(t,x(t,),x'(t))} \\
&\le \frac{\Phi^{-1}\Big(\Phi\Big(a_- \int_{-\infty}^{+\infty} \phi_r(s)ds\Big) + \int_{-\infty}^{+\infty} \phi_r(u)du\Big)}{m}.
\end{aligned}$$

We get that $T\Omega$ is bounded.

(ii) We show that both $\frac{1}{1+\tau(t)}T\Omega$ and $\{\rho(t)(Tx)' : x \in \Omega\}$ are equi-continuous on any closed subinterval $[a, b]$ of $(-\infty, +\infty)$,

For $t_1, t_2 \in [a, b]$ with $t_1 < t_2$, one has

$$\begin{aligned}
&\left|\tfrac{(Tx)(t_1)}{1+\tau(t_1)} - \tfrac{(Tx)(t_2)}{1+\tau(t_2)}\right| \\
&\le \left|\tfrac{1}{1+\tau(t_1)} - \tfrac{1}{1+\tau(t_2)}\right| \int_{-\infty}^{+\infty} |h(s, x(s), x'(s))|ds \\
&+ \Bigg| \tfrac{1}{1+\tau(t_1)} \int_{t_1}^{+\infty} \frac{\Phi^{-1}\Big(\Phi\Big(a_- \int_{-\infty}^{+\infty} g(u,x(u),x'(u))du\Big) + \int_{-\infty}^{s} f(u,x(u),x'(u))du\Big)}{a(s,x(s,),x'(s))\rho(s)} ds \\
&- \tfrac{1}{1+\tau(t_2)} \int_{t_2}^{+\infty} \frac{\Phi^{-1}\Big(\Phi\Big(a_- \int_{-\infty}^{+\infty} g(u,x(u),x'(u))du\Big) + \int_{-\infty}^{s} f(u,x(u),x'(u))du\Big)}{a(s,x(s,),x'(s))\rho(s)} ds \Bigg| \\
&\le |\tau(t_1) - \tau(t_2)| \int_{-\infty}^{+\infty} \phi_r(s)ds \\
&+ \tfrac{1}{1+\tau(t_1)} \int_{t_1}^{t_2} \frac{\Phi^{-1}\Big(\Phi\Big(a_- \int_{-\infty}^{+\infty} g(u,x(u),x'(u))du\Big) + \int_{-\infty}^{s} f(u,x(u),x'(u))du\Big)}{a(s,x(s,),x'(s))\rho(s)} ds \\
&\left|\tfrac{1}{1+\tau(t_1)} - \tfrac{1}{1+\tau(t_2)}\right| \int_{t_2}^{+\infty} \frac{\Phi^{-1}\Big(\Phi\Big(a_- \int_{-\infty}^{+\infty} g(u,x(u),x'(u))du\Big) + \int_{-\infty}^{s} f(u,x(u),x'(u))du\Big)}{a(s,x(s,),x'(s))\rho(s)} ds
\end{aligned}$$

$$\leq |\tau(t_1) - \tau(t_2)| \int_{-\infty}^{+\infty} \phi_r(s)ds$$

$$+ \int_{t_1}^{t_2} \frac{1}{m\rho(s)} ds \Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} \phi_r(s)ds\right) + \int_{-\infty}^{+\infty} \phi_r(s)ds\right)$$

$$\frac{|\tau(t_1)-\tau(t_2)|}{1+\tau(t_2)} \int_{t_2}^{+\infty} \frac{1}{m\rho(s)} ds \Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} \phi_r(s)ds\right) + \int_{-\infty}^{+\infty} \phi_r(s)ds\right)$$

$$\leq |\tau(t_1) - \tau(t_2)| \int_{-\infty}^{+\infty} \phi_r(s)ds$$

$$+ \int_{t_1}^{t_2} \frac{1}{\rho(s)} ds \frac{1}{m} \Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} \phi_r(s)ds\right) + \int_{-\infty}^{+\infty} \phi_r(s)ds\right)$$

$$|\tau(t_1) - \tau(t_2)| \frac{1}{m} \Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} \phi_r(s)ds\right) + \int_{-\infty}^{+\infty} \phi_r(s)ds\right)$$

$$\to 0 \text{ uniformly as } t_1 \to t_2$$

and

$$\begin{aligned}
&|\rho(t_1)(Tx)'(t_1) - \rho(t_2)|(Tx)'(t_2)| \\
= &\left| \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{t_1} f(u, x(u), x'(u))du\right)}{a(t_1, x(t_1), x'(t_1))} \right. \\
&\left. - \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{t_2} f(u, x(u), x'(u))du\right)}{a(t_2, x(t_2), x'(t_2))} \right| \\
\leq &\left| \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{t_1} f(u, x(u), x'(u))du\right)}{a(t_1, x(t_1), x'(t_1))} \right. \\
&\left. - \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{t_2} f(u, x(u), x'(u))du\right)}{a(t_1, x(t_1), x'(t_1))} \right| \\
&+ \left| \frac{1}{a(t_1, x(t_1), x'(t_1))} - \frac{1}{a(t_2, x(t_2), x'(t_2))} \right| \\
&\times \Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{t_2} f(u, x(u), x'(u))du\right) \\
\leq &\left| \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{t_1} f(u, x(u), x'(u))du\right)}{m} \right. \\
&\left. - \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{t_2} f(u, x(u), x'(u))du\right)}{m} \right|
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{m^2}\left|a(t_1,x(t_1),x'(t_1))-a(t_2,x(t_2),x'(t_2))\right| \\
& \times\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\phi_r(s)ds\right)+\int_{-\infty}^{+\infty}\phi_r(s)ds\right) \\
= \quad & \frac{\int_{t_1}^{t_2}\phi_r(s)ds}{m}+\frac{1}{m^2}\left|a(t_1,x(t_1),x'(t_1))-a(t_2,x(t_2),x'(t_2))\right| \\
& \times\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\phi_r(s)ds\right)+\int_{-\infty}^{+\infty}\phi_r(s)ds\right) \\
\to \quad & 0 \text{ uniformly as } t_1\to t_2.
\end{aligned}
$$

It follows that both $\frac{T\Omega}{1+\tau(t)}$ and $\{\rho(t)(Tx)' : x\in\Omega\}$are equi-continuous any subinterval $[a,b]$.

(iii) We show that both $\frac{1}{1+\tau(t)}T\Omega$ and $\{\rho(t)(Tx)' : x\in\Omega\}$ are equi-convergent at $t=-\infty$,

One sees that

$$
\begin{aligned}
& \left|\frac{(Tx)(t)}{1+\tau(t)}-\frac{a_-\int_{-\infty}^{+\infty}g(s,x(s),x'(s))ds}{a(t,x(t),x'(t))}\right| \\
\leq \quad & \frac{\int_{-\infty}^{+\infty}h(s,x(s),x'(s))ds}{1+\tau(t)} \\
& +\left|\frac{\int_t^{+\infty}\frac{\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}g(u,x(u),x'(u))ds\right)+\int_{-\infty}^{s}f(u,x(u),x'(u))du\right)}{a(s,x(s,),x'(s))\rho(s)}ds}{1+\tau(t)}\right. \\
& \left.-\frac{a_-\int_{-\infty}^{+\infty}g(s,x(s),x'(s))ds}{a(t,x(t),x'(t))}\right| \\
\leq \quad & \frac{\int_{-\infty}^{+\infty}\phi_r(s)ds}{1+\tau(t)}+\frac{1}{1+\tau(t)} \\
& +\left|\frac{\int_t^{+\infty}\frac{\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}g(u,x(u),x'(u))ds\right)+\int_{-\infty}^{s}f(u,x(u),x'(u))du\right)}{a(s,x(s,),x'(s))\rho(s)}ds}{1+\tau(t)}\right. \\
& \left.-\frac{\int_t^{+\infty}\frac{\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}g(s,x(s),x'(s))ds\right)\right)}{a(t,x(t),x'(t))\rho(s)}ds}{1+\tau(t)}\right|
\end{aligned}
$$

$$= \frac{\int_{-\infty}^{+\infty}\phi_r(s)ds}{1+\tau(t)} + \frac{1}{1+\tau(t)} + \frac{1}{m}\frac{1}{1+\tau(t)} \times$$
$$\int_t^{+\infty}\left|\frac{\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}g(u,x(u),x'(u))ds\right)+\int_{-\infty}^{s}f(u,x(u),x'(u))du\right)}{\rho(s)}\right.$$
$$\left.-\frac{\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}g(s,x(s),x'(s))ds\right)\right)}{\rho(s)}\right|ds.$$

For any $\epsilon > 0$, since Φ^{-1} is uniformly continuous on $[-\Phi(r_1)-r_2, \Phi(r_1)+r_2]$, there exists $\delta > 0$ such that

$$|\Phi^{-1}(x)-\Phi^{-1}(y)| < \epsilon,\ \ x,y \in [-\Phi(r_1)-r_2, \Phi(r_1)+r_2], |x-y| < \delta.$$

By

$$\left|\Phi\left(a_-\int_{-\infty}^{+\infty}g(u,x(u),x'(u))ds\right)+\int_{-\infty}^{t}f(u,x(u),x'(u))du\right.$$
$$\left.-\Phi\left(a_-\int_{-\infty}^{+\infty}g(s,x(s),x'(s))ds\right)\right|$$
$$\leq \int_{-\infty}^{t}\phi_r(s)ds \to 0 \text{ uniformly as } t\to-\infty,$$

So there exists $T_1 > 0$ such that $t < -T_1$ implies that

$$\left|\Phi\left(a_-\int_{-\infty}^{+\infty}g(u,x(u),x'(u))ds\right)+\int_{-\infty}^{t}f(u,x(u),x'(u))du\right.$$
$$\left.-\Phi\left(a_-\int_{-\infty}^{+\infty}g(s,x(s),x'(s))ds\right)\right| < \delta$$

holds uniformly for $x \in \Omega$. Then $t < -T_1$ implies that

$$\left|\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}g(u,x(u),x'(u))ds\right)+\int_{-\infty}^{s}f(u,x(u),x'(u))du\right)\right.$$
$$\left.-\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}g(s,x(s),x'(s))ds\right)\right)\right| < \epsilon$$

holds uniformly for $x \in \Omega$. It is easy to see that there exists $T_2 > 0$ such that

$$\frac{\int_{-\infty}^{+\infty}\phi_r(s)ds}{1+\tau(t)} + \frac{1}{1+\tau(t)} < \epsilon,\ t < -T_1.$$

Hence $t < -\min\{-T_1, -T_2\}$ implies that

$$\left|\frac{(Tx)(t)}{1+\tau(t)} - \frac{a_-\int_{-\infty}^{+\infty}g(s,x(s),x'(s))ds}{a(t,x(t),x'(t))}\right| < \epsilon + \frac{1}{m}\frac{1}{1+\tau(t)}\int_t^{+\infty}\frac{\epsilon}{\rho(s)}ds < \left(1+\frac{1}{m}\right)\epsilon. \tag{15}$$

Furthermore, we have

$$\begin{aligned}
&\left|\rho(t)(Tx)'(t) + \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right| \\
= &\left|\frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{t} f(u, x(u), x'(u))du\right)}{a(t, x(t,), x'(t))}\right. \\
&\left. - \frac{\Phi^{-1}\left(\Phi\left(a(t, x(t,), x'(t)) \int_{-\infty}^{+\infty} g(s, x(s), x(s), x'(s))ds\right)\right)}{a(t, x(t,), x'(t))}\right| \\
\leq &\frac{1}{m}\left|\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{t} f(u, x(u), x'(u))du\right)\right. \\
&\left. -\Phi^{-1}\left(\Phi\left(a(t, x(t,), x'(t)) \int_{-\infty}^{+\infty} g(s, x(s), x(s), x'(s))ds\right)\right)\right|.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&\left|\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{t} f(u, x(u), x'(u))du\right| \\
&\leq \Phi\left(M \int_{-\infty}^{+\infty} \phi_r(s)ds\right) + \int_{-\infty}^{+\infty} \phi_r(s)ds =: r_4,
\end{aligned}$$

and

$$\left|\Phi\left(a(t, x(t), x'(t)) \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right)\right| \leq \Phi\left(M \int_{-\infty}^{+\infty} \phi_r(s)ds\right) \leq r_4.$$

Since Φ^{-1} is uniformly continuous on $[-r_4, r_4]$, for any $\epsilon > 0$, there exists $\delta_1 > 0$ such that

$$|\Phi^{-1}(x) - \Phi^{-1}(y)| < \epsilon, \ x, y \in [-r_4, r_4], |x - y| < \delta_1.$$

One sees that

$$\begin{aligned}
&\left|\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{t} f(u, x(u), x'(u))du\right. \\
&\left. -\Phi\left(a(t, x(t,), x'(t)) \int_{-\infty}^{+\infty} g(s, x(s), x(s), x'(s))ds\right)\right| \\
\leq &\int_{-\infty}^{t} \phi_r(s)ds + \left|\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right)\right. \\
&\left. -\Phi\left(a(t, x(t,), x'(t)) \int_{-\infty}^{+\infty} g(s, x(s), x(s), x'(s))ds\right)\right|.
\end{aligned}$$

Since

$$\left|a(t, x(t,), x'(t)) \int_{-\infty}^{+\infty} g(s, x(s), x(s), x'(s))ds\right| \leq M \int_{-\infty}^{+\infty} \phi_r(s)ds =: r_3,$$

and Φ is uniformly continuous on $[-r_3, r_3]$, we get that there exists $\delta_2 > 0$ such that

$$|\Phi(u) - \Phi(v)| < \frac{\delta_1}{2},\ v, u \in [-r_3, r_3], |u - v| < \delta_2.$$

It is easy to see that there exists $T_1 > 0$ such that

$$\int_{-\infty}^{t} \phi_r(s)ds < \frac{\delta_1}{2},\ t < -T_1.$$

Since $a(t, x(t), x'(t)) \to a_-$ uniformly as $t \to -\infty$, we get that there exists $T_2 > 0$ such that

$$\left|a_- - a(t, x(t), x'(t))\right| < \frac{\delta_2}{\int_{-\infty}^{+\infty} \phi_r(s)ds}$$

holds uniformly for $t < -T_2$. Then $t < -T_2$ implies that

$$\left|a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds - a(t, x(t), x'(t)) \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right|$$

$$\leq |a(t, x(t), x'(t)) - a_-| \int_{-\infty}^{+\infty} \phi_r(s)ds < \delta_2.$$

Hence

$$\begin{aligned} &\left|\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{t} f(u, x(u), x'(u))du \right. \\ &\left. -\Phi\left(a(t, x(t,), x'(t)) \int_{-\infty}^{+\infty} g(s, x(s), x(s), x'(s))ds\right)\right| \\ < \quad & \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1,\ t < -\max\{T_1, T_2\}. \end{aligned}$$

Then

$$\left|\rho(t)(Tx)'(t) + \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right| < \frac{1}{m}\epsilon,\ t < \max\{T_1, T_2\}. \tag{16}$$

It follows from (2.15) and (2.16) that $\frac{T\Omega}{1+\tau(t)}$ and $\{\rho(t)(Tx)' : x \in \Omega\}$ are equiconvergent at $-\infty$.

(iv) We show that both $\frac{1}{1+\tau(t)}T\Omega$ and $\{\rho(t)Tx)' : x \in \Omega\}$ are equi-convergent at $t = +\infty$.

Similarly we can show that

$$\frac{(Tx)(t)}{1 + \tau(t)} \to \int_{-\infty}^{+\infty} h(u, x(u), x'(u))ds \text{ uniformly as } t \to +\infty$$

and

$$\rho(t)(Tx)'(t)$$

$$\to -\frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x_n(u), x_n'(u))ds\right) + \int_{-\infty}^{+\infty} f(u, x_n(u), x_n'(u))du\right)}{a_+}$$

uniformly as $t \to +\infty$. So $\frac{1}{1+\tau(t)}T\Omega$ and $\{\rho(t)Tx)' : x \in \Omega\}$ are equi-convergent at $t = +\infty$.

Above all discussion implies that $T : X \to X$ is completely continuous. The proof is completed. ∎

3. MAIN RESULTS

In this section, we present the main theorem and its proof. Let Φ be a sup-multiplicative-like function with its supporting function ω. The inverse function of Φ is denoted by Φ^{-1}, whose supporting function is denoted by ν.

For nonnegative functions $\varphi, \psi_1, \psi_2 \in L^1(-\infty, +\infty)$, and $a > 0$, define

$$n = \int_{-\infty}^{+\infty} \psi_2(s)ds + \frac{1}{m}\nu\left(\frac{1}{\omega\left(\frac{1}{a_- \int_{-\infty}^{+\infty} \psi_1(s)ds}\right)} + \int_{-\infty}^{+\infty} \varphi(s)ds\right).$$

Theorem 3.1. *Suppose that there exists a constant $a > 0$ such that*

(H1) *the inequalities*

$$f\left(t, (1+\tau(t))x_1, \frac{1}{\rho(t)}y_1\right) \le f\left(t, (1+\tau(t))x_2, \frac{1}{\rho(t)}y_1\right),$$
$$g\left(t, (1+\tau(t))x_1, \frac{1}{\rho(t)}y_1\right) \le g\left(t, (1+\tau(t))x_2, \frac{1}{\rho(t)}y_1\right),$$
$$h\left(t, (1+\tau(t))x_1, \frac{1}{\rho(t)}y_1\right) \le h\left(t, (1+\tau(t))x_2, \frac{1}{\rho(t)}y_1\right)$$

hold for all

$$t \in (-\infty, +\infty), 0 \le x_1 \le x_2 \le a, |y_1| \le |y_2| \le a.$$

(H2) *there exist nonnegative functions φ, ψ_1, ψ_2 such that $\varphi, \psi_1, \psi_2 \in L^1(0, \infty)$ and*

$$f\left(t, (1+\tau(t))a, \frac{1}{\rho(t)}a\right) \le \Phi\left(\frac{a}{n}\right)\varphi(t),$$
$$g\left(t, (1+\tau(t))a, \frac{1}{\rho(t)}a\right) \le \frac{a}{n}\psi_1(t),$$
$$h\left(t, (1+\tau(t))a, \frac{1}{\rho(t)}a\right) \le \frac{a}{n}\psi_2(t)$$

for all $t \in (-\infty, +\infty)$.

Then BVP(1.6) has two positive solutions w^* and v^* such that $0 < \|w^*\| \le a$ and $\lim_{n\to\infty} w_n = \lim_{n\to\infty} T^n w_0 = w^*$ with

$$w_0(t) = a(1+\tau(t)),\ t \in (0, \infty),$$

and $0 < \|v^*\| \le a$ *with* $\lim_{n\to\infty} v_n = \lim_{n\to\infty} T^n v_0 = w^*$ *and* $v_0(t) \equiv 0$ *on* $(0, \infty)$.

Proof. By Lemmas 2.1 and 2.2, we know that $T : P \to P$ is completely continuous. Denote

$$\overline{P}_a = \{x \in P : \|x\| \le a\}.$$

Step 1. We prove that $T : \overline{P}_a \to \overline{P}_a$. If $x \in \overline{P}_a$, then $\|x\| \le a$. We have

$$r = \max\left\{\sup_{t\in(-\infty,+\infty)} \frac{|x(t)|}{1+\tau(t)}, \sup_{t\in(-\infty,+\infty)} \rho(t)|x'(t)|\right\} < +\infty.$$

From (H2) and (H1), we have

$$\begin{aligned} f(t, x(t), x'(t)) &= f\left(t, (1+\tau(t))\tfrac{x(t)}{1+\tau(t)}, \tfrac{1}{\rho(t)}\rho(t)x'(t)\right) \\ &\le f\left(t, (1+\tau(t))a, \tfrac{1}{\rho(t)}a\right) \\ &\le \Phi\left(\tfrac{a}{n}\right)\varphi(t), \ t \in \mathbb{R}. \end{aligned}$$

Similarly we have

$$g(t, x(t), x'(t)) \le \frac{a}{n}\psi_1(t), \quad h(t, x(t), x'(t)) \le \frac{a}{n}\psi_2(t), t \in \mathbb{R}.$$

Hence, we get from (2.7) and (2.8) that

$$\begin{aligned} \frac{|(Tx)(t)|}{1+\tau(t)} &= \frac{1}{1+\tau(t)}\left|\int_{-\infty}^{+\infty} h(u, x(u), x'(u))du\right. \\ &\left. + \int_t^{+\infty} \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, x(u), x'(u))ds\right) + \int_{-\infty}^{s} f(u, x(u), x'(u))du\right)}{a(s, x(s,), x'(s))\rho(s)} ds\right| \\ &\le \frac{1}{1+\tau(t)} \int_{-\infty}^{+\infty} \frac{a}{n}\psi_2(s)ds \\ &+ \frac{1}{1+\tau(t)} \int_t^{+\infty} \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} \frac{a}{n}\psi_1(s)ds\right) + \int_{-\infty}^{s} \Phi\left(\frac{a}{n}\right)\varphi(s)ds\right)}{m\rho(s)} ds \end{aligned}$$

$$\leq \frac{a}{n}\int_{-\infty}^{+\infty}\psi_2(s)ds$$

$$+\frac{1}{1+\tau(t)}\int_t^{+\infty}\frac{1}{m\rho(s)}ds\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\frac{a}{n}\psi_1(s)ds\right)+\int_{-\infty}^{+\infty}\Phi\left(\frac{a}{n}\right)\varphi(s)ds\right)$$

$$\leq \frac{a}{n}\int_{-\infty}^{+\infty}\psi_2(s)ds+\frac{1}{m}\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\frac{a}{n}\psi_1(s)ds\right)+\int_{-\infty}^{+\infty}\Phi\left(\frac{a}{n}\right)\varphi(s)ds\right)$$

$$\leq \frac{a}{n}\int_{-\infty}^{+\infty}\psi_2(s)ds+\frac{1}{m}\Phi^{-1}\left(\frac{\Phi\left(\frac{a}{n}\right)}{\omega\left(\frac{1}{a_-\int_{-\infty}^{+\infty}\psi_1(s)ds}\right)}+\int_{-\infty}^{+\infty}\Phi\left(\frac{a}{n}\right)\varphi(s)ds\right)$$

$$=\frac{a}{n}\int_{-\infty}^{+\infty}\psi_2(s)ds+\frac{1}{m}\Phi^{-1}\left(\Phi\left(\frac{a}{n}\right)\left(\frac{1}{\omega\left(\frac{1}{a_-\int_{-\infty}^{+\infty}\psi_1(s)ds}\right)}+\int_{-\infty}^{+\infty}\varphi(s)ds\right)\right)$$

$$\leq \frac{a}{n}\int_{-\infty}^{+\infty}\psi_2(s)ds+\frac{1}{m}\frac{a}{n}\nu\left(\frac{1}{\omega\left(\frac{1}{a_-\int_{-\infty}^{+\infty}\psi_1(s)ds}\right)}+\int_{-\infty}^{+\infty}\varphi(s)ds\right)$$

$$=a.$$

Furthermore, we have

$$\begin{aligned}
&\rho(t)|(Tx)'(t)| \\
=&\left|-\frac{\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}g(u,x(u),x'(u))ds\right)+\int_{-\infty}^{t}f(u,x(u),x'(u))du\right)}{a(t,x(t),x'(t))}\right| \\
\leq&\frac{\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\frac{a}{n}\psi_1(s)ds\right)+\int_{-\infty}^{+\infty}\Phi\left(\frac{a}{n}\right)\varphi(s)ds\right)}{m} \\
\leq&\frac{\int_{-\infty}^{+\infty}\frac{a}{n}\psi_2(s)ds+\int_{-\infty}^{+\infty}\frac{a}{n}\psi_1(s)ds}{\int_{-\infty}^{+\infty}\frac{dr}{\rho(r)}} \\
&+\frac{\int_t^{+\infty}\int_s^{+\infty}\frac{dr}{\rho(r)}\frac{a}{n}\varphi(s)ds+\int_{-\infty}^{t}\int_{-\infty}^{s}\frac{dr}{\rho(r)}\frac{a}{n}\varphi(s)ds}{\int_{-\infty}^{+\infty}\frac{dr}{\rho(r)}} \\
\leq&\frac{\int_{-\infty}^{+\infty}\frac{a}{n}\psi_2(s)ds+\int_{-\infty}^{+\infty}\frac{a}{n}\psi_1(s)ds}{\int_{-\infty}^{+\infty}\frac{dr}{\rho(r)}}+\int_{-\infty}^{+\infty}\frac{a}{n}\varphi(s)ds \\
\leq&\frac{1}{m}\Phi^{-1}\left(\frac{\Phi\left(\frac{a}{n}\right)}{\omega\left(\frac{1}{a_-\int_{-\infty}^{+\infty}\psi_1(s)ds}\right)}+\int_{-\infty}^{+\infty}\Phi\left(\frac{a}{n}\right)\varphi(s)ds\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{m}\Phi^{-1}\left(\Phi\left(\frac{a}{n}\right)\left(\frac{1}{\omega\left(\frac{1}{a_-\int_{-\infty}^{+\infty}\psi_1(s)ds}\right)}+\int_{-\infty}^{+\infty}\varphi(s)ds\right)\right)\\
&\leq \frac{1}{m}\frac{a}{n}\nu\left(\frac{1}{\omega\left(\frac{1}{a_-\int_{-\infty}^{+\infty}\psi_1(s)ds}\right)}+\int_{-\infty}^{+\infty}\varphi(s)ds\right)\\
&\leq a.
\end{aligned}
$$

Hence we have show that $T : \overline{P}_a \to \overline{P}_a$.

Step 2. We establish iterative schemes for approximating the solutions.

Choose

$$w_0(t) = a(1+\tau(t)).$$

(i) one sees easily that $\|w_0\| \leq a$. Then $w_0 \in \overline{P}_a$.

(ii) Let $w_1 = Tw_0$ and $w_2 = Tw_1$. Then above discussion implies that $w_1, w_2 \in \overline{P}_a$. We denote $w_{n+1} = Tw_n = T^n w_0$ for $n = 1, 2, \cdots$. Since $T : \overline{P}_a \to \overline{P}_a$, we have $w_n \in \overline{P}_a$ for all $n = 1, 2, 3, \cdots$. It follows from the complete continuity of T that $\{w_n : n = 0, 1, 2, 3, \cdots\}$ is a sequentially compact set.

(iii) We prove that

$$w_{n+1} \leq w_n, \quad n = 1, 2, 3, \cdots.$$

By (H1) and (H2), we get

$$
\begin{aligned}
0 \leq \tfrac{(Tw_0)(t)}{1+\tau(t)} &= \tfrac{1}{1+\tau(t)}\int_{-\infty}^{+\infty} h(u,\omega_0(u),\omega_0'(u))du\\
&+\tfrac{1}{1+\tau(t)}\int_t^{+\infty}\frac{\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty} g(u,\omega_0(u),\omega_0'(u))ds\right)+\int_{-\infty}^{s} f(u,\omega_0(u),\omega_0'(u))du\right)}{a(s,\omega_0(s,),\omega_0'(s))\rho(s)}ds\\
&\leq \tfrac{1}{1+\tau(t)}\int_{-\infty}^{+\infty}\tfrac{a}{n}\psi_2(s)ds\\
&+\tfrac{1}{1+\tau(t)}\int_t^{+\infty}\frac{\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\frac{a}{n}\psi_1(s)ds\right)+\int_{-\infty}^{s}\Phi\left(\frac{a}{n}\right)\varphi(s)ds\right)}{m\rho(s)}ds\\
&\leq \tfrac{a}{n}\int_{-\infty}^{+\infty}\psi_2(s)ds\\
&+\tfrac{1}{1+\tau(t)}\int_t^{+\infty}\tfrac{1}{m\rho(s)}ds\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\tfrac{a}{n}\psi_1(s)ds\right)+\int_{-\infty}^{+\infty}\Phi\left(\tfrac{a}{n}\right)\varphi(s)ds\right)\\
&\leq \tfrac{a}{n}\int_{-\infty}^{+\infty}\psi_2(s)ds+\tfrac{1}{m}\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\tfrac{a}{n}\psi_1(s)ds\right)+\int_{-\infty}^{+\infty}\Phi\left(\tfrac{a}{n}\right)\varphi(s)ds\right)
\end{aligned}
$$

$$\leq \frac{a}{n}\int_{-\infty}^{+\infty}\psi_2(s)ds + \frac{1}{m}\Phi^{-1}\left(\frac{\Phi(\frac{a}{n})}{\omega\left(\frac{1}{a_-\int_{-\infty}^{+\infty}\psi_1(s)ds}\right)} + \int_{-\infty}^{+\infty}\Phi\left(\frac{a}{n}\right)\varphi(s)ds\right)$$

$$= \frac{a}{n}\int_{-\infty}^{+\infty}\psi_2(s)ds + \frac{1}{m}\Phi^{-1}\left(\Phi\left(\frac{a}{n}\right)\left(\frac{1}{\omega\left(\frac{1}{a_-\int_{-\infty}^{+\infty}\psi_1(s)ds}\right)} + \int_{-\infty}^{+\infty}\varphi(s)ds\right)\right)$$

$$\leq \frac{a}{n}\int_{-\infty}^{+\infty}\psi_2(s)ds + \frac{1}{m}\frac{a}{n}\nu\left(\frac{1}{\omega\left(\frac{1}{a_-\int_{-\infty}^{+\infty}\psi_1(s)ds}\right)} + \int_{-\infty}^{+\infty}\varphi(s)ds\right) = a.$$

Then

$$\omega_1(t) = (T\omega_0)(t) \leq (1+\tau(t))a = \omega_0(t).$$

Furthermore, we have

$$\begin{aligned}
\rho(t)|(Tw_0)'(t)| &\leq \frac{\Phi^{-1}\left(\Phi\left(a_-\int_{-\infty}^{+\infty}\frac{a}{n}\psi_1(s)ds\right) + \int_{-\infty}^{+\infty}\Phi\left(\frac{a}{n}\right)\varphi(s)ds\right)}{m} \\
&\leq \frac{\int_{-\infty}^{+\infty}\frac{a}{n}\psi_2(s)ds + \int_{-\infty}^{+\infty}\frac{a}{n}\psi_1(s)ds}{\int_{-\infty}^{+\infty}\frac{dr}{\rho(r)}} \\
&\quad + \frac{\int_t^{+\infty}\int_s^{+\infty}\frac{dr}{\rho(r)}\frac{a}{n}\varphi(s)ds + \int_{-\infty}^{t}\int_{-\infty}^{s}\frac{dr}{\rho(r)}\frac{a}{n}\varphi(s)ds}{\int_{-\infty}^{+\infty}\frac{dr}{\rho(r)}} \\
&\leq \frac{\int_{-\infty}^{+\infty}\frac{a}{n}\psi_2(s)ds + \int_{-\infty}^{+\infty}\frac{a}{n}\psi_1(s)ds}{\int_{-\infty}^{+\infty}\frac{dr}{\rho(r)}} + \int_{-\infty}^{+\infty}\frac{a}{n}\varphi(s)ds \\
&\leq \frac{1}{m}\Phi^{-1}\left(\frac{\Phi\left(\frac{a}{n}\right)}{\omega\left(\frac{1}{a_-\int_{-\infty}^{+\infty}\psi_1(s)ds}\right)} + \int_{-\infty}^{+\infty}\Phi\left(\frac{a}{n}\right)\varphi(s)ds\right) \\
&= \frac{1}{m}\Phi^{-1}\left(\Phi\left(\frac{a}{n}\right)\left(\frac{1}{\omega\left(\frac{1}{a_-\int_{-\infty}^{+\infty}\psi_1(s)ds}\right)} + \int_{-\infty}^{+\infty}\varphi(s)ds\right)\right)
\end{aligned}$$

$$\leq \frac{1}{m}\frac{a}{n}\nu\left(\frac{1}{\omega\left(\frac{1}{a_-\int_{-\infty}^{+\infty}\psi_1(s)ds}\right)} + \int_{-\infty}^{+\infty}\varphi(s)ds\right) \leq a.$$

Then

$$\rho(t)|(Tw_0)'(t)| \le a = \rho(t)|\omega_0'(t)|,\ |\omega_1'(t)| = |(Tw_0)'(t)| \le |\omega_0'(t)|.$$

Hence

$$\omega_1 \le \omega_0.$$

By the monotonic property of f, g, and h and the definition of T, we get

$$w_2(t) = (Tw_1)((t) \le (Tw_0)(t) = w_1(t), |\omega_2'(t)| = |(Tw_1)'(t)| \le |\omega_1'(t)|\ t \in (0, \infty).$$

So $\omega_2 \le \omega_1$. By induction, we get

$$w_{n+1} \le w_n,\ n = 1, 2, 3, \cdots .$$

Thus there exists $w^* \in \overline{P}_a$ such that $w_n \to w^*$ as $n \to \infty$. Applying the continuity of T and $w_{n+1} = Tw_n$, we get that $w^* = Tw^*$.

Choose

$$v_0(t) = 0 \text{ for all } t \in (-\infty, \infty).$$

Then $v_0 \in \overline{P}_a$. Let $v_1 = Tv_0$ and $v_2 = Tv_1$. One can prove that $v_1 \in \overline{P}_a$ and $v_2 \in \overline{P}_a$. We denote $v_{n+1} = Tv_n = T^n v_0$ for $n = 1, 2, 3, \cdots$. Since $T : \overline{P}_a \to \overline{P}_a$, we have $v_n \in \overline{P}_a$. for all $n = 1, 2, 3, \cdots$. It follows from the complete continuity of T that $\{v_n\}$ is a sequentially compact set.

Since $v_1 = Tv_0 \in \overline{P}_a$, we have

$$v_1(t) = (Tv_0)(t) \ge 0 \equiv v_0,\ t \in (-\infty, \infty).$$

On the other hand, we have

$$\begin{aligned} & \rho(t)|v_1'(t)| = \rho(t)|(Tv_0)'(t)| \\ = & \frac{\Phi^{-1}\left(\Phi\left(a_- \int_{-\infty}^{+\infty} g(u, v_0(u), v_0'(u))ds\right) + \int_{-\infty}^{t} f(u, v_0(u), v_0'(u))du\right)}{a(t, v_0(t), v_0'(t))} \\ \ge & \ 0 = \rho(t)|v_0'(t)|,\ \ t \in (-\infty, \infty). \end{aligned}$$

It follows that $|v_1'(t)| \ge |v_0'(t)|,\ t \in (-\infty, \infty)$. Then $v_1 \ge v_0$. By induction, we get

$$v_{n+1} \ge v_n,\ \ n = 1, 2, 3, \cdots .$$

Thus there exists $v^* \in \overline{P}_a$ such that $v_n \to v^*$ as $n \to \infty$. Applying the continuity of T and $v_{n+1} = Tv_n$, we get that $v^* = Tv^*$.

Since $f(t, 0, 0) \not\equiv 0$, we see that the zero function is not the solution of BVP(1.6). Thus both w^* and v^* are positive solution of the operator equation $x = Tx$ in $\overline{P}_a$.

It is well known that each fixed point of T in P is a solution of BVP(1.6). Hence w^* and v^* are two positive solutions of BVP(1.6). The proof is complete. ∎

Remark 3.1. *The iterative schemes in Theorem 3.1 are* $w_0(t) = a(1 + \tau(t))$ *and* $w_{n+1}(t) = (Tw_n)(t)$ *for* $n = 0, 1, 2, 3, \cdots$ *and* $v_0(t) \equiv 0$, $v_{n+1}(t) = (Tv_n)(t)$ *for* $n = 0, 1, 2, 3, \cdots$ *. They start off with a known simple function and the zero function respectively. This is convenient for application.*

Remark 3.2. *One sees that the ideas of the proof of the main theorem are also obtainable from the metric version of Theorem 3 in [12] (rephrased as Theorem 8 in [13]).*

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