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RADICALS AND EMBEDDINGS OF MOUFANG LOOPS IN ALTERNATIVE LOOP ALGEBRAS

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Abstract The paper defines the notion of alternative loop algebra $F[Q]$ for any nonassociative Moufang loop Q as being any non-zero homomorphic image of the loop algebra FQ of a loop Q over a field F. For the class $\mathcal{M}$ of all nonassociative alternative loop algebras $F[Q]$ and for the class $\mathcal{L}$ of all nonassociative Moufang loops Q are defined the radicals $\mathcal{R}$ and $\mathcal{S}$, respectively. Moreover, for classes $\mathcal{M}$, $\mathcal{L}$ is proved an analogue of Wedderburn Theorem for finite dimensional associative algebras. It is also proved that any Moufang loop Q from the radical class $\mathcal{R}$ can be embedded into the loop of invertible elements $U(F[Q])$ of alternative loop algebra $F[Q]$. The remaining loops in the class of all nonassociative Moufang loops $\mathcal{L}$ cannot be embedded into loops of invertible elements of any unital alternative algebras.

Keywords: Moufang loop, alternative loop algebra, circle loop, loop of invertible elements, radical, analog of Wedderburn Theorem, embedding.

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1. INTRODUCTION

Embedding of a Moufang loop into a loop of invertible elements $U(A)$ of an alternative algebra with unit A (see, for example, [11], [34] is one of the major questions in the Moufang loop theory. In general, the answer to this question is negative [34], [29]. Nevertheless, many authors look into such Moufang loops assuming that they can be embedded into a loop of type $U(A)$ (see, Section 6). The question on embedding a Moufang loop into a loop of type $U(A)$ is fully solved in this paper.

To solve this question (section 6) the notion of alternative loop algebra $F[Q]$ for any Moufang loop Q is introduced. The algebra $F[Q]$ is alternative and it is a nonzero homomorphic image of the loop algebra FQ for the loop Q over a field F (Section 2). Moreover, the radicals $\mathcal{R}$ and $\mathcal{S}$ are introduced for the class $\mathcal{M}$ of all alternative loop algebras $F[Q]$ and for the class $\mathcal{L}$ of all Moufang loops (section 4). It is also introduced the class $\mathcal{R}_A$ into the alternative loop algebras. Let $A \in \mathcal{M}$ and let $A = R^\sharp = R \oplus Fe$, i.e A is obtained by adjoining the exterior unit element e to R. Then $A \in \mathcal{R}_A$ when and only when $R \in \mathcal{R}$. Algebras of type $A = R^\sharp$ are considered in Section 3.

The semisimple classes $\mathcal{P}$ and $\mathcal{S}$ corresponding to radicals $\mathcal{R}$ and $\mathcal{S}$ respectively are considered in section 5.

Proposition 7.3 and Theorem 7.4 are the crucial structure results for the examination of Moufang loops. These statements are similar with Wedderburn Theorem for associative algebras, which is regarded as the beginning of radical theory. In 1908 he proved that every finite dimensional associative algebra is an extension of the direct sum of full matrix algebras over corps with the help of nilpotent algebra.

Proposition 7.3. *Let $F[Q]$ be an alternative loop algebra from the class $\mathcal{M}$ and let $\mathcal{R}(F[Q])$ be its radical. Then algebra $(\mathcal{R}(F[Q]))^\sharp = F[G]$, $G \subseteq Q$, $F[G] \in \mathcal{R}_A$, is nonassociative antisimple with respect to nonassociativity or, equivalently, it does not contain subalgebras that are nonassociative simple algebras and the quotient-algebra $F[Q]/\mathcal{R}(F[Q])$ is a direct sum of Cayley-Dickson algebras over their centre.*

Theorem 7.4. *Let Δ be a prime field, let P be its algebraic closure, and let F be a Galois extension over Δ in P. Then the radical $\mathcal{S}(Q)$ of a Moufang loop Q is nonassociative antisimple with respect to nonassociativity or, equivalently, it does not contain subloops that are nonassociative simple loops and quotient-loop $Q/\mathcal{S}(Q)$ is isomorphic to a direct product of matrix Paige loops $M(F)$.*

For the basic properties of Moufang loops see [2], [5], and of alternative algebras see [36].

The Cayley-Dickson algebras (simple alternative algebras) and the Paige loops (simple Moufang loops) are quite well explored, see [36], [33].

Let $Q \in \mathcal{L}$. By definition $Q \in \mathcal{S}$ if and only if $F[Q] \in \mathcal{R}_A$. Then from Proposition 7.3 and Theorem 7.4 it follows that the construction of nonassociative Moufang loops Q by module of nonassociative simple Moufang loops is limited to the examination of alternative loop algebras from radical class $\mathcal{R}_A$. Such algebras are described in Propositions 4.3, 4.4, Corollary 4.5.

According to Lemma 7.1 and Proposition 7.2 the following statements are equivalent for a nonassociative Moufang loop $Q \in \mathcal{L}$:

r1) $Q \in \mathcal{S}$;

r2) Q is antisimple with respect to nonassociativity of loop;

r3) the loop Q does not have subloops that are simple loops. Then the contrary statements hold:

nr1) $G \notin \mathcal{S}$, i.e. $G \in \mathcal{L}\backslash\mathcal{S}$;

nr2) G are not antisimple with respect to nonassociativity loop;

nr3) the loop G contains subloops that are simple loops hold for any nonassociative Moufang loop $G \in \mathcal{L}\backslash\mathcal{S}$.

From the definition of class of alternative loop algebras $\mathcal{R}_A$, the definition of class of loops $\mathcal{S}$ and Theorem 4.2 it follows that if a nonassociative Moufang loop Q satisfies the condition r1) then the loop Q can be embedded into the loop of invertible elements $U(F[Q])$ of alternative loop algebra $F[Q]$. On the other hand, in [29], it was proved that if a nonassociative Moufang loop G satisfies the condition nr2) then the loop Q is not imbedded into the loop of invertible elements $\mathcal{U}(A)$ for a suitable unital alternative F-algebra A, where F is an associative commutative ring with unit. As

$\mathcal{S} \bigcap (\mathcal{L} \backslash \mathcal{S}) = \varnothing$ then the main result of this paper follows from the above-mentioned statements.

Theorem 7.5. *Any nonassociative Moufang loop Q that satisfies one of the equivalent conditions r1) - r3) can be embedded into a loop of invertible elements $U(F[Q])$ of alternative loop algebra $F[Q]$. The remaining loops in the class of all nonassociative Moufang loops $\mathcal{L}$, i.e. the loops $G \in \mathcal{L}$ that satisfy one of the equivalent conditions nr1 - nr3 cannot be embedded into loops of invertible elements of any unital alternative algebras.*

From Corollary 6.10 and Theorem 7.5 follows

Corollary 7.6. *Any commutative Moufang loop Q can be embedded into a loop of invertible elements $U(F[Q])$ of alternative loop algebra $F[Q]$.*

Recently a series of papers have been published, which look into the Moufang loops with the help of the powerful instrument of group theory, in particular finite group theory (see, for example, [7], [10], [9], [13], [16]). For this purpose the correspondence between Moufang loops and groups with triality [7] is used. The proofs based on the correspondence are complex and cumbersome.

This paper offers another simpler approach method: to use the Theorem 7.5, Corollary 7.6 instead of the the correspondence between Moufang loops and groups with triality. Such examples are presented into the end of section 6 and section 7. In section 7 this is proved on the basis of Theorem 7.5 of the known results from [9]: every finite Moufang p-loop is centrally nilpotent. Paper [9] introduces the notion of group with triality. We note that Theorem 7.5 was also used in [32] for proving the next statements.

If three elements a, b, c of Moufang loop Q are tied by the associative law $ab \cdot c = a \cdot bc$, then they generate an associative subloop (Moufang Theorem).

The intersection of the terms of the lower central series of a free Moufang loop $L_X(\mathfrak{M})$ is the unit loop.

Any finitely generated free Moufang loop is Hopfian.

We will examine only nonassociative Moufang loops and nonassociative alternative algebras over a fixed field F. Particularly, the alternative loop algebra $F[Q]$ corresponding to nonassociative loop Q is nonassociative. If the loop Q is commutative then algebra $F[Q]$ is also commutative. Then, in the commutative case, we will consider that char $F = 0$ or 3 as there are no nonassociative commutative alternative algebras over fields of characteristic $\neq 0; 3$ [36].

Any algebra A with unit e is always considered nontrivial by definition, therefore all such algebras A contain one dimensional central subalgebra $Fe = \{\alpha a | \alpha \in F\}$ (with the same unit $e \neq 0$, which allows to identify Fe and F).

If J is an ideal of algebra A and the quotient algebra is an algebra with unit e, then J is a proper ideal of A ($J \neq A$) and $e \notin J$. Besides, by definition, the homomorphisms of algebras with unit e is always *unital*, i.e. keep the unit. Hence, if $\varphi : A \to B$

is a homomorphism of algebras with unit e, then $\ker\varphi$ is a proper ideal of A, as $\varphi e = e \neq 0$.

Let A be an associative algebra. By $I(A)$ we denote the set of such elements $u \in A$ that

$$u + v + uv = 0, \quad u + v + vu = 0 \tag{1}$$

for some $v \in A$ and by $J(A)$ we denote the set of all *quasiregular elements* of A, i.e. the set of such elements $a \in A$ that

$$a + b - ab = 0, \quad a + b - ba = 0 \tag{2}$$

for some $b \in A$. In the past, almost simultaneously, with various goals on the elements of the set $I(A)$ different authors (see., for example, [8], [18], [24]) have introduced the group operation ($\otimes$):

$$u \otimes v = uv + u + v. \tag{3}$$

However, at present the so-called circle operation ($\circ$):

$$a \circ b = a + b - ab \tag{4}$$

on the elements of the set $J(Q)$, as in such a case the strong instrument of the theory of quasiregular associative algebras (see., for example, [15], [25]) can be used to consider operation ($\circ$) (we mentioned that this paper is influenced by [25]).

The operations ($\otimes$) and ($\circ$) on the sets $I(A)$ and $J(A)$ respectively are groups. Thus, this paper established by analogy a link between alternative algebras and Moufang loops. In [12] this link is established with the help of the operation, defined by (3), but we believe it is not successful. In such a form it is impossible to use the developed theory of quasiregular alternative algebras, though in [12] the elements defined by (1) are wrongly called quasiregular. According to [36] a quasiregular alternative algebra can be characterized (defined) as alternative algebra A satisfying the property that the set A form a loop with respect to circle operation ($\circ$), defined by (4). Then the set $J(A) \subseteq A$ define the Zhevlakov (quasiregular) radical of alternative algebra A, the analog of Jacobson radical of associative algebra theory. In [12] the notion of Zhevlakov radical is defined with respect to operation ($\otimes$) that not correspond [36]. However, the loops $(I(A), \otimes)$ and $(J(A), \circ)$ are isomorphic by Corollary 2.7.

Let A be an alternative algebra. Unlike [12], this paper establishes a link between alternative algebras and Moufang loops with the help of relation (4). In such a case the developed theory of quasiregular alternative algebras can be used. For example, in [12, Theorem 1], it is quite cumbersomely proved that groupoid $(I(A), \otimes)$ is a Moufang loop, but this paper proves such a result for $(J(A), \circ)$ (Proposition 2.4) quite easily. We will call loop $(J(A), \circ)$ *circle loop* of algebra $J(A)$. The paper also gives a full answer to the modified question from [12] about embedding a Moufang loop into circle loop of a suitable alternative algebra (Corollaries 7.8, 7.9).

2. CIRCLE MOUFANG LOOPS

Let A be an algebra over a field F. Let's consider that the field F is a module over itself. The unit e of F is the generating element of the F-module Fe. We consider the direct sum $A^\sharp = A \oplus Fe$ of the modules A and Fe and define on it the multiplication:

$$(a + \alpha \cdot e)(b + \beta \cdot e) = (ab + \alpha b + \beta a) + \alpha\beta \cdot e$$

where $a, b \in A,\ \alpha, \beta \in F$. It is easy to see that e is the unit of algebra $A^\sharp$ and A is an ideal of $A^\sharp$. $A^\sharp$ is called *the algebra obtained by adjoining the exterior unit element e to A*.

Consequently, it is always possible to pass, from any algebra R to algebra $R^\sharp = R \oplus F \cdot e$ with externally attached unit e and R be an ideal of algebra $R^\sharp$. In general, it is not always possible to restore algebra R from the algebra $R^\sharp$: it is possible that $R_1^\sharp = A = R_2^\sharp$, though the algebras R_1 and R_2 are not isomorphic. However, if the algebras are given R and $A = R^\sharp$ then for any algebra B with unit ϵ every homomorphism $\varphi : R \to B$ unequivocally proceeds up to homomorphism $\varphi : A \to B$ by rule: $\varphi(\alpha e + r) = \alpha\epsilon + \varphi r$. Particularly, the homomorphism $\pi : A \to F$, defined by $\pi(\alpha e + r) = \alpha$, will be the only unital homomorphism of algebra $A = R^\sharp$ in algebra $F \equiv Fe$, continuing the null homomorphism of algebra R. Moreover, hold.

Lemma 2.1. *An algebra A with unit e will be an algebra with externally adjoined unit (i.e. $A = R^\sharp$ for some algebra R) when and only when there exists a homomorphism $\pi : A \to \pi A = F$ of algebra A. In such a case $A = R^\sharp = R \oplus Fe$, where $R = \ker \pi$.*

Proof. For the homomorphism π with $\ker \pi = R$ we have $A/R \cong F$. Besides, π is identical on $Fe \equiv F$ and according to decomposition $a = \pi a + (a - \pi a) = \alpha e + r$ of elements $a \in A$ and equality $\pi A = Fe$ we have $A = R^\sharp$. On the other hand, $R \bigcap Fe = 0$ for any proper ideal R and hence if $A = R^\sharp = R \oplus Fe$ then $A \to A/R \cong F$ will be the only unital homomorphism $R^\sharp \to F$, continuing the null homomorphism of algebra R. This completes the proof of Lemma 2.1. ∎

An alternative algebra is an algebra in which $x \cdot xy = x^2 y$ and $yx \cdot x = yx^2$ are identities. Any alternative algebra satisfies *the Moufang identity*

$$(x \cdot yx)z = x(y \cdot xz). \tag{5}$$

The loop, satisfying the identity (5), is called *Moufang loop*.

Let A be an alternative algebra with unit e. The element $a \in A$ is said to have an *inverse*, if there exists an element $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = e$. It is well known that for an alternative algebra A with the unit e the set $U(A)$ of all invertible elements of A forms a Moufang loop with respect to multiplication [21].

Lemma 2.2 [36]. *Let A be an alternative algebra. Then, the following statements are equivalent:*

a) the elements a and b are invertible;
b) the elements ab and ba are invertible.

The element a of alternative algebra A is called *quasiregular* if it satisfies the relation (2). The element b of (2) is called *quasiinverse* of a. An alternative algebra is called *quasiregular* if any of its elements is quasiregular.

Lemma 2.3 [36]. *The following statements are equivalent:*
a) the element a of the alternative algebra A is quasiregular with quasiinverse b;
b) the element $e - a$ of the algebra $A^\sharp$ is inverse with the inverse element $e - b$.

Quasiregularity is a fundamental concept in algebra theory because it allows to define one of the most important radicals. An ideal is called *quasiregular* if it consists entirely of quasiregular elements. Every alternative algebra A has the largest quasiregular ideal $J(A)$ such that $A/J(A)$ has no non-zero quasiregular ideals. This ideal $J(A)$ is called the *Zhevlakov radical* and it is, of course, like the Jacobson radical of associative algebra theory [36].

Proposition 2.4. *Let A be an alternative algebra and let $J(A)$ be its Zhevlakov radical. Then the set $J(A)$ forms a Moufang loop with respect to operation $x \circ y = x + y - xy$.*

Proof. We suppose that x and y are quasiregular elements with quasiinverses a and b respectively. We denote $u = e - x, v = e - y$ where e is the unit of algebra $A^\sharp$. From Lemma 2.3 it follows that $u^{-1} = e - a,\ v^{-1} = e - b$, where $uu^{-1} = e,\ vv^{-1} = e$, and from Lemma 2.2 it follows that $(uv)(v^{-1}u^{-1}) = e$. From here we get that $((e-x)(e-y))((e-b)(e-a)) = e, (e-x-y+xy)(e-b-a+ba) = e, (e-x\circ y)(e-b\circ a) = e$. Hence the element $e - x \circ y$ is inverse with the element $e - b \circ a$ and from Lemma 2.3 it follows that $x \circ y$ is quasiregular with quasiinverse $b \circ a$. Consequently, the set $J(A)$ is closed under the operation $(\circ)$.

It is easy to see that the 0 element of A is an unit for $(\circ)$. To prove that the set $J(A)$ forms a loop under $(\circ)$, it sufficient to show that $(x \circ y) \circ b = x$ and similarly that $a \circ (x \circ y) = y$. Indeed, by Lemma 10.5 from [6]?? $xy \cdot b = x \cdot yb$ and as $-yb + y + b = 0$ then $(x \circ y) \circ b = (-xy + x + y) \circ b = xy \cdot b - xb - yb - xy + x + y + b = xy \cdot b - xb - xy + x = x \cdot yb - xb - xy + x = x(yb - b - y) + x = x \cdot 0 + x = x$. In this manner it is proved that $a \circ (x \circ y) = y$. Hence $(J(A), \circ)$ is a loop.

Finally, in order to prove the validity of Moufang identity (5) in the loop $(J(A), \circ)$ it is sufficient to evaluate the difference $((x \circ y) \circ x) \circ z) - x \circ (y \circ (x \circ z))$ by (4) and by using the identity (5) for the algebra A, diassociativity of A and the identity $xy \cdot z + yx \cdot z = x \cdot yz + y \cdot xz$ obtained through linearization of algebra identity $xx \cdot z = x \cdot xz$. As a result we have obtained that this difference is 0. This completes the proof of Proposition 2.4. ∎

Corollary 2.5. *Let $J(A)$ be the Zhevlakov radical of the alternative algebra A. Then the Moufang loop $(J(A), \circ)$ is isomorphic to $(I(A), \otimes)$.*

Proof. Let $x \in J(A)$. $J(A)$ is a module, then $-x \in J(A)$. Let $-y$ be a quasiinverse for $-x$. By (2) we have $(-x) + (-y) - (-x)(-y) = 0, -x - y - xy = 0, -(x + y + xy) = 0, x + y + xy = 0$. Hence by (1) $x \in I(A)$, i.e. $J(A) \subseteq I(A)$. Inversely, let $x \in I(A)$ and let $x + y + xy = 0$. Then $(-x) + (-y) - (-x)(-y) = 0$, and by (1) $-x \in J(A)$, $x \in J(A)$, i.e. $I(A) \subseteq J(A)$. Hence $J(A) = I(A)$.

Now for $x \in J(A)$ we define $\varphi(x) = -x$. φ is a one-to-one map of $J(A)$ onto $I(A)$. Moreover,

$$\varphi(x \circ y) = -(x + y - xy) = -x + (-y) + (-x)(-y) = \varphi(x) \otimes \varphi(y),$$

so φ is an isomorphism of $(J(A), \circ)$ onto $(I(A), \otimes)$, as required. ∎

Further the Moufang loop $(J(A), \circ)$, considered in Proposition 2.4 will be called the *circle loop of algebra* A and denoted by $U^*(A)$. If A contains the unit e then the correspondence $e - a \to a$ maps the multiplicative loop of simple inverse elements of A upon $U^*(A)$ and, in this case, the circle operation does not offer anything new. Therefore, we will assume further that algebra A is without unit element.

Let now A be an arbitrary alternative algebra with externally adjoined unit e, i.e. by Lemma 2.1 with clearly distinguished one-dimensional subalgebra $Fe \equiv F$ with same unit e. We define the mapping $\eta : A \to A$ by the rule $\eta a = e - a \in A$ for any $a \in A$ (particularly, $\eta e = 0 \neq e = \eta 0$). Then, from the definition of circle loop (Proposition 2.4), we get the equality

$$(e - a)(e - b) = e - a \circ b, \tag{6}$$

which, by replacing of type $x \to \eta x$, it is rewritten as follows:

$$(e - a) \circ (e - b) = e - ab. \tag{7}$$

Obviously, $\eta = \eta^{-1}$. By replacing of type $c \to e - c$ in Lemma 2.3, we get that an element $a \in A$ is invertible if and only if the element $e - a \in A$ is quasiregular. Then from (6), (7) and Proposition 2.4 it follows that η is an isomorphism, which connects the group of invertible elements $U(A)$ of algebra A with circle group of quasiregular elements $U^*(A)$ by rule

$$U^*(A) = \eta(U(A)) = \{a \in A | e - a \in U(A)\}. \tag{8}$$

Hence the rule

$$a \in U^*(A) \to e - (e - a)^{-1} = -a(e - a)^{-1} = a^* \in U^*(A) \tag{9}$$

defines on A, by isomorphism η, the operation $a \to a^*$ *of taking the quasiinverse*, defined on $U^*(A)$ and corresponding precisely to the operation of taking the inverse, defined on $U(A)$, according to equality $a \circ a^* = 0 = a^* \circ a$ and the isomorphism

$$U^*(A) \cong U(A). \tag{10}$$

By definition, an alternative algebra is quasiregular if any of its elements is quasiregular. Consequently, from definitions of Zhevlakov radical $J(A)$ and circle loop $U^*(A)$ it follows that for any alternative algebra A the Zhevlakov radical $J(A)$ is a loop with respect to circle operation $(\circ)$ and

$$J(A) = U^*(A). \tag{11}$$

In particular, an alternative algebra A is quasiregular if and only if the algebra A coincides with its circle loop $U^*(A)$. According to (9), it means that, on algebra A, there also exists the unique operation $x \rightsquigarrow x^*$, of taking the quasiinverse, related with the basic operations of identity $x + x^* = xx^* = x^*x$ (i.e. $r \circ r^* = 0 = r^* \circ r$ for all $r \in A$ by the construction of loop $U^*(A)$).

Hence the class of all quasiregular algebras $\mathcal{K}^*$ form a variety, if considered with an additional unitary operation $x \rightsquigarrow x^*$ of taking the quasiinverse. Then, by Birkhoff Theorem, the class $\mathcal{K}^*$ is closed with respect to the taking of quasiregular subalgebras, of direct product of quasiregular subalgebras and of homomorphic images of homomorphisms of quasiregular subalgebras. But the class $\mathcal{K}^*$ is also closed in respect to usual homomorphisms, i.e. to homomorphisms of algebras. Indeed, the following result holds.

Lemma 2.6. *Let φ be a homomorphism of algebra $A \in \mathcal{K}^*$. Then $\varphi A \in \mathcal{K}^*$.*

Proof. Let $a \in A$. From definition of quasiregular elements it follows that φa is a quasiregular element. Let a^*, $(\varphi a)^*$ be their quasiinverse elements. The homomorphism φ is unital. Hence from (9) it follows that $(\varphi a)^* = \varphi a^*$. Then the homomorphism $\alpha\varphi$ saves the identity $x + x^* = xx^* = x^*x$, distinguishing quasiregular algebras. Consequently, $\varphi A \in \mathcal{K}^*$, as required. ∎

From equalities $ab + (a \circ b) = a + b$ it follows easily that a subspace R of algebra A is it subalgebra then and only then R is a subgroupoid of the groupoid $(A, \circ)$. Hence the *circle subgroupoid* $(R, \circ)$ of circle groupoid $(A, \circ)$, isomorphic to multiplicative subgroupoid $e - R$ of algebra A is linked with any subalgebra R of alternative algebra A by rule

$$e - R = \{e - r | r \in R\} = \eta R. \tag{12}$$

Then, according with (6) - (11), with

$$U^*(R) = \eta(U(e - R)) \tag{13}$$

and with the definition of multiplicative loop of algebra, we get the circle loop $U^*(R)$ of algebra R, which is a subloop of loop $U^*(A)$ and it is isomorphic to multiplicative loop $U(e - R)$ of loop $U(A) \cong U^*(A)$ by (10). In particular, as $F \equiv Fe$ then the multiplicative group $U(F) = F\backslash\{0\}$ of field F and circle group $U^*(F) = F\backslash\{0\}$ are central subloops of loops $U(A)$ and $U^*(A)$, respectively.

From equalities $a + x + ax = x + a + xa = 0$ it follows that the set of all elements of some subalgebra R of algebra A, that have quasiinverses in R, is a subloop of loop $U^*(A)$, $U^*(R) \subseteq U^*(A)$. By (11) an alternative algebra A is quasiregular if A coincides with circle loop $U^*(A)$, $J(A) = U^*(A)$. The Zhevlakov radical $J(A)$ of any alternative algebra A is hereditary, i.e. $J(R) = R \bigcap J(A)$ for any ideal R of A. Hence

$$U^*(R) = R \bigcap U^*(A) = \{r \in R | e - r \in U(A)\}. \tag{14}$$

The following result holds, too.

Proposition 2.7. *Let A be an alternative algebra and R be an ideal of A. Then $U^*(R)$ is a normal subloop of $U^*(A)$ and $U^*(A)/U^*(R) \cong U^*(A/R)$.*

Proof. Let $x, y \in U^*(A), u \in U^*(R)$ and let a, b will be the quasiinverses of x, y respectively. In the proof of Proposition 2.4 it is shown that the element $x \circ y$ is quasiregular with a quasiinverse $b \circ a$. By definition the subloop $U^*(R)$ is normal in $U^*(A)$ if $x \circ U^*(R) = U^*(R) \circ x, x \circ (y \circ U^*(R)) = (x \circ y) \circ U^*(R), (U^*(R) \circ x) \circ y = U^*(R) \circ (x \circ y)$ for any $x, y \in U^*(A)$. Any Moufang loop is an IP-loop, i.e. it satisfies the identities $x^{-1} \cdot xy = y, yx \cdot x^{-1} = y$. Then to show that $U^*(R)$ is normal in $U^*(A)$ it is sufficient to show that $t_1 = (x \circ u) \circ a \in U^*(R)$ and $t_2 = ((u \circ x) \circ y) \circ (b \circ a) \in U^*(R)$. From the aforementioned we have that $t_1, t_2 \in U^*(A)$. Further, $u \in R$, then by (4) $t_1 = xu \cdot a - xa - ua - xu + x + u + a = xu \cdot a - ua - xu + u \in R$ since $-xa + x + a = 0$. We similarly have $t_2 = r - xy \cdot ba + xy \cdot b + xy \cdot a + x \cdot ba - xb + y \cdot ba - ya - xy - ba$, where $r \in R$. We denote $\bar{x} = e - x$, $\bar{y} = e - y$, $\bar{a} = e - a$, $\bar{b} = e - b$, where e is the unit of algebra $A^\sharp$. Let us express t_2 in terms of x, y, a, b over $\bar{x}, \bar{y}, \bar{a}, \bar{b}$ respectively. We get that $t_2 = r - \bar{x}\bar{y} \cdot \bar{b}\bar{a} + \bar{x}\bar{a} + \bar{y}\bar{b} - e$. By Lemmas 2.2 and 2.3 $\bar{x}\bar{y} \cdot \bar{b}\bar{a} = \bar{x}\bar{a} = \bar{y}\bar{b} = e$. Hence $t_2 = r$. Then $t_1, t_2 \in U^*(R)$ and, consequently, the subloop $U^*(R)$ is normal in $U^*(A)$. We proved that the ideal R of algebra A induced the normal subloop $U^*(R)$ of loop $U^*(A)$.

As noted above, the homomorphic image of circle loop $U^*(A)$ under homomorphism $A \to A/R$ is a circle loop. Hence the quotient loop $U^*(A)/U^*(R)$ is a circle loop. By (10) $U^*(A/R) = J(A/R)$. The Zhevlakov radical $J(A/R)$ of a maximal ideal of A/R. Hence $U^*(A/R)$ is a maximal subloop of multiplicative groupoid of algebra A/R.

We will show that the quotient loop $U^*(A)/U^*(R)$ is isomorphic to the corresponding subloop of circle loop $U^*(A/R)$. Indeed, if x_1 and x_2 belong to the same coset of $U^*(A)$ modulo $U^*(R)$, then $x_1 = x_2 \circ r$ where r is a quasiregular element of R. But $x_2 \circ r = -x_2 r + x_2 + r$, consequently, $x_1 - x_2 = r - x_2 r \in R$. Conversely, if $x_1 - x_2 \in R$ and a is a quasiinverse for x_2, i.e. $a = x_2^{-1}$, then $x_1 = x_2 + r$ $(r \in R)$, $x_2 + a - ax_2 = 0$, $x_2^{-1} \circ x_1 = a + x_1 - ax_1 = a + x_2 + r - ax_1 = ax_2 + r - ax_1 = ax_2 + r - ax_1 = r - a(x_1 - x_2) = r - ar \in R$. Hence $x_2^{-1} \circ x_1 \in U^*(R)$.

We proved that $U^*(A)/U^*(R) \subseteq U^*(A/R)$ or, by (10), $J(A)/J(R) \subseteq J(A/R)$. If $B/J(R) = J(A/R)$, then by [36, Lemma 13, cap. 10] it follows that $B \subseteq J(A)$. Con-

sequently, $J(A)/J(R) \cong J(A/R)$ or, by (19), $U^*(A)/U^*(R) \cong U^*(A/R)$, as it was required. ∎

Corollary 2.8. *Let R be an arbitrary non-zero alternative algebra and let $A = R^\sharp = Fe \oplus R$. Then the following results hold*:

(i) *the circle loop $U^*(A)$ is a direct product of the central subloop $U^*(F)$ and the normal subloop $U^*(R)$*

(ii) *the loop of invertible elements $U(A)$ is a direct product of central subloop $U(F)$ and normal subloop $U(e - R)$.*

Proof. By Proposition 2.7 and (10) and the above-mentioned result as well, $U^*(R)$, $U^*(F)$ are normal subloops of $U^*(A)$ and $U(e - R)$, $U(F)$ are normal subloops of $U(A)$. As R is a proper ideal of algebra A then

$$U^*(R) \bigcap U^*(F) = \{0\}, \quad U(e - R) \bigcap U(F) = \{e\}. \tag{15}$$

Besides, as $e \notin R$ then $R \bigcap U(F) = \varnothing = R \bigcap U(A)$. Hence

$$U(A) \subseteq A \backslash R = \{\alpha e + r \mid 0 \neq \alpha \in F,\ r \in R\}, \tag{16}$$

$$U(A) = \{\alpha(e - u) \mid \alpha \in U(F),\ u \in U^*(R)\}. \tag{17}$$

Now, Corollary 2.8 follows from (14), (15) and (17). ∎

According to Lemma 2.1 the peculiarity of algebras with externally adjoined unit $A = R^\sharp$ is linked with the homomorphisms $\pi : A \to F$ of the considered algebras on the one-dimensional algebra $F \equiv Fe$. An algebra A with unit e will be algebra with externally adjoined unit (i.e. $A = R^\sharp$ for some algebra R) when and only when $A \neq \bigcap\{J \lhd A | A/J \cong F\}$ or, equally, the set $S_F(A) = \bigcap\{J \lhd A | A/J \cong F\}$ is non-empty. As indicated in the beginning of the section, it could be the case that $R_1, R_2 \in S_F(A)$, i.e. that $R_1^\sharp = A = R_2^\sharp$, though algebras R_1 and R_2 are quite different. But for circle loops $U^* \cong U(e - R)$ from Corollary 2.8 the following result holds.

Corollary 2.9. *If $R_1, R_2 \in S_F(A)$ then $U^*(R_1) \cong U^*(R_2)$.*

3. ALTERNATIVE LOOP ALGEBRAS

Let F be a field (with unit 1) and Q be a Moufang loop with unit e. We remind that, by its definition, the *loop algebra* $FQ \equiv F(Q)$ is a free F-module with the basis $\{g | g \in Q\}$ and the product of the elements of this basis is just their product in the loop Q. Any element $g \in Q$ is identified with the element $1g$, and any element $\lambda \in F$ is identified with the element λe. In particular, the unit of algebra FQ may be considered both as unit of field F and as unit of loop Q. In this case, every homomorphism φ of algebra FQ must be *unital*, i.e. it has to maintain the unit, $\varphi e = e$. Since $\varphi e = e \neq 0$ then $\ker \varphi$ is a proper ideal of FQ.

Let H be a normal subloop of the loop Q and let $\omega H \equiv \omega(H)$ be the ideal of the loop algebra FQ, generated by the elements $e - h$ ($h \in H$). If $H = Q$, then ωQ will be called the *augmentation ideal* of loop algebra FQ. In [1, Lemma 1] it is proved that

$$F(Q/H) \cong FQ/\omega H. \tag{18}$$

By definition the Moufang loop Q satisfies the Moufang identity $(xy \cdot x)z = x(y \cdot xz)$. It is easy to see that the loop algebra FQ does not always satisfy the Moufang identity if the loop Q is nonassociative. This is an equivalent to the fact that the equalities

$$(a, b, c) + (b, a, c) = 0, \quad (a, b, c) + (a, c, b) = 0 \quad \forall a, b, c \in Q, \tag{19}$$

where the notation $(a, b, c) = ab \cdot c - a \cdot bc$ means that the associator in algebra, does not always hold in loop algebra FQ. This means that algebra FQ is not alternative. We remind that algebra A is called *alternative* if the identities $(x, x, y) = (y, x, x) = 0$ hold in it.

Let $I(Q)$ denote the ideal of algebra FQ, generated by all elements of the left part of equalities (19). It follows from the definition of loop algebra and diassociativity of Moufang loops that $FQ/I(Q)$ will be an alternative algebra. Further for the alternative algebra $FQ/I(Q)$ we use the notation $F[Q]$ and we call them *alternative loop algebra*.

In [31], [32] it is proved that a free Moufang loop L is isomorphically embedded under homomorphism $\eta : FL \to F[L]$ into loop of invertible elements of algebra $F[L]$. If the image L is identified with L then the following holds.

Lemma 3.1. *Any free Moufang loop L is a subloop of the loop of invertible elements $U(F[L])$ of the alternative loop algebra $F[L]$.*

From the definition of loop algebra FL and Lemma 3.1 it follows.

Corollary 3.2. *Any element u of the alternative loop algebra $F[L]$ of any free Moufang loop L is a finite sum $u = \sum_{i=1}^{k} \alpha_i g_i$, where $\alpha_i \in F$, $g_i \in L$.*

Further we will use the following statement proved in [31], [32].

Lemma 3.3. *Let A be an alternative algebra and let Q be a subloop of the loop of invertible elements $U(A)$. Then the restriction of any homomorphism of algebra A upon Q will be a loop homomorphism. Consequently, any ideal J of A induces a normal subloop $Q \bigcap (e + J)$ of loop Q.*

Let H be a normal subloop of free Moufang loop L with unit e. We denote the ideal of algebra $F[L]$, generated by the elements $e - h$ ($h \in H$) by $\omega[H]$. If $H = L$, then $\omega[L]$ will be called the *augmentation ideal* of the alternative loop algebra $F[L]$.

For a Moufang loop Q, let L be a free Moufang loop such that the loop Q has a presentation $Q = L/H$. We consider the mapping $\bar{\mu} : FL \to FQ$ induced by homomorphism $\mu : L \to L/H = Q$ by

$$\bar{\mu}(\sum_{g \in L}^{(FL)} \alpha_g g) = \sum_{g \in L}^{(FL)} \alpha_g \mu(g) = \sum_{a \in Q}^{(FQ)} \alpha_a a, \tag{20}$$

where $a = \mu(g)$, $\alpha_g, \alpha_a \in F$ and $\sum^{(FL)}$ means sum in F-module FL, $\sum^{(FQ)}$ means sum in F-module FQ. The mapping $\bar{\mu}$ is defined correctly because FL is an F-module with basis $\{g \mid \forall g \in L\}$, FQ is an F-module with basis $\{a|\forall a \in Q\}$ and μ is an epimorphism. Moreover, as FL is a free module, then $\bar{\mu}$ is an epimorphism of F-modules. Further, let $x = \sum_{g\in L}^{(FL)} \alpha_g g$, $y = \sum_{h\in L}^{(FL)} \beta_h h$. Then $\bar{\mu}(xy) = \bar{\mu}(\sum_{g,h\in L}^{(FL)} \alpha_g\beta_h(gh)) = \sum_{g,h\in L}^{(FL)} \alpha_g\beta_h\mu(gh) = \sum_{g,h\in L}^{(FL)} \alpha_g\beta_h\mu(g)\mu(h) = \sum_{g\in L}^{(FL)} \alpha_g\mu(g) \cdot \sum_{h\in L}^{(FL)} \beta_h\mu(h) = \bar{\mu}(x)\bar{\mu}(y)$. Consequently, $\bar{\mu} : FL \to FQ$ is a homomorphism of algebras, and by (18)

$$\ker\bar{\mu} = \omega H, \quad FQ = FL/\omega H. \tag{21}$$

Let μ_Q be the homomorphism of the alternative loop algebra $F[L]$ induced by homomorphism $\bar{\mu}$ of loop algebra FL, $\mu_Q(F[L]) = \bar{\mu}(FL)/\bar{\mu}(I(L)) = FQ/\bar{\mu}(I(L))$. To exclude the null homomorphisms we will consider that the induced homomorphism μ_Q is unital.

The algebra $F[L]$ is alternative and the algebra $FQ/\bar{\mu}(I(L))$ is alternative, as well. In this case $I(Q) \subseteq \bar{\mu}(I(L))$. Further, by definition, the ideal $I(L)$ of the loop algebra FL is generated by the set $\{(u, v, w) + (v, u, w), (u, v, w) + (u, w, v)|\forall u, v, w \in L\}$. Since $\bar{\mu}((u, v, w) + (v, u, w)) = (\mu(u), \mu(v), \mu(w)) + (\mu(v), \mu(u), \mu(w))$, $\bar{\mu}((u, v, w) + (u, w, v)) = (\mu(u), \mu(v), \mu(w)) + (\mu(u), \mu(w), \mu(v))$ and $\mu(u), \mu(v), \mu(w) \in Q$, then $\bar{\mu}(I(L) \subseteq I(Q)$ and $\bar{\mu}(I(L)) = I(Q)$. Consequently,

$$\mu_Q(F[L]) = FQ/I(Q) = F[Q]. \tag{22}$$

Further, according to (21) and homomorphism theorems it follows $I(Q) = \overline{\mu}_Q(I(L)) = (I(L) + \omega H)/\omega H \cong \omega H/(\omega H \bigcap I(L))$, i.e.

$$I(Q) \cong \omega H/(\omega H \bigcap I(L)).$$

We denote by " - " the difference in loop algebra FL, by " $\ominus$ " we denote the difference in alternative loop algebra $F[L]$ and by θ - the restriction on ideal ωH of natural homomorphism $\eta : FL \to FL/I(L) = F[L]$. It is obvious that $\ker\theta = \omega H \bigcap I(L)$ and $\theta(\omega H) = \omega H/(\omega H \bigcap I(L))$.

By definition, the ideal ωH is generated by set $\{e - h|\forall h \in H\}$. From Lemma 3.1, it follows that $\eta(H) = H$. Then the algebra $\theta(\omega H)$ is generated by the set $\{e \ominus h|\forall h \in H\}$. We have $\theta(\omega H) = \eta(\omega H)$. Recall that we have above proved the equality $\theta(\omega H) = \omega H/(\omega H \bigcap I(L))$. By the homomorphisms theorem it results $\omega H/(\omega H \bigcap I(L)) \cong (\omega H + I(L))/I(L)$. Hence the ideal $(\omega H + I(L))/I(L)$ of algebra $FL/I(L)$ is generated by the set $\{e \ominus h|\forall h \in H\}$. Consequently,

$$(\omega H + I(L))/I(L) = \omega[H]. \tag{23}$$

Now, by (21), homomorphism theorems and (23) it follows

$$\mu_Q(F[L]) = \mu_Q(FL/I(L)) = \bar{\mu}(FL)/\bar{\mu}(I(L)) =$$

$$((FL+\omega H)/\omega H)/((I(L)+\omega H)/\omega H) \cong (FL+\omega H)/(I(L)+\omega H) =$$

$$FL/(I(L)+\omega H) \cong (FL/I(L))/((I(L)+\omega H)/I(L)) = F[L]/\omega[H].$$

According to (22), it results $\mu_Q(F[L]) = F[Q]$, i.e.

$$\ker \mu_Q = \omega[H]. \tag{24}$$

The homomorphism of alternative loop algebras $\mu_Q : F[X] \to F[Q]$ is induced by homomorphism of loop algebras $\bar{\mu} : FX \to FQ$ which is induced, in its turn, by the homomorphism of loops $\mu : X \to Q$. Then, from (20), it follows that any homomorphism of loops $\mu : X \to Q$ induces a homomorphism of alternative loop algebras $\mu_Q : F[X] \to F[Q]$, defined by

$$\mu_Q(\sum_{g\in L}^{(F[L])} \alpha_g g) = \sum_{a\in Q}^{(F[Q])} \alpha_a \mu(g) = \sum_{a\in Q}^{(F[Q])} \alpha_a a, \tag{25}$$

where $a = \mu(g)$, $\alpha_g, \alpha_a \in F$ and $\sum^{(F[L])}$ means the sum in the F-module $F[L]$, $\sum^{(F[Q])}$ means the sum in the F-module $F[Q]$.

We remind that in order to exclude the case $ker\mu_Q = F[Q]$ we assume that the homomorphism μ_Q is unital. Then from (24) the following results.

Proposition 3.4. *Let L be a free Moufang loop, let Q be a Moufang loop which has the presentation $Q = L/H$ such that the homomorphism μ_Q, induced by (25) by homomorphism $\mu : L \to Q$, is unital. Then the alternative loop algebra $F[Q]$ has the presentation $F[Q] = F[L]/\omega[H]$.*

Corollary 3.5. *The alternative loop algebra $F[Q]$ of a Moufang loop Q is generated as an F-module by the set $\{q|q \in Q\}$.*

The statement follows from Proposition 3.4 and (25).

Now we consider a homomorphism ρ of the alternative loop algebra $F[L]$. From Lemma 3.1, it follows that $F[L]$ is generated as F-module by set $\{g \mid g \in L\}$. Then the F-module $\rho(F[L])$ is generated by set $\{\rho(g)|g \in L\}$. Hence any element $x \in \rho(F[L])$ has a form $x = \sum_{g\in L} \alpha_g \rho(g)$.

By Lemma 3.3 ρ induces a normal subloop H of loop L. From (25) it follows that the homomorphism $\mu : L \to L/H = Q$ induces a homomorphism of alternative loop algebras $\mu_Q : F[L] \to F[Q]$, defined by

$$\mu_Q(\sum_{g\in L}^{(F[L])} \alpha_g g) = \sum_{a\in Q}^{(F[Q])} \alpha_a \mu(g).$$

Since $\mu(g) = gH = \rho(g)$, it follows $\eta(F[X]) = \mu_Q(F[X]) = F[Q]$. Hence we proved the next result.

Proposition 3.6. *Let L be a free Moufang loop. The homomorphic images of the form $\mu_Q(F[L]) = F[Q]$ are the only alternative loop algebras. The homomorphisms μ_Q are unital and are induced by homomorphisms of loops $\mu : L \to Q$ by rules (25).*

Corollary 3.7. *Let φ be an unital homomorphism of alternative loop algebra $F[Q]$. Then the homomorphic image $\varphi(F[Q])$ is a non-zero alternative loop algebra.*

Proof. We consider the homomorphism $\mu_Q : F[L] \to F[Q]$ from Proposition 3.6 and let φ be a homomorphism of alternative loop algebra $F[Q]$. The product $\varphi\mu_Q$ is a homomorphism of alternative loop algebra $F[L]$ on algebra $\varphi(F[Q])$. By Proposition 3.6, $\varphi(F[Q])$ is an alternative loop algebra, as it was required. ∎

Let L be a free Moufang loop with unit e. By Corollary 3.2 any element $a \in F[L]$ has a form $a = \sum_{i=1}^{k} \alpha_i u_i$, where $\alpha_i \in F$, $u_i \in L$. Let H be a normal subloop of loop L and let $\varphi : L \to L/H$ be the natural homomorphism. It is easy to see that the mapping $\overline{\varphi} : F[L] \to F[L/H]$, defined by rule

$$\overline{\varphi}(\sum_{g\in L} \alpha_g g) = \sum_{g\in L} \alpha_g \varphi(g) = \sum_{g\in L} \alpha_g gH \tag{26}$$

is a homomorphism. Then it necessarily follows

$$F[L/H] \cong F[L]/\ker\overline{\varphi}. \tag{27}$$

We assume that $F[L]/\ker\overline{\varphi}$ is an algebra with externally adjoined unit. Then $\overline{\varphi}$ is an unital homomorphism, i.e. $\overline{\varphi}(e) = e \neq 0$. In such a case $e \notin \ker\overline{\varphi}$.

Lemma 3.8. *Let $\overline{\varphi}$ be a homomorphism defined in (26) and we assume that $F[L]/\ker\overline{\varphi}$ is an algebra with externally adjoined unit. Then*

1) $h \in H$ if and only if $e - h \in \omega[H]$,

2) $F[L/H] \cong F[L]/\omega[H]$,

3) $\omega[H] = \ker\overline{\varphi}$.

Proof. 1). As the mapping $\overline{\varphi}$ is F-linear, then for $u \in F[L]$ and $h \in H$

$$\overline{\varphi}((e-h)u) = (\varphi e - \varphi h)\varphi u = (e - H)(uH) = uH - uH = 0$$

and

$$\omega[H] \subseteq \ker\overline{\varphi}. \tag{28}$$

If $g \notin H$ then $gH \neq H$ and $\overline{\varphi}(e - g) = H - gH \neq (0)$. Hence $e - g \notin \ker\overline{\varphi} \supseteq \omega[H]$ by (28), i.e. $e - g \notin \omega[H]$.

2). Let the ideal $\omega[H]$ of algebra $F[L]$ induces, by Lemma 3.3, the normal subloop $K = L\bigcap(e - \omega[H])$ of loop L and, hence, $F[L/K] \equiv F[L]/\omega[H]$. From the first relation, we get $1 - K \subseteq \omega[L]$. By item 1) $K = H$, hence $F[L/H] \equiv F[L]/\omega[H]$.

3). The isomorphism $\xi : F[L]/\omega[H] \to F[L]/\ker\overline{\varphi}$ follows from (27) and item 2). For any element $u \in F[L]$ we denote by $\overline{u}$ the image of u into $F[L]/\omega[H]$ and by $\overline{\overline{u}}$ we

denote the image of u into $F[L]/\ker\overline{\varphi}$. Let $0 \neq u \in \ker\overline{\varphi}\backslash\omega[H]$. As $a \notin \omega[H]$ then $\overline{0} \neq \overline{u}$. Hence $\xi(\overline{0}) \neq \xi(\overline{u})$, $\overline{\overline{0}} \neq \overline{\overline{u}}$. But as $u \in \ker\overline{\varphi}$ then $\overline{\overline{0}} = \overline{\overline{u}}$ what is a contradiction. Hence $\omega[H] = \ker\overline{\varphi}$. This completes the proof of Lemma 3.8. ∎

4. ALTERNATIVE LOOP ALGEBRAS WITH EXTERNALLY ADJOINED UNIT

Let now $\omega[L]$ be the augmentation ideal of the alternative loop algebra $F[L]$ of the free Moufang loop $L \neq \{e\}$. According to Corollary 3.2 any element $a \in F[L]$ has the form $a = \sum_{i=1}^{k} \alpha_i u_i$, where $\alpha_i \in F$, $u_i \in L$. We denote $R = \{\sum_{u\in L} \lambda_u u | \sum_{u\in L} \lambda_u = 0\}$. Obviously, $\omega[L] \subseteq R$. Conversely, if $r \in R$ and $r = \sum_{u\in L} \lambda_u u$, then $-r = -\sum_{u\in L} \lambda_u u = (\sum_{u\in L} \lambda_u)e - \sum_{u\in L} \lambda_q q = \sum_{u\in L} \lambda_u(e-u) \in \omega L$, i.e. $R \subseteq \omega[L]$. Hence

$$\omega[L] = \{\sum_{u\in L} \lambda_u u | \sum_{u\in L} \lambda_u = 0\}. \tag{29}$$

From (29) it follows that $\omega[L] \bigcap L = \{\emptyset\}$. Then the algebra $\omega[L]$ will be non-zero when and only when $L \neq \{e\}$, i.e. when $F[L] \neq Fe = 0^{\sharp}$. In such case for any $t \in L$ the equalities $Ft \bigcap \omega[L] = 0$, $Ft + \omega[L] = F[L]$ hold and, by (29), $t - s \in \omega[L]$. Then, the set $B_t(\omega[L]) = \{t - s \mid s \in L,\ t \neq s\}$ generates the F-module $\omega[L]$ for any $t \in L$ as $L \neq \{t\}$ and the set $\{t\} \bigcup B_t(\omega[L])$ generates the F-module $F[L]$ by Corollary 3.2. In particular, the set $\{e\} \bigcup B_e(\omega[L])$ generates the F-module $F[L]$ and the set $B_e(\omega[L])$ generates the F-module $\omega[L]$. Then $F[L]/\omega[L] \cong Fe$ and, by Lemma 2.1,

$$F[L] = Fe \oplus \omega[L] = (\omega[L])^{\sharp}. \tag{30}$$

If $u \in L$ then by Lemma 2.3 $e - u$ is a quasiregular element. The set $B_e(\omega[L])$ generates the F-module $\omega[L]$. By [36, Lemma 10.4.12], in an alternative algebra, the sum of quasiregular elements is a quasiregular element. Hence the augmentation ideal $\omega[L]$ is a quasiregular algebra and from (30) it follows that $\omega[L]$ coincides with Zhevlakov radical $J(F[L])$, $\omega[L] = J(F[L])$.

Hence we have proved the next result.

Lemma 4.1. *Let L be a free Moufang loop and let $\omega[L]$ be the augmentation ideal of the alternative loop algebra $F[L]$. Then*

1) $\omega[L]$ is generated as an ideal of the algebra $F[L]$, as well as an F-module, by set $L^0 = \{e - u \mid for\ all u \in L\}$,

2) $\omega[L]$ is a quasiregular algebra, i.e. $\omega[L] = J(F[L])$, where $J(F[L])$ is the Zhevlakov radical of the algebra $F[L]$.

Theorem 4.2. *Let Q be a Moufang loop with unit e such that the alternative loop algebra $F[Q]$ is an algebra with externally adjoined unit e. Then the loop Q can be embedded into the loop of invertible elements $U(F[Q])$ of the alternative loop algebra $F[Q]$.*

Proof. Let $Q = L/H$, where L is a free Moufang loop, let $\varphi : L \to L/H$ be the natural homomorphism and let $\overline{\varphi} : F[L] \to F[L/H] = F[L]/\ker\overline{\varphi}$ be the homomorphism defined in (20) by

$$\overline{\varphi}(\sum_{g\in L}\alpha_g g) = \sum_{g\in L}\alpha_g\varphi(g) = \sum_{g\in L}\alpha_g gH.$$

By item 3) of Lemma 3.8, $F[L]/\ker\overline{\varphi} \cong F[L]/\omega[L]$ and, by Proposition 3.4, $F[L]/\omega[L] \cong F[Q]$. Hence $F[Q] \cong F[L]/\ker\overline{\varphi}$.

According to (30) $F[L] = Fe \oplus \omega[L]$. Fe is a field, then from $F[L]/\omega[L] \cong Fe$ it follows that $\omega[L]$ is a maximal proper ideal of $F[L]$. Further, $\omega[H] \subseteq \omega[L]$. Then it is easy to see that $\overline{\varphi}(\omega[L]) = \Delta$ is a maximal proper ideal of $F[Q] \cong F[L]/\omega[H]$. Since the homomorphism $\overline{\varphi}$ is unital, then $\overline{\varphi}(Fe) = Fe$ and $F[Q]/\Delta \cong Fe$. By Lemma 2.1 $F[Q] = \Delta^\sharp = Fe \oplus \Delta$.

We denote $B(\Delta) = \{e - q | e \neq q \in Q\}$, $Q^0 = B(\Delta)\bigcup\{0\} = \{e - q \mid q \in Q\}$. By item 1) of Lemma 4.1 the augmentation ideal $\omega[L]$ of algebra $F[L]$ is generated as ideal of $F[L]$ and as F-module by set $B(\omega[L]) = \{e - g \mid g \in L,\ e \neq g\}$. Further, $\overline{\varphi}B(\omega[L]) = \{\overline{\varphi}(e - g) \mid e \neq \overline{\varphi}g \in \overline{\varphi}L\} = \{e - q \mid e \neq q \in Q\} = B(\Delta) = Q^0\backslash\{0\}$. Hence the ideal Δ is generated, as an ideal of the algebra $F[Q]$ and as well as an F-module, by the set $B(\Delta) = Q^0\backslash\{0\}$. From $F[Q] = Fe \oplus \Delta$ it follows that the algebra $F[Q]$ is generated by set Q. According to Corollary 3.5, any element $a \in F[Q]$ have the form

$$\sum_{i=1}^{k}\alpha_i q_i, \qquad (31),$$

where $\alpha_i \in F$, $q_i \in Q$.

By item 2) of Lemma 4.1 the ideal $\omega[L]$ is a quasiregular and $\omega[L] = J(F[L])$. From $\omega[H] \subseteq \omega[L]$ it follows that $\omega[H]$ is a quasiregular ideal. Then $\omega[H] = J(\omega[H])$. According to (11) and Propositions 2.7, 3.4, $J(F[Q]) = J(F[L]/\omega[H]) = J(F[L])/J(\omega[H]) = \omega[L]/\omega[H] = \overline{\varphi}(\omega[L]) = \Delta$. Hence $J(F[Q]) = \Delta$ and, by (11), Δ coincide with circle loop $U^*(F[Q])$, i.e.

$$\Delta = U^*(F[Q]). \qquad (32)$$

We consider the mapping $\eta : F[Q] \to F[Q]$ defined by the rule $\eta u = e - u, \forall u \in F[Q]$. From (11) and (32) it follows that the rule $\eta_Q : \eta_Q b = e - b$, $\forall b \in \Delta$, defines an isomorphism between the circle loop $(\Delta, \circ)$ and the loop of invertible elements $U(F[Q])$ of algebra $F[Q]$ because $\eta^{-1} = \eta$. Particularly, the restriction $\overline{\eta}_Q$ of η_Q on Q^0 is an isomorphism of subloop $(Q^0, \circ) \subseteq (\Delta, \circ)$ and loop Q defined by rule: $\overline{\eta}_Q b = e - b^0$, $\forall b^0 \in Q^0$. Consequently, the given Moufang loop Q is a subloop of multiplicative loop of invertible elements $U(F[Q])$ of algebra $F[Q]$. This completes the proof of Theorem 4.2. ∎

Now we define the class $\mathcal{R}_A$ of alternative F-algebras. Any alternative loop algebra with externally adjoined unit $F[Q]$ belong to class $\mathcal{R}_A$. Remind that if $F[Q] \in \mathcal{R}_A$

then the loop Q, the field F and the algebra $F[Q]$ have the same unit e and $\varphi(e) = e$ for any homomorphism φ of $F[Q]$. Let $F[Q] \in \mathcal{R}_A$. Then, from Theorem 4.2, it follows that $Q \subseteq U(F[Q])$. This fact suggest us to give the following definition. Let $F[Q]$ be an alternative loop algebra and let H be a normal subloop of loop Q such that $H \subseteq U(F[Q])$. In such a case, we denote by $\omega[H]$ the ideal of $F[Q]$ generated by the set $\{e - h | h \in H\}$. If $H = Q$, then $\omega[Q]$ will be called an *augmentation ideal* of the alternative loop algebra $F[Q]$.

Proposition 4.3. *Let $\omega[Q]$ be the augmentation ideal of an alternative loop algebra $F[Q]$ with externally adjoined unit e, i.e. let $F[Q] \in \mathcal{R}_A$. Then:*

1) any element $a \in F[Q]$ has the form $\sum_{i=1}^{k} \alpha_i q_i$, where $\alpha_i \in F$, $q_i \in Q$;

2) $F[Q] = (\omega[Q])^\sharp = \omega[Q] \oplus Fe$;

3) if $F[Q] = R^\sharp$, then $R = \omega[Q]$;

4) $\omega[Q]$ is generated as F-module by the set $B(\omega[Q]) = \{e - q | e \neq q \in Q\}$;

5) any isomorphism φ of algebra $F[Q]$ induces the identical isomorphism on loop Q and on ideal $\omega[Q]$, as well;

6) $\omega[Q] = \{\sum_{q \in Q} \alpha_q q | \sum_{q \in Q} \alpha_q = 0\}$;

7) the algebra $\omega[Q]$ is quasiregular and coincides with the Zhevlakov radical $J(F[Q])$, $\omega[Q] = J(F[Q])$;

8) $\omega[Q]$ coincides with the circle loop $U^*(F[Q])$ ($\omega[Q] = U^*(F[Q])$), i.e., by (9), on the algebra $\omega[Q]$ there exists and it is unique, the unary operation $x \rightsquigarrow x^*$ of taking the quasiinverse, that is connected with the basic operations by identity $x + x^* = xx^* = x^*x$ (i.e. $r \circ r^* = 0 = r^* \circ r$ for all $r \in \omega[Q]$ from the construction of loop $U^*(\omega[Q])$). The circle loop $U^*(F[Q])$ is isomorphic to the loop $U(F[Q])$ of invertible elements under the isomorphism $\eta : u \to e - u$, $\forall u \in \omega[Q]$. The subloop $Q^0 = B(\omega[Q]) \bigcup \{0\} = \{e - q \mid q \in Q\}$ of the loop $U^*(F[Q])$ is isomorphic to the given loop Q, i.e. $\eta(Q^0) = Q$;

9) $F[Q] \backslash \omega[Q] = U(F[Q])$, i.e. the algebra $\omega[Q]$ coincides with set of all non-invertible elements of algebra $F[Q]$;

10) $J(I) = I \bigcap J(F[Q])$, $U^*(I) = I \bigcap U^*(F[Q])$ for all ideals I of $F[Q]$.

Proof. The item 1) was already proved (see the proof of Theorem 4.2 for the equality (31)).

Theorem 4.2 is proved by showing that the ideal Δ coincides with augmentation ideal $\omega[Q]$. Then the statements 2), 4), 7), 8) are contained in proof of Theorem 4.2.

3) Let $F[Q] = R^\sharp = R \oplus Fe$. By items 2), 7), 8) $F[Q] = (\omega[Q])^\sharp = \omega[Q] \oplus Fe$, $\omega[Q] = J(Q)$, $(\omega[Q], \circ) = U^*(\omega[Q])$ and by Corollary 2.9 $U^*(\omega[Q]) \cong U^*(R)$. An alternative algebra is quasiregular if it coincides with its circle loop. Hence the ideal R is quasiregular. Fe is a field. Then from relation $F[Q]/R \cong Fe$ it follows that R is a maximal ideal of $F[Q]$. Hence $R = J(R)$. As $J(R) = J(\omega[Q])$ then $R = J(R) = J(\omega[Q]) = \omega[Q]$, as required.

5) As φ is an isomorphism, then $\ker\varphi = \{0\}$. By Lemma 3.3 it follows that φ induces the normal subloop $Q \bigcap (e + \ker\varphi) = e$ of loop Q. Hence φ induces on Q the

identical isomorphism ϵ. By item 4) the ideal $\omega[Q]$ is generated as F-module by set $B(\omega[Q]) = \{e - q | e \neq q \in Q\}$. But $\varphi(e - q) = e - \epsilon q = e - q$. Consequently, again by item 4), $\varphi(\omega[Q]) = \omega[Q]$.

Using the item 1) the statement 6) is proved similarly as equality (29).

By item 7) $U^*(\omega[Q]) = \omega[Q]$. Then the item 9) follows from the description of the loop $U(A)$ by equalities (16), (17).

According to (11), the item 10) is just the equality (14). This completes the proof of Proposition 4.3. ∎

Let H be a normal subloop of the Moufang loop Q and let $F[Q] \in \mathcal{R}_A$. By item 2) of Proposition 4.3, $e \notin \omega[Q]$. Then $e \notin \omega[H]$ and hence $F[H] \in \mathcal{R}_A$.

Let us determine the homomorphism of F-algebras φ: $F[Q] \to F[Q/H]$ by the rule $\varphi(\sum \alpha_q q) = \sum \alpha_q Hq$. The following result holds.

Proposition 4.4. *Let $F[Q] \in \mathcal{R}_A$ and let H, H_1, H_2 be normal subloops of the loop Q. Then:*

1) $\ker\varphi = \omega[H]$*;*

2) $e - h \in \omega[H]$ if and only if $h \in H$;

3) if the family of elements $\{h_i\}$ generates the subloop H, then the family of elements $\{e - h_i\}$ generates the ideal $\omega[H]$;

4) if $H_1 \neq H_2$, then $\omega[H_1] \neq \omega[H_2]$; if $H_1 \subset H_2$, then $\omega[H_1] \subset \omega[H_2]$; if $H = \{H_1, H_2\}$, then $\omega[H] = \omega[H_1] + \omega[H_2]$;

5) $F[Q]/\omega[H] \cong F[Q/H]$, $\quad \omega[Q]/\omega[H] \cong \omega[Q/H]$.

Proof. The statement 1) is proved similarly with item 3 of Lemma 3.8. To prove it is necessary only to use Theorem 4.2 instead of Lemma 3.1 and, in particular, to use the item 1) of Proposition 2 instead of Corollary 3.2.

2). If $q \notin H$, then $Hq \neq H$. Consequently, $\varphi(e - q) = H - Hq \neq 0$, i.e., by 1), $e - q \notin \ker\varphi = \omega[H]$.

3), 4). Let elements $\{h_i\}$ generate subloop H and let I be the ideal, generated by the elements $\{e - h_i\}$. Obviously $I \subseteq \omega[H]$. Conversely, let $g \in H$ and let $g = g_1 g_2$, where g_1, g_2 are words from $\{h_i\}$. We suppose that $e - g_1, e - g_2 \in I$. Then $e - g = (e - g_1)g_2 + e - g_2 \in I$, i.e. $\omega[H] \subseteq I$ and $I = \omega[H]$. Let $H_1 \neq H_2$ (respect. $H_1 \subset H_2$) and $g \in H_1, g \notin H_2$. Then, by item 2), $e - g \in \omega[H_1]$, but $e - g \notin \omega[H_2]$. Hence $\omega H_1 \neq \omega H_2$ (respect. $\omega H_1 \subset \omega H_2$). If $H = \{H_1, H_2\}$, then by the first statement of 3), $\omega H = \omega H_1 + \omega H_2$.

5). Mapping $\varphi : F[Q] \to F[Q/H]$ is the homomorphism of alternative loop algebras and, as by item 1), $\ker\varphi = \omega[H]$, then $F[Q/\omega[H] \cong F[Q/H]$. The mapping $\omega[Q] \to \omega[Q]/\omega[H]$ save the sum of coefficients then from item 5) of Proposition 4.3 it follows that $\omega L/\omega H \cong \omega(L/H)$. ∎

Corollary 4.5. *For a normal subloop H of a Moufang loop Q with unit e the following statements are equivalent:*

1) $F[H] \in \mathcal{R}_A$*;*
2) $F[H] = \omega[H] \oplus F1$*;*
3) $e \notin \omega[H]$*;*
4) $\omega[H]$ *is a proper ideal of algebra* $F[Q]$*;*
5) $\omega[H] = \{\sum_{q\in H} \alpha_q q | \sum_{q\in H} \alpha_q = 0\}$.

Proof. The equivalence of items 1), 2) follows from Lemma 2.1 and item 3) of Proposition 4.3. The implications 2) $\Leftrightarrow$ 3), 3) $\Leftrightarrow$ 4) are obvious. The implication 1) $\Leftrightarrow$ 5) follows from items 2), 6) of Proposition 4.3. ∎

5. RADICALS IN ALTERNATIVE LOOP ALGEBRAS AND MOUFANG LOOPS

Let $\mathcal{M}$ denote the class of all alternative loop algebras and its augmentation ideals. If φ is a non-zero homomorphism of algebra $F[Q] \in \mathcal{M}$ then, by Corollary 3.7, the homomorphic image $\varphi(F[Q])$ will be an alternative loop algebra. Hence $\varphi(F[Q]) \in \mathcal{M}$.

Let now $\omega[Q]$ be the ideal of the alternative loop algebra $F[Q] \in \mathcal{M}$ with unit e, defined above, and let ψ be a homomorphism of $\omega[Q]$. If $e \in \omega[Q]$ then $\omega[Q] = F[Q]$ and ψ is the zero homomorphism. We suppose that $e \notin \omega[Q]$. Then $F[Q] \in \mathcal{R}$ and, by Corollary 4.5, $F[Q] = \omega[Q] \otimes Fe$. We extend ψ to the homomorphism φ of $F[Q]$ considering that $\varphi(e) = e$. Let $\varphi(F[Q]) = F[G]$. Then $F[G] = \psi(\omega[Q]) \otimes \varphi(Fe) = \psi(\omega[Q]) \otimes Fe$. By Lemma 2.1 it follows $F[G] \in \mathcal{R}$. Then by item 3) of Proposition 4.3 $\psi(\omega[Q] = \omega[G]$. Consequently, we proved that the class $\mathcal{M}$ is closed with respect to homomorphic images.

Let J be an ideal of the alternative loop algebra $F[Q]$. By Lemma 3.3 J induces the normal subloop $H = Q \bigcap (e + J)$ of loop Q. In its turn H induces the homomorphism $\varphi : F[Q] \to F[Q]/H$ defined by $\varphi(\sum \alpha_q q) = \sum \alpha_q qH$, $\alpha_q \in F$, $q \in Q$. We have $\ker\varphi = J$. According to item 1) of Proposition 4.4 $J = \ker\varphi = \omega[H]$. If J_1 is another ideal of $F[Q]$, $J_1 \neq J$, then $\omega[H] = J \neq J_1 = \omega[H_1]$, where $H_1 = Q \bigcap (e + J_1)$. By item 3) of Proposition 4.4, it follows that $H_1 \neq H$. Consequently, various ideals of algebra $F[Q]$ induce various normal subloops of the loop Q.

Conversely, let $H \neq H_1$ be normal subloops of loop Q. The subloops H, H_1 induce the homomorphisms φ, φ_1 of algebra $F[Q]$ and, by items 1), 3) of Proposition 4.4, $\ker\varphi = \omega[H] \neq \omega[H_1] = \ker\varphi_1$. Hence various normal subloops of the loop Q induce various ideals of algebra $F[Q]$. The proper ideals of algebra $F[Q]$ have the form $\omega[H]$, where H is a normal subloop of the loop Q and $\omega[H]$ is the augmentation ideal without unit of the alternative loop algebra $F[H]$. Consequently, the correspondence $\omega[H] \to H$ is an one-to-one mapping between all normal subloops H of the loop Q and all ideals of the algebra $F[Q]$. Further let's consider, that all considered algebras belong to class $\mathcal{M}$, i.e. they have the form $F[Q]$, any of its ideals $J \neq F[Q]$ has the form $\omega[H]$, where $\omega[H]$ is the augmentation ideal of some alternative loop algebra $F[H]$ where H is a normal subloop of loop Q.

Now we consider the class of algebras $\mathcal{R}_A \subseteq \mathcal{M}$, defined in section 3. This class was analysed in Propositions 4.3, 4.4 and Corollary 4.5. The class $\mathcal{R}_A$ is characterized by the property that any algebra from $\mathcal{R}_A$ is an alternative loop algebra $F[Q]$ such that its ideals $J \neq F[Q]$ are augmentation ideals $\omega[H]$ without unit of some alternative loop algebras $F[H]$ from $\mathcal{R}_A$, where $H \subset Q$ is a normal subloop of the loop Q.

We denote by $\mathcal{R}$ the class of such augmentation ideals $\omega[Q]$ of its alternative loop algebra $F[Q] \in \mathcal{R}_A$.

An ideal I of algebra $A \in \mathcal{M}$ will be called $\mathcal{R}$-*ideal* if it belongs to class $\mathcal{R}$. An alternative loop algebra $F[Q] \in \mathcal{M}$, containing non-zero ideals will be called $\mathcal{R}_A$ – *algebra* if its augmentation ideal $\omega[Q]$ belongs to class $\mathcal{R}$, i.e. the ideal $\omega[Q]$ is without unit and $(\omega[Q])^\sharp = F[Q]$.

Lemma 5.1. *Any algebra A of class $\mathcal{M}$ contains a unique maximal $\mathcal{R}$-ideal $\mathcal{R}(A)$. The ideal $\mathcal{R}(A)$ coincides with augmentation ideal $\omega[K]$ of some alternative loop algebra $F[K] \in \mathcal{R}_A$.*

Proof. Let Σ denote the set of all $\mathcal{R}$-ideals of algebra A. The set Σ is non-empty, since the ideal $(0) \in \Sigma$. By Zorn Lemma the set Σ contains a maximal $\mathcal{R}$-ideal $\mathcal{R}(A)$.

Let's show that $\mathcal{R}(A)$ is an unique maximal $\mathcal{R}$-ideal. Let $\mathcal{R}(B)$ also be a maximal $\mathcal{R}$-ideal and let $x \in \mathcal{R}(A)$, $y \in \mathcal{R}(B)$. As $\mathcal{R}(A), \mathcal{R}(B) \in \mathcal{R}$, then, by item 5) of Corollary 4.5, $x = \sum_{g\in Q} \alpha_g g$ with $\sum_{g\in Q} \alpha_g = 0$ and $y = \sum_{g\in Q} \beta_g g$ with $\sum_{g\in Q} \beta_g = 0$. Then $x + y = \sum_{g\in Q} \gamma_g g$ with $\sum_{g\in Q} \gamma_g = 0$. Hence $\mathcal{R}(A) + \mathcal{R}(B) \in \mathcal{R}$. If $\mathcal{R}(A) \neq \mathcal{R}(B)$ then $\mathcal{R}(A) + \mathcal{R}(B)$ strictly contain $\mathcal{R}(A)$. Contradiction. Consequently, $\mathcal{R}(A)$ is the unique maximal $\mathcal{R}$-ideal.

The second statement of lemma follows from relation $\mathcal{R}(A) \in \mathcal{R}$ and the construction of ideals of class $\mathcal{R}$. This completes the proof of Lemma 5.1. ∎

Theorem 5.2. *The class $\mathcal{R}$ of all augmentation ideals without unit is radical in class $\mathcal{M}$ of all alternative loop F-algebras and its ideals.*

Proof. According to the definition of radical [36] should prove the statements:

(a) any homomorphic image of any $\mathcal{R}$-ideal is an $\mathcal{R}$-ideal;

(b) each algebra A from $\mathcal{M}$ contains an $\mathcal{R}$-ideal $\mathcal{R}(A)$, containing all $\mathcal{R}$-ideals of algebra A;

(c) the quotient-algebra $A/\mathcal{R}(A)$ does not contain any non-null $\mathcal{R}$-ideals.

Really, let $\omega[H] \in \mathcal{R}$ and let $x = \sum_{h\in H} \alpha_h h \in \omega[H]$. By item 5) of Corollary 4.5, $\sum_{h\in H} \alpha_h = 0$. Any homomorphism φ of the ideal $\omega[H]$ does not change the sum of coefficients, $\sum \alpha_h$. From here, it follows that $\varphi(\omega[H]) \in \mathcal{R}$ and the statement (a) is proved.

The Lemma 5.1 is just the statement (b).

Let $A \in \mathcal{M}$. The homomorphism $\varphi : A \to A/\mathcal{R}(A)$ maintains the sum of coefficients. Hence if $J \neq (0)$ is an $\mathcal{R}$-ideal of $A/\mathcal{R}(A)$ then the inverse image $\varphi^{-1}J$ will be an $\mathcal{R}$-ideal and $\mathcal{R}(A) \subset \varphi^{-1}J$. But this contradicts the maximality of $\mathcal{R}$-ideal $\mathcal{R}(A)$.

Consequently, $J = (0)$ and the statement (c) is proved. This completes the proof of Theorem 5.2. ∎

Let $A \in \mathcal{M}$. The mapping $A \to \mathcal{R}(A)$ is called radical, defined in class of algebra $\mathcal{M}$; denote it by $\mathcal{R}$.

Let now introduce some notions, derived from the general concepts of the theory of radicals [36]. *The ideal* $\mathcal{R}(A)$ of algebra $A \in \mathcal{M}$ is called its $\mathcal{R}$*-radical. An alternative loop algebra* $F[Q] \in \mathcal{M}$ will be called $\mathcal{R}$*-radical* if $F[Q] \in \mathcal{R}_A$ and $\omega[Q] = \mathcal{R}(F[Q])$. *Non-zero algebras* $F[Q] \in \mathcal{M}$, whose radical is null, will be called $\mathcal{R}$*-semisimples*. The class $\mathcal{P}$ of all $\mathcal{R}$-semisimple algebras of class $\mathcal{M}$ is called *semisimple class of radical* $\mathcal{R}$.

Let $A \in \mathcal{M}$. According to Lemma 5.1 the radical $\mathcal{R}(A)$ coincides with the augmentation ideal $\omega[K]$ of some alternative loop algebra $F[K] \in \mathcal{R}_A$. Then, by item 7) of Proposition 4.3, $\mathcal{R}(A)$ coincides with the Zhevlakov radical $J(F[K])$ which, by item 10) of Proposition 4.3, is hereditary. Remind that the radical $\mathcal{R}$ in the class of algebras $\mathcal{A}$ is called hereditary if $\mathcal{R}(J) = J \bigcap \mathcal{R}(A)$ for any algebra $A \in \mathcal{A}$ and any it ideal J. Then the following holds.

Corollary 5.3. *The radical* $\mathcal{R}$ *is hereditary in class* $\mathcal{M}$.

Corollary 5.4. *Let* A *be an algebra of class* $\mathcal{M}$ *and let* J *be an ideal of* A. *The following results hold:*

(1) if $A \in \mathcal{R}$ *then* $J \in \mathcal{R}$*;*

(2) if $A \in \mathcal{P}$ *then* $J \in \mathcal{P}$.

It follows from Corollary 1 and [36, Theorem 3, cap. 8].

Let $F[Q]$ be the alternative loop algebra of a Moufang loop Q with unit e. According to Corollary 3.5 any element $a \in F[Q]$ is a finite sum $a = \sum_{q \in Q} \alpha_q q$, where $\alpha_q \in F$. Then we may define the ideal of $F[Q]$ generated by set $\{e - q | q \in Q\}$. We denote it by $\omega[Q]$. Note that in a similar way we have above defined the augmentation ideal $\omega[Q]$ of algebra $F[Q] \in \mathcal{R}_A$.

Corollary 5.5. *For any algebra* A *of class* $\mathcal{P}$ *the following statements hold:*

1) if $A = \omega[Q]$ *then* $\omega[Q] = F[Q]$*;*

2) if $x \in A$ *and* $x = \sum_{g \in Q} \alpha_g g$ *then* $\sum_{g \in Q} \alpha_g \neq 0$*;*

3) any ideal J *of algebra* A *has the form* $J = F[H]$ *and, if* $J \neq 0$*, then* J *is nonassociative.*

Proof. If $A \in \mathcal{M}$, then the from definition of class $\mathcal{M}$, it follows that $A = F[Q]$ for some alternative loop algebra $F[Q]$. By Lemma 5.1, $\mathcal{R}(A) = \omega[H]$. If $A \in \mathcal{P}$ then $\mathcal{R}(A) = \{0\}$. From here it follows that the algebra A does not have non-zero proper ideals. Then $\omega[Q] = F[Q]$.

The item 2) follows from item 6) of Proposition 4.3.

Further, $A \in \mathcal{P}$ implies $J \in \mathcal{P}$ for any ideal J of algebra A, by Corollary 5.4. We have above proved that any ideal I of A have the form $I = \omega[H]$. Then, by item 1), it follows $J = F[H]$. If $F[H]$ is associative then H is a group and $F[H]$ is a group algebra. From the definition of group algebra it follows that $F[H]$ is a free F-module with bases $\{h \in H\}$. Then $e \notin \omega[H]$ and $F[H] \neq \omega[H]$. Contradiction. Consequently, the ideal $J \neq 0$ cannot be associative. This completes the proof of Corollary 5.5. ∎

In the beginning of the section we showed that for any alternative loop algebra $F[Q]$ the mapping $\omega[H] \to H$ is an one-to-one mapping between all normal subloops H of loop Q and all ideals of algebra $F[Q]$. Moreover, the following statement holds.

Lemma 5.6. *Let $F[Q]$ be an alternative loop algebra and let H, H_1, H_2 be normal subloops of loop Q. Then:*

1) $e - h \in \omega[H]$ if and only if $h \in H$;

2) if the elements $\{h_i\}$ generate the subloop H, then the elements $\{1 - h_i\}$ generate the ideal $\omega[H]$; if $H_1 \neq H_2$, then $\omega[H_1] \neq \omega[H_2]$; if $H_1 \subset H_2$, then $\omega[H_1] \subset \omega[H_2]$; if $H = \{H_1, H_2\}$, then $\omega[H] = \omega[H_1] + \omega[H_2]$.

Proof. If $e \notin \omega[H]$ then $F[H] \in \mathcal{R}_A$ and the statement 1) is the statement 2) of Proposition 4.4. If $e \in \omega[H]$ then $\omega[H] = F[H]$ and statement 1) follows, from property (31), that $F[H]$ is generated as an F-module by the set H.

The statement 2) is proved similarly as item 3) of Proposition 4.4. We have to use the item 1), only.

By $\mathcal{L}$ denote the class of all Moufang loops and by $\mathcal{S}$ denote the class of Moufang loops G such that $F[G] \in \mathcal{R}_A$ or, equivalently, $e \notin \omega[G]$ (by Proposition 4.3). Any loop from the class $\mathcal{L}$ (respect. $\mathcal{S}$) will be called $\mathcal{L}$-*loop* (respect. $\mathcal{S}$-*loop*). Now, let $Q \in \mathcal{L}$ be a Moufang loop, $F[Q] \in \mathcal{M}$ be it alternative loop algebra and, according to Theorem 5.2, let $\mathcal{R}(F[Q])$ be the $\mathcal{R}$-radical of $F[Q]$. By Lemma 5.1, $\mathcal{R}(F[Q]) = \omega[\mathcal{S}(Q)]$, where $\omega[\mathcal{S}(Q)]$ is the augmentation ideal of some alternative loop algebra $F[\mathcal{S}(Q)] \in \mathcal{R}_A$. By Theorem 5.2 the mapping $\mathcal{R} : F[Q] \to \mathcal{R}(F[Q]) = \omega[\mathcal{S}(Q)]$ is a radical of class $\mathcal{M}$. Obviously, $\mathcal{R}$ induces the mapping $\mathcal{S} : Q \to \mathcal{S}(Q)$.

Note that, with the help of Lemma 3.3, it is easy to see that from $\mathcal{R}(F[Q]) = 0$ it follows $\mathcal{S}(Q) = e$. Further, we will show that the mapping $\mathcal{S}$ is a radical of the class $\mathcal{L}$ of loops. For this, the class of loops $\mathcal{S}$ should satisfy the following conditions:

- the homomorphic image of any $\mathcal{S}$-loop is a $\mathcal{S}$-loop;
- each $\mathcal{L}$-loop Q contains a normal $\mathcal{S}$-subloop $\mathcal{S}(Q)$, containing all normal $\mathcal{S}$-subloops of the loop Q;
- the quotient loop $Q/\mathcal{S}(Q)$ does not contain non-unitary normal $\mathcal{S}$-subloops. ∎

Theorem 5.7. *The class $\mathcal{S}$ is radical in the class $\mathcal{L}$ of all Moufang loops.*

Proof. Let $G \in \mathcal{S}$. Then $F[G] \in \mathcal{R}_A$. Any homomorphism φ of loop G induces a homomorphism $\overline{\varphi} : F[G] \to F[\varphi G]$ defined by rules $\overline{\varphi}(\sum_{g\in G} \alpha_g g) = \sum_{g\in G} \alpha_g \varphi g$, $\overline{\varphi}(F[G]) = F[\varphi G]$. By Theorem 5.2 $\overline{\varphi}(F[G]) \in \mathcal{R}_A$. Then $F[\varphi G] \in \mathcal{R}$. Hence $\varphi G \in \mathcal{S}$. Consequently, the class $\mathcal{S}$ is closed under homomorphisms.

Let now Q be a Moufang loop and let, by Theorem 5.2, $\mathcal{R}(Q)$ be the $\mathcal{R}$-radical of $F[Q]$. By Lemma 3.3 the ideal $\mathcal{R}(Q)$ of $F[Q]$ induces the normal subloop $\mathcal{S}(Q) = Q \bigcap (e + \mathcal{R}(Q))$ of the loop Q. Let $\psi : F[Q] \to F[Q/\mathcal{S}(Q)]$ be the homomorphism defined by: $\sum_{g\in Q} \alpha_g g \to \sum_{g\in Q} \alpha_g (g\mathcal{S}(Q))$. Then, $F[Q]/\ker\psi \cong F[Q/\mathcal{S}(Q)]$ and $\mathcal{R}(Q) \subseteq \ker\psi$.

If $e \in \ker\psi$, then $\ker\psi = F[Q]$ and $F[Q/\mathcal{S}(Q)] \cong F[Q]/\ker\psi = F[Q]/F[Q] = 0$. We get a contradiction because $e \in F[Q/\mathcal{S}(Q)]$. Hence $\ker\psi$ is a proper ideal of $F[Q]$. Then, as it was shown in the beginning of section, $\ker\psi$ is an augmentation ideal. The radical $\mathcal{R}(Q)$ is a maximal augmentation ideal of $F[Q]$. Hence $\ker\psi = \mathcal{R}(Q)$ and $F[Q/\mathcal{S}(Q)] \cong F[Q]/\mathcal{R}(Q)$, $\mathcal{R}(F[Q/\mathcal{S}(Q)]) \cong \mathcal{R}(F[Q]/\mathcal{R}(Q))$. By Theorem 5.2, $\mathcal{R}(Q))$ is a maximal ideal of $F[Q]$ such that the $\mathcal{R}$-radical of the quotient-algebra $F[Q]/\mathcal{R}(Q)$ is zero. Hence $\mathcal{R}(F[Q/\mathcal{S}(Q)]) = 0$ and $\mathcal{S}(Q/\mathcal{S}(Q)) = 0$. Consequently, the normal subloop $\mathcal{S}(Q)$ of loop Q is maximal and such that $\mathcal{S}(Q/\mathcal{S}(Q)) = 0$. This completes the proof of Theorem 5.7. ∎

Let $Q \in \mathcal{L}$. By Theorem 5.7, the mapping $Q \to \mathcal{S}(Q)$ is a radical defined in the class of loops $\mathcal{L}$; denote it by $\mathcal{S}$. The normal subloop $\mathcal{S}(Q)$ of loop Q will be called its $\mathcal{S}$-*radical*. *A loop* coinciding with its $\mathcal{S}$-radical will be called $\mathcal{S}$-*radical*, and *the non-unitary loops*, whose radical is equal to unit, will be called $\mathcal{S}$-*semisimples*. The class $\mathcal{T}$ of all $\mathcal{S}$-semisimples algebras in class $\mathcal{L}$ will be called *semisimple class of radical* $\mathcal{S}$.

Proposition 5.8. *The radical $\mathcal{S}$ is hereditary in the class $\mathcal{L}$ of all Moufang loops, i.e. for any loop $Q \in \mathcal{L}$ and its normal subloop H, $\mathcal{S}(H) = H \bigcap \mathcal{S}(Q)$.*

Proof. By Theorem 5.7, the radical $\mathcal{S}(G)$ is a maximal normal subloop H of the loop G with respect to property $\omega[H] \in \mathcal{R}$. From item 3) of Proposition 4.4 it follows that $\omega[\mathcal{S}(G)]$ is a maximal ideal $\omega[H]$ of algebra $\omega[G]$ with respect to property $\omega[H] \in \mathcal{R}$. Then by Theorem 5.2 $\omega[\mathcal{S}(H)] = \mathcal{R}(\omega[H])$.

Let now H be a normal subloop of loop Q. Then $\omega[H]$ will be a normal subloop of loop $\omega[Q]$. By Corollary 5.3 $\mathcal{R}(\omega[H]) = \omega[H] \bigcap \mathcal{R}(\omega[Q])$ and by the precious equality $\omega[\mathcal{S}(H)] = \omega[H] \bigcap \omega[\mathcal{S}(Q)]$. Then from item 3) of Proposition 5.3 it follows from here that $\omega[\mathcal{S}(H)] = \omega[H \bigcap \mathcal{S}(Q)]$, $\mathcal{S}(H) = H \bigcap \mathcal{S}(Q)$. This completes the proof of Proposition 5.8. ∎

Corollary 5.9. *Let Q be a Moufang loop and let K be a normal subloop of Q. Then the following statements hold:*

(i) if $Q \in \mathcal{S}$ then $K \in \mathcal{S}$;

(ii) if $Q \in \mathcal{T}$ then $K \in \mathcal{T}$.

These follow from Proposition 3 and [36, Theorem 3, cap. 8].

6. SEMISIMPLE ALTERNATIVE LOOP ALGEBRAS AND SEMISIMPLE MOUFANG LOOPS

Let A be an algebra. *The sum of ideals* $\{I_s | s \in S\}$ of algebra A is called the ideal I of A generated by reunion $\bigcup_{s\in S} I_s$. The ideal I consists of elements x, presented in the form $x = x_1 + \ldots + x_k$, where $x_j \in I_{s_j}$ for some $s_j \in S$ and denote $I = \sum_{s\in S} I_s$. The sum is called *direct* if $I_s \bigcap \sum_{s\neq t\in S} I_t = 0$. Denote $I = \sum^{\oplus} I_s$ and $I = I_1 \oplus \ldots \oplus I_k$ for finite sum of ideals.

By analogy, *the product N of normal subloops* $\{N_s | s \in S\}$ of the loop Q consists of elements x, presented in the form $x = x_1 \cdot \ldots \cdot x_k$, where $x_j \in N_{s_j}$ for some $s_j \in S$ and denote $N = \prod_{s\in S} N_s$. The product is called *direct* if $N_s \bigcap \prod_{s\neq t\in S} N_t = 1$. Denote $N = \prod^{\otimes} N_s$ and $N = N_1 \otimes \ldots \otimes N_k$ for a finite factor product.

An ideal J of the algebra A is called *simple* if J does not have other ideals of A besides the null and *ideal J* itself and it is called *principal* if it is generated by one element. *A normal subloop N* of loop Q will be called *simple* if N does not have other normal subloops of Q besides the unitary subloop and loop N itself.

Lemma 6.1. *The following statements are equivalent for the simple normal subloops $\{N_s | s \in S\}$ of a loop Q:*

1) $Q = \prod_{s\in S} N_s$;

2) $Q = \prod^{\otimes}_{s\in S} N_s$.

Lemma 6.2. *The following statements are equivalent for the family of simple ideals $\{I_s | s \in S\}$ of algebra A:*

1) $A = \sum_{s\in S} I_s$;

2) $A = \sum^{\oplus}_{s\in S} I_s$.

Proof. Lemmas 6.1, 6.2 have similar proofs. Let us prove Lemma 6.2.

Let T be a maximal subset of S such that the sum $\sum_{t\in T} I_t$ is direct. The sum $\sum_{t\in T} I_t$ is an ideal of A. Let us show that this sum coincides with A. For this it is enough to show that each ideal I_j is contained in this sum. The intersection of our sum with I_j is an ideal in A and, consequently, equals 0 or I_j. If it equals 0, then subset T is not maximal, as we can add j to it. Consequently, I_j is contained in the sum $\sum_{t\in T} I_t$. This completes the proof of Lemmas 6.2. ∎

Let A be an F-algebra and let $I(M)$ be the ideal of A generated by set $M \subseteq A$. The ideal $I(M)$ consists of all possibly types of finite sums of elements of form

$$\varphi(x_1, \ldots, x_j, a, x_{j+1}, \ldots, x_n)_\alpha, \tag{33}$$

where $\varphi \in F$, $a \in M$, $x_i \in A$, α is a certain distribution of parenthesis.

Let $a, b \in A$. Then from (33) it follows that

$$I(a+b) \subseteq I(a) + I(b). \tag{34}$$

Now, we consider an ideal $\omega[Q]$ of the alternative loop algebra $F[Q]$ of a Moufang loop Q. If $a, b \in Q$ then $e - ab = (e-a) + (e-b) - (e-a)(e-b)$. Denote $e - u = \overline{u}$. By (34), $I(\overline{ab}) = I(\overline{a} + \overline{b} - \overline{a}\overline{b})$, $I(\overline{ab}) \subseteq I(\overline{a}) + I(\overline{b}) - I(\overline{a}\overline{b})$. From (33), it follows that $I(\overline{a}\overline{b}) \subseteq I(\overline{a})$. Then

$$I(\overline{ab}) \subseteq I(\overline{a}) + I(\overline{b}). \tag{35}$$

Moreover, the following result holds.

Lemma 6.3. *Let consider a principal ideal $I(\overline{a})$, for $a \in Q$, of ideal $\omega[Q]$ which is not simple. Then, there exists an element $b \in Q$ such that $I(\overline{a}) = I(\overline{b}) + I(\overline{c})$, where $a = bc$, and $I(\overline{b})$ is a proper ideal of algebra $I(\overline{a})$.*

Proof. Let J be a proper ideal of $I(\overline{a})$. By Lemma 3.3 the normal subloops B and A of the loop Q correspond to the ideals J, $I(\overline{a})$ and, by item 2) of Lemma 5.6, $B \subset A$. Let $b \in B$ and let $a = bc$. Then $b, c \in A$ and, by item 1) of Lemma 5.6, $e - b, e - c \in \omega[A] \subseteq I(\overline{a})$. Hence $I(\overline{b}) \subseteq I(\overline{a})$, $I(\overline{c}) \subseteq I(\overline{a})$. By (35), $I(\overline{a}) \subseteq I(\overline{b}) + I(\overline{c})$. Then $I(\overline{a}) = I(\overline{b}) + I(\overline{c})$, as required. ∎

Let $F[Q] \in \mathcal{P}$ and let $I(\overline{a})$, where $\overline{a} = e - a$, for $a \in Q$, be a principal ideal of $F[Q]$. As $F[Q] \in \mathcal{P}$ then, by item 2) of Corollary 5.4, $I(\overline{a}) \in \mathcal{P}$. Let A be the normal subloop of the loop Q induced, via Lemma 3.3, by the ideal $I(\overline{a})$. Then, by item 3) of Corollary 5.5, $I(\overline{a}) = F[A]$ and any element of $I(\overline{a})$ has the form

$$\sum_{i=1}^{n} \alpha_i u_i, \quad \sum_{i=1}^{n} \alpha_i \neq 0, \tag{36}$$

where $\alpha_i \in F$, $u_i \in Q$.

Further, for principal ideals $I(\overline{a_i})$, $I(\overline{b_j}), \ldots$ we use the notations $F[A_i] = I(\overline{a_i})$, $F[B_j] = I(\overline{b_j}), \ldots$ The symbols $\overline{FY}$ $\overline{F[A]}$ will denote the F-modules FY, $F[A]$.

Lemma 6.4. *Let $F[Q] \in \mathcal{P}$ and let $I(\overline{a})$, where $\overline{a} = e - a$, $a \in Q$, be a principal ideal of $F[Q]$. Then, there exists an element $b \in Q$ such that $I(\overline{a}) = I(\overline{b}) + I(\overline{c})$, where $a = bc$, $I(\overline{b})$ is a proper ideal of algebra $I(\overline{a})$ and $\overline{F[A]} = \overline{F[B]} \oplus M[K]$, where $M[K]$ denotes the F-submodule of $\overline{F[C]}$ generated by set $K = A \backslash B$.*

Proof. Let $Q = L/H$, where L is a free Moufang loop. We consider the homomorphisms $\varphi : LX \to LX/I = F[X]$, $\psi : F[X] \to F[L]/\omega[H] = F[Q]$ (see Proposition 3.4). From item 5) of Corollary 5.5, it follows that any element in $\omega[H]$ has the form

$$\sum_{j=1}^{m} \beta_j h_j, \quad \sum_{j=1}^{m} \beta_j = 0, \tag{37}$$

where $\beta_j \in F$, $h_j \in H$. By Lemma 6.3, $I(\overline{a}) = I(\overline{b}) + I(\overline{c})$ and $I(\overline{b})$ is a proper ideal of $I(\overline{a})$.

If we denote $\psi^{-1}(A) = X_A$, $\psi^{-1}(B) = X_B$, $\psi^{-1}(C) = X_C$, then $\psi^{-1}(I(\overline{a})) = \psi^{-1}(F[A]) = F[X_A]$, $\psi^{-1}(I(\overline{b})) = F[X_B]$, $\psi^{-1}(I(\overline{c})) = F[X_C]$. By (37), the homomorphism $F[X] \to F[X]/\omega[H]$ maintains the sum of coefficients, thus any element in $F[X_A]$, $F[X_B]$, $F[X_C]$ has the form

$$\sum_{i=1}^{k} \gamma_i x_i, \quad \sum_{i=1}^{k} \gamma_i \neq 0, \tag{38}$$

where $\gamma_i \in F, x_i \in X$. Then from (37), (38) it follows that $F[X_A] \bigcap \omega[H] = \{0\}$, $F[X_B] \bigcap \omega[H] = \{0\}$, $F[X_C] \bigcap \omega[H] = \{0\}$. Consequently, $F[A] = \psi(F[X_A]) = (F[X_A] + \omega[H])/\omega[H] \cong F[X_A]/(F[X_A] \bigcap \omega[H]) = F[X_A]/\{0\} = F[X_A]$, i.e. $F[X_A] \cong F[A]$. Similarly, $F[X_B] \cong F[B]$, $F[X_C] \cong F[C]$.

According to Lemma 3.1 $\varphi^{-1}(X_A) = X_A$, $\varphi^{-1}(X_B) = X_B$, $\varphi^{-1}(X_C) = X_C$. Hence $\varphi^{-1}(F[X_A]) = FX_A$, $\varphi^{-1}(F[X_B]) = FX_B$, $\varphi^{-1}(F[X_C]) = FX_C$.

From the definition of an ideal I of the loop algebra FX it follows that any element of I has the form $\sum_{j=1}^{n} \beta_j x_j$ with $\sum_{j=1} \beta_j = 0$, where $\beta_j \in F, x_j \in X$. Then from (38) it follows that $F[X_A] \bigcap I = \{0\}$, $F[X_B] \bigcap I = \{0\}$, $F[X_C] \bigcap I = \{0\}$ and $FX[A] = \varphi(FX_A) = (FX_A + I)/I \cong FX_A/(FX_A \bigcap I) = F[X_A]/\{0\} = F[X_A]$. Hence $FX_A \cong F[X_A]$. Before we have proved that $F[X_A] \cong F[A] = I(\overline{a})$. Consequently, $FX_A \cong I(\overline{a})$. Similarly, $FX_B \cong F[B] = I(\overline{b})$, $FX_C \cong F[C] = I(\overline{c})$.

The inverse image of equality $I(\overline{a}) = I(\overline{b}) + I(\overline{c})$ regarding homomorphism $\varphi\psi$, is the equality $FX_A = FX_B + FX_C$ of the loop algebra FX. The loop algebra FX is a free F-module with basis $\{x | x \in X\}$. Then $\overline{FX_A} = \overline{FX_B} \oplus M(X_A \backslash X_B)$, $M(X_A \backslash X_B) \subseteq \overline{FX_C}$. Hence $\overline{F[A]} = \overline{F[B]} \bigcap M[A \backslash B]$, $M[A \backslash B] \subseteq \overline{F[C]}$. This completes the proof of Lemma 6.4. ∎

Proposition 6.5. *Let $F[Q] \in \mathcal{P}$ and let $I(\overline{a})$, where $\overline{a} = e - a$, for $a \in Q$, be a principal ideal of $F[Q]$. Then $I(\overline{a})$ decomposes into a direct sum of finite number of simple nonassociative principal ideals $I(\overline{a}) = I(\overline{b}_1) \oplus \ldots \oplus I(\overline{b}_n)$.*

Proof. Inductively we construct two series

$$I(\overline{b}_1) \supset I(\overline{b}_2) \supset \ldots \supset I(\overline{b}_n) \supset \ldots,$$
$$I(\overline{d}_1) \subseteq I(\overline{d}_2) \subseteq \ldots \subseteq I(\overline{d}_n) \subseteq \ldots \tag{39}$$

of proper non-zero ideals of the algebra $I(\overline{a})$ such that $I(\overline{a}) = I(\overline{b}_n) + I(\overline{d}_n)$ and a series

$$M[K_1] \subset M[K_2] \subset \ldots M[K_n] \subset \ldots \tag{40}$$

of F-submodules of the F-module $\overline{F[A]}$ such that $M[K_i] \subseteq M[D_i]$ and $\overline{F[A]} = \overline{F[B_i]} \oplus M[K_i]$, $K_i = A \backslash B_i$. The inductive process stops if an ideal $I(\overline{b}_n)$ is simple for some integer n.

Let the ideal $I(\overline{a})$ be non-simple. Then by Lemma 6.4 $I(\overline{a}) = I(\overline{b}_1) + I(\overline{c}_1)$, where $I(\overline{b}_1)$ is a proper ideal of $I(\overline{a})$ and $\overline{F[A]} = \overline{F[B_1]} \oplus M[D_1]$, $D_1 = A \backslash B_1$, $M[D_1] \subseteq \overline{F[C_1]}$. If at least one of the ideals $I(\overline{b}_1)$, $I(\overline{c}_1)$ is simple then the inductive process ends. Conversely, let us consider that the ideal $I(\overline{b_1})$ is non-simple. By Lemma 6.4, let $I(\overline{b}_1) = I(\overline{b}_2) + I(\overline{c}_2)$, where $I(\overline{b}_2)$ is an ideal of $F[A]$ and is a proper ideal of $I(\overline{b}_1)$. Again by Lemma 6.4 $I(\overline{a}) = I(\overline{b}_2) + I(\overline{d}_2)$ and $\overline{F[A]} = \overline{F[B_2]} \oplus M[K_2]$, $K_2 = A \backslash B_2$, $M[K_1] \subseteq \overline{F[D_2]}$.

Let us continue the inductive process. Let $I(\overline{a}) = I(\overline{b}_n) + I(\overline{d}_n)$, $\overline{F[A]} = \overline{F[B_n]} \oplus M[K_n]$, $K_n = A \backslash B_n$, $M[K_n] \subseteq \overline{F[C_n]}$ and let the ideal $I(\overline{b}_n)$ be non-simple. By Lemma 6.4 $I(\overline{b}_n) = I(\overline{b}_{n+1}) + I(\overline{c}_{n+1})$ and $I(\overline{a}) = I(\overline{b}_{n+1}) + I(\overline{d}_{n+1})$ and $\overline{F[A]} = \overline{F[B_{n+1}]} \oplus M[K_{n+1}]$, $K_{n+1} = A \backslash B_{n+1}$. From $I(\overline{b}_n) \supset I(\overline{b}_{n+1})$ it follows that $I(\overline{d}_n) \subset I(\overline{d}_{n+1})$ and $K_n \subset K_{n+1}$, $M[K_n] \subset M[K_{n+1}]$. Consequently, the series (39), (40) with property $\overline{F[A]} = \overline{F[B_n]} \oplus M[K_n]$, $n = 1, 2, \ldots$ are defined.

The modules $M[K_n]$ in the ascending series (40) satisfy the property $\overline{b}_1 \notin M[K_n]$. Then, by Zorn Lemma, this series have a maximal proper ideal J in $I(\overline{a})$ such that $\overline{b}_1 \notin J$. Let $\overline{b} \in I(\overline{a}) \setminus J$. As J is a maximal ideal of $I(\overline{a})$ then $I(\overline{a}) = I(\overline{b}) + J$.

Let the ideal $I(\overline{b})$ be non-simple. Then, by Lemma 6.4, $I(\overline{b}) = I(\overline{b}_1) + I(\overline{b}_2)$, where $I(\overline{b}_1)$ is a proper non-zero ideal of $I(\overline{a})$, $I(\overline{a}) = I(\overline{b}_1) + I(\overline{b}_2) + J$ and the ideal $I(\overline{b}_2) + J$ strictly contain the maximal ideal J. Contradiction. Hence the ideal $I(\overline{b})$ is simple.

By Lemma 6.4 $I(\overline{a}) = I(\overline{b}) + I(\overline{d})$ for some proper ideal $I(\overline{d})$ of $I(\overline{a})$. The ideal $I(\overline{b})$ is simple. Let the ideal $I(\overline{d})$ is non-simple. Then $I(\overline{a}) = I(\overline{b}_1) + I(\overline{d}_1)$, where $I(\overline{d}_1) = I(\overline{d})$ and $I(\overline{b}_1) = I(\overline{b})$ is a simple ideal. Further, $I(\overline{a}) = I(\overline{b}) + I(\overline{b}_2) + \ldots + I(\overline{b}_k) + \ldots$, where $I(\overline{b}_i)$, $i = 1, 2, \ldots$, is a simple ideal. Then, by Lemma 6.2, $I(\overline{a}) = I(\overline{b}) \oplus I(\overline{b}_2) \oplus \ldots \oplus I(\overline{b}_k) \oplus \ldots$, where each $I(\overline{b}_i)$, $i = 1, 2, \ldots$, is a simple ideal.

It is known that any element of the direct sum is written unequivocally as the sum of a finite number of non-zero elements, taken one from some ideals $I(\overline{b}_{i_j})$. Let $\overline{a} = \overline{b}_1 + \ldots + \overline{b}_n$, where $\overline{b}_j \in I(\overline{b}_{i_j})$. Then by (34) $I(\overline{a}) \subseteq I(\overline{b}_{i_1}) \oplus \ldots \oplus I(\overline{b}_{i_n})$ and, consequently, $I(\overline{a}) = I(\overline{b}_{i_1}) \oplus \ldots \oplus I(\overline{b}_{i_n})$. As $F[Q] \in \mathcal{P}$ then by [36, Theorem 3, cap. 8] $I(\overline{b_{i_j}}) \in \mathcal{P}$ and by item 3) of Corollary 5.5 the ideals $I(\overline{b_{i_j}})$ are nonassociative. This completes the proof of Proposition 6.5. ∎

Lemma 6.6. *Let $Q \in \mathcal{T}$ be a nonassociative semisimple Moufang loop, let $F[Q] \in \mathcal{P}$ be it corresponding alternative loop algebra and let A, B be normal subloops of the loop Q. Then $A \subset B$ (respect. $A = B$) when and only when $\omega[A] \subset \omega[B]$ (respect. $\omega[A] = \omega[B]$).*

Proof. Let $Q = L/H$, where L is a free Moufang loop and let $\varphi : LX \to LX/I = F[X]$, $\psi : F[X] \to F[L]/\omega[H] = F[Q]$ be the homomorphisms considered in proof of Lemma 6.4. Let $\psi^{-1}(A) = X_A$, $\psi^{-1}(B) = X_B$, $X_A, X_B \subseteq X$. It is proved that $FX_A \cong F[X_A] \cong F[A] = \omega[A]$, $FX_B \cong F[X_B] \cong F[B] = \omega[B]$. From (36) – (38), it follows easily that the restrictions of homomorphism ψ on X_A and on X_B are isomorphisms of loops X_A, A and X_B, B, respectively.

From the mentioned isomorphisms, it follows that the inclusions $F[A] \subset F[B]$ in the alternative loop algebra $F[Q]$ and $FX_A \subset FX_B$ in the loop algebra FX are equivalent. The loop algebra FX is a free F-module with basis $\{x|x \in X\}$. Then the inclusion $FX_A \subset FX_B$ is equivalent to inclusion $X_A \subset X_B$ of subloops in the loop X. Further, from isomorphisms of loops X_A, A and X_B, B it follows that the inclusion $X_A \subset X_B$ is equivalent to inclusion $A \subset B$. Consequently, the inclusion $F[A] \subset F[B]$ in the alternative loop algebra $F[Q]$ is equivalent to inclusion $A \subset B$ of subloops in the loop Q. The facts that the equalities $F[A] = F[B]$ and $A = B$ are equivalent are analogously proved. ∎

Proposition 6.7. *Let $Q \in \mathcal{T}$ be nonassociative semisimple Moufang loop and let $F[Q] \in \mathcal{P}$ be its corresponding alternative loop algebra. Then for any element $a \in Q$ the normal subloop $N(a)$ of Q generated by element a decompose into a direct product of finite number of nonassociative simple loops.*

Proof. By Proposition 6.5 $I(\overline{a}) = I(\overline{a}_1) \oplus \ldots \oplus I(\overline{a}_k)$, where each $I(\overline{a}_i)$ is a simple ideal of $F[Q]$ generated by element $\overline{a}_i = e - a_i$, for $a_i \in Q$, $i = 1, \ldots, k$. By Lemma 3.3, the ideal $I(\overline{a}_i)$ induces in Q the normal subloop $H_i = Q \bigcap (e + I(\overline{a}_i))$. Let $N(a_i)$ denote the normal subloop of Q generated by the element $a_i \in Q$. It is clear that $a_i \in H_i$. Then $N(a_i) \subseteq H_i$. If $N(a_i) \subset H_i$ (strictly) then by Lemma 6.6 $\omega[N(a_i)] \subset \omega[H_i]$ (strictly). But $\omega[H_i] = I(\overline{a})$. Hence $\omega[N(a_i)] \subset I(\overline{a})$ (strictly), i.e. $\omega[N(a_i)]$ is a proper ideal of $I(\overline{a}_i)$. We get a contradiction because $I(\overline{a}_i)$ is a simple ideal. Consequently, $\omega[N(a_i)] = I(\overline{a}_i)$.

If K is a proper normal subloop of $N(a_i)$ then, by Lemma 6.6, $\omega[K]$ is a proper ideal of $\omega[N(a_i)] = I(\overline{a}_i)$. Again we get a contradiction. Hence the normal subloops $N(a_i)$, $i = 1, \ldots, k$, are simple. We have

$$I(\overline{a}) = I(\overline{a}_1) \oplus \ldots \oplus I(\overline{a}_k) \tag{41}$$

or $\omega[N(a)] = \omega[N(a_1)] + \ldots + \omega[N(a_k)]$. Then, by item 2) of Lemma 6.6, $\omega[N(a)] = \omega[N(a_1)] \cdot \ldots \cdot N(a_k)]$ and, by Lemma 6.6, $N(a) = N(a_1)] \cdot \ldots \cdot N(a_k)$. The subloops $N(a_i)$, $i = 1, \ldots, k$, are simple. Then, by Lemma 6.1,

$$N(a) = N(a_1) \otimes \ldots \otimes N(a_k). \tag{42}$$

This completes the proof of Proposition 6.7. ∎

Corollary 6.8. *Let $Q \in \mathcal{T}$ be a nonassociative semisimple Moufang loop and let $F[Q] \in \mathcal{P}$ be it corresponding alternative loop algebra. Then:*

1) any nonassociative simple subloop of the loop Q has the form $H = I(\overline{a})$, where $I(\overline{a})$ is a normal subloop of $F[Q]$, with $\overline{a} = e - a$ for some $a \in Q$;

2) any nonassociative simple subalgebra of algebra $F[Q]$ has the form $F[H] = \omega[H]$, where $H = I(\overline{a})$, with $\overline{a} = e - a$, for some $a \in Q$;

3) a nonassociative subalgebra $F[H]$ of algebra $F[Q]$ is simple when and only when the nonassociative normal subloop H of loop Q is simple.

The corollary follows from (41), (42) and Lemma 6.6.

Lemma 6.9. *Any algebra A of semisimple class $\mathcal{P}$ of radical $\mathcal{R}$ decomposes into a direct sum of nonassociative simple algebras.*

Proof. By item 3) of Corollary 5.5, any algebra A of semisimple class $\mathcal{P}$ has the form $A = F[Q]$ and $F[Q] = \omega[Q]$. The ideal $\omega[Q]$ is generated as ideal by set $\{e-g|g \in Q\}$. Let $g_1 \in Q$. As $(e-g)g_1 = (e-gg_1)-(e-g_1)$, $g_1(e-g) = (e-g_1g)-(e-g_1)$ then $\omega[Q]$ is generated as F-module by elements of form $e-g$, where $g \in Q$. Denote $e-g = \overline{g}$ and let $I(\overline{g})$ be the (principal) ideal generated by element $\overline{g} \in F[Q]$. Then

$$F[Q] = \sum_{g \in Q} I(\overline{g}). \tag{43}$$

As $F[Q] \in \mathcal{P}$ then by item 2) of Corollary 5.4 $I(\overline{g}) \in \mathcal{P}$. By Proposition 6.5, $I(\overline{g}) = I(\overline{b}_1) \oplus \ldots \oplus I(\overline{b}_k)$, where $I(\overline{b}_i)$ is a simple ideal of $F[Q]$. Then, from (43), it follows that $F[Q] = \sum I(\overline{b}_i)$ and, by Lemma 6.2, $F[Q] = \sum^{\oplus} I(\overline{b}_i)$, where $I(\overline{b}_i)$ are simple ideals of algebra $F[Q]$. This completes the proof of Lemma 6.9. ∎

Corollary 6.10. *Let $charF = 0$ or $charF = 3$. Then any nonassociative commutative Moufang loop Q is $\mathcal{S}$-radical and any it alternative loop algebra $F[Q]$ is $\mathcal{R}$-radical.*

Proof. According to Theorem 5.2 $F[Q]/\mathcal{R}(F[Q]) = P(F[Q])$, where $P(F[Q]) \in \mathcal{P}$. We assume that $P(F[Q]) \neq Fe$. As the loop Q is commutative then by (31) the algebra $P(F[Q])$ also is commutative. From Lemma 6.9 it follows that $P(F[Q])$ decomposes into a direct sum of nonassociative simple algebras. But any commutative simple alternative algebra is a field [36, pag. 172]. We get a contradiction. Hence $P(F[Q]) = Fe$. Then, by using the definitions, $F[Q] \in \mathcal{R}$ and $Q \in \mathcal{S}$, as required. ∎

Lemma 6.11. *Any nonassociative semisimple Moufang loop $Q \in \mathcal{T}$ decomposes into a direct product of nonassociative simple loops.*

The statement follows from (41 - 43) and Lemmas 6.1, 6.6.

7. MAIN RESULTS

Let us consider the following notions. In the beginning of the paper, we have mentioned that, in the literature, an algebra is called antisimple, if none of its two-sided ideals allows homomorphism on a simple algebra.

An alternative loop algebra $F[Q] \in \mathcal{M}$ will be called *antisimple with respect to nonassociativity* if for any its ideal $\omega[H]$ the algebra $(\omega[H])^{\sharp} = F[H]$ does not allow homomorphism on a simple nonassociative algebra.

Analogously, a loop $Q \in \mathcal{L}$ will be called *antisimple with respect to nonassociativity* if none of its normal subloops allows homomorphism on a simple nonassociative loop.

Let $\mathcal{A}$ be a class of algebras, let $\mathcal{B}$ be its radical class and let $\mathcal{C}$ be it semisimple class. By the definition of radical $\mathcal{B}$ any homomorphic image of $\mathcal{B}$-algebra is a $\mathcal{B}$-algebra. In [36, Proposition 1, pag. 184] it is proved that the radical class $\mathcal{B}$ of $\mathcal{A}$ is the totality of algebras from $\mathcal{A}$, not reflected homomorphically on the algebras of class $\mathcal{C}$. Then from Lemmas 6.9, 6.11 it follows the next result.

Lemma 7.1. *The class of all antisimple with respect to nonassociativity alternative loop algebras $F[Q]$ coincides with the radical class $\mathcal{R}_A$ of all alternative loop algebras of type $F[Q] = (\omega[Q])^\sharp$. The class of all antisimple with respect to nonassociativity Moufang loops coincides with the radical class $\mathcal{S}$ of Moufang loops.*

Proposition 7.2 *An alternative loop algebra $F[Q]$ is antisimple with respect to nonassociativity when and only when $F[Q]$ does not have subalgebras that are nonassociative simple algebras. A Moufang loop Q is antisimple with respect to nonassociativity when and only when Q does not have subloops that are nonassociative simple loops.*

Proof. Let $F[Q] \in \mathcal{M}$ and, according to Theorem 5.2, let $F[Q]/\mathcal{R}(F[Q]) = P(F[Q])$, where $P(F[Q]) \in \mathcal{P}$. By Lemma 7.1, the algebra $F[Q]$ is antisimple with respect to nonassociativity when and only when $P(F[Q]) = Fe$.

We assume that $P(F[Q]) \neq Fe$. By Lemma 3.3, the ideal $\mathcal{R}(F[Q])$ of algebra $F[Q]$ induces the normal subloop $R = Q \bigcap (e + \mathcal{R}(F[Q]))$ of loop Q. As $P(F[Q]) \neq Fe$ then $\mathcal{R}(F[Q])$ is a proper ideal of $F[Q]$ and, according to the one-to-one mapping among all ideals of $F[Q]$ and all normal subloops of Q the normal subloop R is proper and $F[Q/R] \cong F[Q]/\mathcal{R}(F[Q]) = P(F[Q])$. By Lemma 6.9, $P(F[Q]) \neq Fe$ decomposes into a direct sum of nonassociative simple algebras. According to Corollary 6.8, let $F[\overline{H}] = \omega[\overline{H}]$, where $\overline{H}$ is the normal subloop of loop Q/R generated by one element $hR = h + \mathcal{R}(F[Q])$, be one of such nonassociative simple algebras. We denote by H the normal subloop of the loop Q generated by the element $h \in Q$.

Clearly, the inverse image of subalgebra $F[\overline{H}]$ under the natural homomorphism $F[Q] \to F[Q]/\mathcal{R}(F[Q]) = P(F[Q])$ is $F[H] + \mathcal{R}(F[Q]$. For $a \in \mathcal{R}(F[Q])$, we have $a = \sum_{q \in Q} \alpha_q q$ with $\sum_{q \in Q} \alpha_q = 0$, by item 5) of Corollary 4.5, and for $b \in F[\overline{H}]$, we have $b = \sum_{\overline{g} \in \overline{H}} \beta_{\overline{g}} \overline{g}$ with $\sum_{\overline{g} \in \overline{H}} \beta_{\overline{g}} \neq 0$, by item 2) of Corollary 5.5. The extension $F[Q]/\mathcal{R}(F[Q])$ does not change the sum of coefficients. Hence if $c \in F[H]$ then $c = \sum_{g \in H} \gamma_g g$ with $\sum_{g \in H} \gamma_g \neq 0$. Then $F[H] \bigcap \mathcal{R}(F[Q]) = (0)$ and, by homomorphism theorems, it follows $F[\overline{H}] \cong (F[H] + \mathcal{R}(F[Q]))/\mathcal{R}(F[Q]) \cong F[H]/(F[H] \bigcap \mathcal{R}(F[Q]) = F[H]$. Hence, the subalgebra $F[H]$ of the algebra $F[Q]$ is a nonassociative simple algebra. Then, by item 3) Corollary 6.8, H is a nonassociative simple loop. Consequently, if $P(F[Q]) \neq (0)$ then:

(i) the algebra $F[Q]$ contains a nonassociative simple algebra $F[H]$;

(ii) the loop Q contains a nonassociative simple loop H.

If the algebra $F[Q]$ does not contain a nonassociative simple loop then, from item 2) of Corollary 5.5, it follows that $\sum_{q\in Q}\alpha_q = 0$ for any element $a = \sum_{q\in Q}\alpha_q q \in \omega[Q]$. In such a case, $F[Q] = (\omega[Q])^\sharp$ and, from Lemma 5.1, it follows that $P(F[Q]) = Fe$. This completes the proof of the first statement.

Now, let $Q \in \mathcal{L}$ be a Moufang loop and, according to Theorem 5.7, let $Q/\mathcal{S}(Q) = T(Q)$, where $T(Q) \in \mathcal{T}$. By Lemma 7.1, the loop Q is antisimple with respect to nonassociativity when and only when $T(Q) = \{e\}$. We prove that the equality $T(Q) = \{e\}$ is equivalent to the property that the loop Q does not contain a subloop isomorphic to simple nonassociative loop.

Indeed, we assume that $T(Q) \neq \{1\}$. Then, from the relation $Q/\mathcal{S}(Q) = T(Q)$, it follows that $Q \neq \mathcal{S}(Q)$ and, from the definition of the class $\mathcal{S}$, it follows that $\mathcal{R}(F[Q]) \neq F[Q]$ and $P(F[Q]) \neq Fe$, in accordance with the relation $F[Q]/\mathcal{R}(F[Q]) = P(F[Q])$. In such a case, the loop Q contains a nonassociative simple loop H, by statement (ii).

Now let the loop Q not contain any nonassociative simple loop as subloop. Then, from item 3) of Corollary 6.8, it follows that the alternative loop algebra $F[Q]$ does not contains nonassociative simple algebra as subalgebra. Thus, by the first case, $P(F[Q]) = Fe$ or $\mathcal{R}(F[Q]) = F[Q]$, $\mathcal{S}(Q) = Q$, $T(Q) = \{e\}$. This completes the proof of Proposition 7.2. ∎

Let us consider the analogue for alternative loop algebras of the Wedderburn Theorem for finite dimensional associative algebras.

By Kleinfeld Theorem [36] any nonassociative simple alternative algebra is a Cayly-Dickson algebra over its centre. Then, from Theorem 5.2 and Lemmas 6.9, 7.1, it follows the next result.

Proposition 7.3. *Let $F[Q]$ be an alternative loop algebra from class $\mathcal{M}$ and let $\mathcal{R}(F[Q])$ be its radical. Then algebra $(\mathcal{R}(F[Q]))^\sharp = F[G]$, $G \subseteq Q$, $F[G] \in \mathcal{R}_A$, is a nonassociative antisimple with respect to nonassociativity or, equivalently, it does not contain subalgebras that are nonassociative simple algebras and the quotient-algebra $F[Q]/\mathcal{R}(F[Q])$ is a direct sum of Cayley-Dickson algebras over their centre.*

As it was above mentioned, the nonassociative antisimple with respect to nonassociativity algebras are considered in Propositions 4.3, 4.4, 7.2 and Corollary 4.5.

Let now $Q \in \mathcal{L}$ be a nonassociative Moufang loop. According to Theorem 5.7, $Q/\mathcal{S}(Q) = T(Q)$, where $T(Q) \in \mathcal{T}$, $\mathcal{S}$ is the radical class, $\mathcal{T}$ is the semisimple class for class loop $\mathcal{L}$. Further, as a rule in the theory of algebraic systems, in order to study the loops of class $\mathcal{L}$ we will consider the loops of classes $\mathcal{S}$ and $\mathcal{T}$ separately.

To describe class $\mathcal{T}$, we remind the description of nonassociative simple Moufang loops from [33]. Let $M(F)$ denote the *matrix Paige loop* constructed, over the field F, as in [22]. That is, $M(F)$ consists of vector matrices

$M^*(F) = \begin{pmatrix} \alpha_1 & \alpha_{12} \\ \alpha_{21} & \alpha_2 \end{pmatrix}$, where $\alpha_1, \alpha_2 \in F$, $\alpha_{12}, \alpha_{21} \in F^3$,
$\det M^* = \alpha_1\alpha_2 - (\alpha_{12}, \alpha_{21}) = 1$, and where M^* is identified with $-M^*$.
The multiplication in $M(F)$ coincides with the Zorn matrix multiplication

$$\begin{pmatrix} \alpha_1 & \alpha_{12} \\ \alpha_{21} & \alpha_2 \end{pmatrix}\begin{pmatrix} \beta_1 & \beta_{12} \\ \beta_{21} & \beta_2 \end{pmatrix} =$$

$$\begin{pmatrix} \alpha_1\beta_1 + (\alpha_{12}, \beta_{21}) & \alpha_1\beta_{12} + \beta_2\alpha_{12} - \alpha_{21} \times \beta_{21} \\ \beta_1\alpha_{21} + \alpha_2\beta_{21} + \alpha_{12} \times \beta_{12} & \alpha_2\beta_2 + (\alpha_{21}, \beta_{12}) \end{pmatrix},$$

where, for vectors $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, $\delta = (\delta_1, \delta_2, \delta_3) \in A^3$, $(\gamma, \delta) = \gamma_1\delta_1 + \gamma_2\delta_2 + \gamma_3\delta_3$ denotes their scalar product and $\gamma \times \delta = (\gamma_2\delta_3 - \gamma_3\delta_2, \gamma_3\delta_1 - \gamma_1\delta_3, \gamma_1\delta_2 - \gamma_2\delta_1)$ denotes the cross vector product.

Let Δ be a prime field and let P be its algebraic closure. In [33], it was proved that only and only the Paige loops $M(F)$, where F is a Galois extension over Δ in P are, up to an isomorphism, nonassociative simple Moufang loops. [33] also describes the finite nonassociative simple Moufang loops and the set of generators, the group of automorphisms of nonassociative simple Moufang loops.

An analogue for loops of the Wedderburn Theorem (for finite-dimensional associative algebras), follows from Theorem 5.7, Lemmas 6.10, 7.1 and Proposition 7.2.

Theorem 7.4. *Let Δ be a prime field, P be its algebraic closure, and F be a Galois extension over Δ in P. Then the radical $\mathcal{S}(Q)$ of a Moufang loop Q is nonassociative antisimple with respect to nonassociativity or, equivalently, it does not contain any subloops that are nonassociative simple loops and quotient-loop $Q/\mathcal{S}(Q)$ is isomorphic to the direct product of matrix Paige loops $M(F)$.*

Let G be a finite Moufang loop. Obviously, from the finiteness of G it follows that for any subloop H of G there exists a normal subloop K of H such that the composition factor H/K is a simple loop. According to [14], a finite Moufang loop G it said to be a loop of *group type* if all composition factors of G are groups. It is clear that the homomorphic image of loop of group type is a loop of group type and the product of two normal subloops of group type is again a loop of group type. Hence every finite Moufang loop has a unique maximal normal subloop of group type ([14, Proposition 1]). We denote this maximal normal subloop of group type by $Gr(G)$. It is obvious that $Gr(G/Gr(G)) = e$. Hence $Gr(G)$ is a radical of G. By [14] $Gr(G)$ is called the *group-type radical* of G. Obviously, for any finite Moufang loop G, the definition of normal subloop $Gr(G)$ is equivalent with the condition: $Gr(G)$ does not contain any subloops that are nonassociative simple loops, by Theorem 7.4. Hence, for finite loops, the group-type radical Gr coincides with radical $\mathcal{S}$.

In the proof of the main result from [14] about the existence of quasi-p-Sylow subloops in every finite Moufang loop, the following structural Theorem B is used essentially: *every finite Moufang loop M contains uniquely determined normal subloops*

$Gr(M)$ and M_0 such that $Gr(M) \leq M_0$, M/M_0 is an elementary abelian 2-group, $M_0/Gr(M)$ is the direct product of simple Paige loops $M(q)$ (where q may vary), the composition factors of $Gr(M)$ are groups and $Gr(M/Gr(M)) = 1$.

The proof of Theorem B is based on the correspondence between Moufang loops and groups with triality [7]. The proof of Theorem B is quite cumbersome and uses deep results from finite groups. Moreover, the Theorem B in such a version does not hold true. For example, the case when M is a simple loop leads to a contradiction with the condition that M/M_0 is an elementary abelian 2-group. In reality, M/M_0 is the unitary group, i.e. $M = M_0$. In such a case Theorem B is a particular case of Theorem 7.4. Hence if the corresponding results from this paper are used in the proofs of the main results from [14], then these proofs become as it is shown below.

From Theorem 7.4, it follows that the loops from semisimple class $\mathcal{T}$ are well described. However, unlike the class $\mathcal{T}$, much less is known about the qualities and construction of loops from the radical class $\mathcal{S}$. A new approach is suggested for the study of the loops in the class $\mathcal{S}$ (though some authors made some attempts earlier):

a) by using the one-to-one mapping between loops $Q \in \mathcal{S}$ and alternative loop algebras $F[Q] \in \mathcal{R}$, below indicated in Theorem 7.5;

b) by using the developed theory of alternative algebras, in particular, of the algebras with externally adjoined unit, of Zhevlakov radicals, of circle loops and others.

According to Lemma 7.1 and Proposition 7.2 the following statements are equivalent for a nonassociative Moufang loop $Q \in \mathcal{L}$:

r1) $Q \in \mathcal{S}$;

r2) Q is an loop antisimple with respect to nonassociativity;

r3) the loop Q does not have any subloops that are simple loops.

Then the opposite statements

nr1) $G \notin \mathcal{S}$, i.e. $G \in \mathcal{L}\backslash\mathcal{S}$;

nr2) G is not antisimple with respect to nonassociativity loop;

nr3) the loop G has subloops that are simple loops

hold for any nonassociative Moufang loop $G \in \mathcal{L}\backslash\mathcal{S}$.

From the definition of class of alternative loop algebras $\mathcal{R}_A$, definition of class of loops $\mathcal{S}$ and Theorem 4.2, it follows that if a nonassociative Moufang loop Q satisfies the condition r1) then the loop Q can be embedded into the loop of invertible elements $U(F[Q])$ of the alternative loop algebra $F[Q]$. On the other hand, [29] proves that if a nonassociative Moufang loop G satisfies the condition nr2) then the loop Q is not imbedded into the loop of invertible elements $\mathcal{U}(A)$ for a suitable unital alternative F-algebra A, where F is an associative commutative ring with unit. As $\mathcal{S} \bigcap (\mathcal{L}\backslash\mathcal{S}) = \varnothing$ then the main result of this paper follows from the above statements.

Theorem 7.5. *Any nonassociative Moufang loop Q that satisfies one of the equivalent conditions r1) - r3) can be embedded into a loop of invertible elements $U(F[Q])$ of alternative loop algebra $F[Q]$. The remaining loops of class of all nonassociative Moufang loops $\mathcal{L}$, i.e.the loops $G \in \mathcal{L}$ that satisfy one of the equivalent conditions*

nr1) - nr3) cannot be embedded into a loop of invertible elements of any unital alternative algebras.

From Corollary 6.10 and Theorem 7.5 the following Corollary follows.

Corollary 7.6. *Any commutative Moufang loop Q can be embedded into a loop of invertible elements $U(F[Q])$ of the commutative alternative loop algebra $F[Q]$.*

Corollary 7.7. *Any finite Moufang p-loop Q can be embedded into a loop of invertible elements $U(F[Q])$ of alternative loop algebra $F[Q]$.*

Proof. According to [33], with up to an isomorphism, only Paige loops $M(q)$ over a finite field F_q are finite simple Moufang loops. By [22] the order of $M(Q)$ is $(1/d)q^3(q^4-1)$, where $d = gcd(2, q-1)$, and is the product of two coprime numbers $q^3(q^2-1)$ and $(1/d)q^3(q^2+1)$. Q is a finite p-loop if the order of elements from Q is a power of p (for Moufang loops, this is equivalent to the condition that the order of Q be a power of p). From here it follows that the finite p-loop satisfies the condition r3). By Theorem 7.5 the Corollary 7.7 is proved. ∎

From Theorem 7.5, Corollary 7.6 and (10) the following Corollary follows.

Corollary 7.8. *Any nonassociative Moufang loop Q that satisfy one of the equivalent conditions r1) - r3) can be embedded into a circle loop $(U^\star(F[Q]), \circ)$ of alternative loop algebra $F[Q]$. The remaining loops of class of all nonassociative Moufang loops $\mathfrak{L}$, i.e.the loops $G \in \mathfrak{L}$ that satisfy one of the equivalent conditions nr1) - nr3) cannot be embedded into circle loops of any unital alternative algebras.*

Corollary 7.9. *Any commutative Moufang loop Q can be embedded into a circle loop $(U^\star(F[Q]), \circ)$ of the alternative loop algebra $F[Q]$.*

In [12], the circle loops are examined under condition that the underlying sets defining the alternative algebra and the loop are identical. Under this supposition it is proved that for any prime p, there are no Moufang circle loops of order dividing p^4 which are not associative, though non associative Moufang loops of order p^4 exist [4]. It is also proved that no commutative Moufang which is not associative is the circle loop of an alternative nil ring of index 2. But from Corollaries 7.6, 7.7 the following Corollary follows.

Corollary 7.10. *Any commutative Moufang loop Q can be embedded into a circle loop $(U(F[Q], \circ, 0)$ of the alternative loop algebra $F[Q]$.*

Corollary 7.11. *Any finite Moufang p-loop Q can be embedded into a circle loop of invertible elements $(U(F[Q], \circ, 0)$ of the alternative loop algebra $F[Q]$.*

Now let us present some examples that were proved on the basis on the correspondence between commutative Moufang loops and loops of invertible elements of commutative alternative algebras (Corollary 6.6).

1^0. **Bruck's Theorem.** This theorem is one of the profound results in the theory of commutative Moufang loops: *a commutative Moufang loop with n ($n \geq 2$) generators*

is centrally nilpotent of class at most $n-1$ [2], Chap. VIII]. The proof of this assertion is very cumbersome; it is based on a complicated inductive process and uses several hundred nonassociative identities. In [19, Chap. 1] Manin used group methods to prove a weaker assertion, namely that any finite commutative Moufang loop of period 3 is centrally nilpotent. Although less calculative, his proof is by no means simple; it uses deep facts from finite group theory. The supremum of the central nilpotence class of a commutative Moufang loop with n generators is equal to $n-1$ [17].

In [32], the relationship between commutative Moufang loops and alternative commutative algebras, i.e. the Corollary 7.6, (in [32] the proof is not very convincing) is used to prove (rather simply) that any finitely generated commutative Moufang loop is centrally nilpotent. In the proof, we use the fact that any alternative commutative nil-algebra of index 3 is locally nilpotent, only.

In [23], a Moufang loop E is called *special*, if it can be embedded in the loop $U(A)$ of invertible elements of an alternative algebra A with unit. The Bruck Theorem is proved in a quite transparent manner (and an accurate appraisal is made) for special commutative Moufang loops. In the proof the assertion that the commutator ideal of the multiplication algebra of a free commutative alternative algebra with n free generators is nilpotent of index $n-1$ is transferred on such loops. Consequently, according to Corollary 7.5, in [23] the Bruck Theorem (and the accurate appraisal) is proved for any commutative Moufang loops.

2^0. **Infinite independent system of identities.** In [6] Slin'ko has formulated the question: if any variety of solvable alternative algebras would be finitely based. Umirbaev has got an affirmative answer to this question for alternative algebras over a field of characteristic $\neq 2, 3$ (see [35]), while Medvedev [20] gave a negative answer for characteristic 2. The topic of work [28] is the transfer of infinite independent systems of a commutative Moufang loop, constructed in [26] on solvable alternative commutative algebra over a field of characteristic 3 (another example was constructed by Badeev, see [1]), provided that holds the Corollary 7.6. Consequently, the last result together with the former results, completes the statement of Slin'ko problem for solvable alternative algebras.

3^0. **The order of free commutative Moufang loops of exponent 3.** Let L_n be the free commutative Moufang loop on n generators of exponent 3 with unit e and let $\mid L_n \mid = 3^{\delta(n)}$. The Manin problem asks to calculate $\delta(n)$ [19]. One of the main results of paper [13] is that $\delta(3) = 4$, $\delta(4) = 12$, $\delta(5) = 49$, $\delta(6) = 220$, $\delta(7) = 1014$ or 1035 and $\delta(7) = 1014$ if and only if L_n can be embedded into a loop of invertible elements of a unital alternative commutative algebra. It is also proved that the free loop L_n on $n < 7$ generators is embedded into a loop of invertible elements $U(A)$ for a unital alternative commutative algebra A. Moreover, L_7 may be embedded in $U(A)$ if and only if the following identity is true for commutative Moufang loops

$$\begin{aligned}&((((a,x,y),z,b),t,c),b,c)((((a,x,z),y,b),t,c),b,c)\\&((((a,x,t),y,b),z,c),b,c)^{-1}((((a,x,b),y,z),t,c),b,c)\end{aligned} \tag{44}$$

$$((((a,x,c),y,z),t,b),b,c)((((a,x,b),y,c),z,t),b,c) = e.$$

Here $(x,y,z) = (xy)z \cdot (x(yz))^{-1}$. According to Corollary 7.6, from here it follows that $\delta(7) = 1014$ and the identity (44) is true for any commutative Moufang loop.

8. FINITE MOUFANG *P*-LOOPS

Let Q be a loop with unit e. The set $\{z \in Q \mid zx = xz,\ zx \cdot y = z \cdot xy,\ xz \cdot y = x \cdot zy,\ xy \cdot z = x \cdot yz \quad \forall x, y \in Q\}$ is a subloop $Z(Q)$ of Q, the *centre*. $Z(Q)$ is an abelian group, and every subgroup of $Z(Q)$ is a normal subloop of Q. If $Z_1(Q) = Z(Q)$, then the normal subloops $Z_{i+1}(Q) : Z_{i+1}(Q)/Z_i(Q) = Z(Q/Z_i(Q))$ are inductively determined. A loop Q is called *centrally nilpotent of class n*, if its *upper central series* have the form

$$\{e\} \subset Z_1(Q) \subset \ldots \subset Z_{n-1} \subset Z_n(Q) = Q.$$

If N is a normal subloop of Q, there is a unique smallest normal subloop M of Q such that N/M is part of the centre of Q/M, and we write $M = [N, Q]$. The *lower central series* of Q is defined by $Q_1 = Q, Q_{i+1} = [Q_i, Q]$ $(i \geq 1)$. The loop Q is centrally nilpotent of class n if and only if its lower central series have the form $Q \supset Q_1 \supset \ldots \supset Q_{n-1} \supset Q_n = \{e\}$ [2].

The *associator* (x,y,z) and *commutator* (x,y) of elements $x, y, z \in Q$ are defined by the equalities $xy \cdot z = (x \cdot yz)(x,y,z)$ and $xy = (yx)(x,y)$ for an arbitrary loop Q. The *commutator-associator of weight n* is defined inductively:

1) any associator (x,y,z) and any commutator (x,y), where $x, y, z \in Q$, are commutator-associator of the weight 1;

2) if a is a commutator-associator of weight i, then (a,x,y) or (a,x), where $x, y \in Q$, is a commutator-associator of the weight $i+1$.

Lemma 8.1 [3]. *The subloops Q_i $(i = 1, 2, \ldots)$ of the lower central series of a Moufang loop Q are generated by all commutator-associators of weight i of Q.*

If A is an F-algebra, then its n degree A^n is an F-module with a basis, consisting of products from any of its n elements with any bracket distribution. Algebra A is called *nilpotent* if $A^n = (0)$ for a certain n.

Lemma 8.2. *Let Q be a finite Moufang p-loop and F be a field of characteristic p. Then, the ideal $\omega[Q]$ of the alternative loop algebra $F[Q]$ is nilpotent.*

Proof. In accordance with Corollary 7.7 we assume that $Q \subseteq F[Q]$. By Theorem 2.2 from [2, pag. 92] in a finite Moufang loop Q the order of any of its element divides the order of Q. Hence $g^k = e$, where $k = p^n$, for $g \in Q$. We have $(e-g)^k = e - C_k^1 g + \ldots + (-1)^i C_k^i g^i + \ldots + (-1)^k g^k$. All binomial coefficients C_k^i can be divided by p, therefore $(e-g)^k = e + (-1)^k g^k$. If $p = 2$, then $(e-g)^k = e + g^k = e + e = 2 = 0$, because F is a field of characteristic 2. But if $p > 2$, then $(e-g)^k = e - g^k = e - e = 0$. Then we can apply the following statement to algebra ωQ: any alternative F-algebra,

generated as an F-module by a finite set of nilpotent elements, is nilpotent [36, pages 144, 408]. Consequently the augmentation ideal $\omega[Q]$ is nilpotent, as required. ∎

Let now A be an alternative F-algebra with unit e and B be a subalgebra from A, satisfying the law

$$x^m = 0. \tag{45}$$

Then $e - B = \{e - b | b \in B\}$ will be a loop and $(e - b)^{-1} = e + b + \ldots + b^{m-1}$. We remind that inscription $(a, b, c) = ab \cdot c - a \cdot bc, (a, b) = ab - ba$ means the associator and commutator in algebra, but $[a, b, c] = (a \cdot bc)^{-1} \cdot (ab \cdot c), [a, b] = a^{-1}b^{-1} \cdot ab$ are *associator* and *commutator* in IP-loops.

Lemma 8.3. *Let A be an alternative algebra with unit e and B its subalgebra, satisfying the law (45). Then for $u, v, w \in B$ $[e - u, e - v, e - w] = e - ((e + w + \ldots + w^{m-1})(e + v + \ldots + v^{m-1}) \cdot (e + u + \ldots + u^{m-1}))(u, v, w)$, $[u, v] = (e + u + \ldots + u^{m-1})(e + v + \ldots + v^{m-1})(u, v)$.*

Proof. We denote $e - u = a,\ e - v = b,\ e - w = c$. Then we have $[e - u, e - v, e - w] = (a \cdot bc)^{-1}(ab \cdot c) = (a \cdot bc)^{-1}(ab \cdot c) - (a \cdot bc)^{-1}(a \cdot bc) + e = e + (a \cdot bc)^{-1}(a, b, c) = e + (((e - w)^{-1} \cdot (e - v)^{-1})(e - u)^{-1})(e - u, e - v, e - w) = e - ((e - w)^{-1}(e - v)^{-1} \cdot (e - u)^{-1})(u, v, w) = e - ((e + w + \ldots + w^{m-1})(e + v + \ldots + v^{m-1}) \cdot (e + u + \ldots + u^{m-1}))(u, v, w)$. The second equality is analogously proved. ∎

Lemma 8.4. *Let Q be a Moufang loop and let the ideal $\omega[Q]$ of the alternative loop algebra $F[Q]$ be nilpotent. Then the loop Q is centrally nilpotent.*

Proof. It follows, from the definition of ideal $\omega[Q]$, that $Q = Q_0 \subseteq e - \omega[Q] = e - \omega[Q]^{2 \cdot 0+1}$. We suppose that $Q_{i-1} \subseteq e - (\omega[Q])^{2 \cdot i+1}$. Then it follows from Lemma 8.4 that $Q_i \subseteq e - (\omega[Q])^{2 \cdot (i+1)+1}$. The algebra $\omega[Q]$ is nilpotent; we suppose that $(\omega[Q])^{2 \cdot k+3} = (0)$. Then $Q_k = e$. Hence the loop Q is centrally nilpotent, as required. ∎

The following Proposition follows from Lemmas 8.2 and 8.5.

Proposition 8.5. *Any finite Moufang p-loop is centrally nilpotent.*

Theorem 8.6. *Let a Moufang loop Q belong to radical class $\mathcal{S}$. Then the following statements are equivalent:*

1) the augmentation ideal $\omega[Q]$ of alternative loop algebra $F[Q]$ is nilpotent;

2) Q is a p-loop and the field F has a characteristic p;

3) ωQ is artinian.

Proof. Let the algebra $\omega[Q]$ be nilpotent, for example, of index r and let $0 \neq x \in (\omega[Q])^{r-1}$. By (31) the element x will be written in form $x = \alpha_1 g_1 + \cdots + \alpha_k g_k$, where

$\alpha_i \in F$, $g_i \in Q$. We suppose that $g_i \neq g_j$ if $i \neq j$. As $Q \in \mathcal{S}$ then from definition of the class $\mathcal{S}$ it follows that $F[Q] \in \mathcal{R}$ and by item 4) of Proposition 4.3, we have $x(e-u) = 0$ for any $u \in Q$. Hence $\alpha_1 g_1 + \cdots + \alpha_k g_k = \alpha_1 g_1 u + \cdots + \alpha_k g_k u$ in the alternative loop algebra $F[Q]$. But $F[Q] = FQ/I$. Thus $\alpha_1 g_1 + \cdots + \alpha_k g_k = \alpha_1 g_1 u + \cdots + \alpha_k g_k u (\mathrm{mod} I)$ in the loop algebra FQ. We suppose that the loop Q is infinite. Then, there exist $u \in Q$ such that $\alpha_1 g_1 u \notin \{\alpha_1 g_1, \ldots, \alpha_k g_k\}$. By the definition the loop algebra FQ is a free F-module with the basis $\{g | g \in Q\}$. Then $\alpha_1 g_1 u \in I$. From here it follows that $I = FQ$. But this contradicts Theorem 7.5. Hence the loop Q is finite.

By [2, pag. 92], in the finite Moufang loop Q the order of any of its element divides the order of Q. If $e \neq g \in Q$ is an element of simple order p then, by item 6) of Proposition 4.3, $a = p - (e + g + g^2 + \ldots + g^{p-1}) \in \omega[Q]$. We have $(e + g + g^2 + \ldots + g^{p-1})(e + g + g^2 + \ldots + g^{p-1}) = p(e + g + g^2 + \ldots + g^{p-1})$. Then $a^2 = p^2 - p(e + g + g^2 + \ldots + g^{p-1})$ and, by induction of n, it is easy to show that $a^{2^n} = p^{2^n} - p^{2^n - 1}(e + g + g^2 + \ldots + g^{p-1})$. We choose an n such that $(\omega[Q])^{2^n} = (0)$. Then $a^{2^n} = 0$. We suppose that F does not have the characteristic p. It follows, from the equalities $0 = p^{2^n} - p^{2^n - 1}(e + g + g^2 + \ldots + g^{p-1}) = p^{2^4 - 1}(p - (e + g + g^2 + \ldots + g^{p-1})$, that $0 = p - (e + g + g^2 + \ldots + g^{p-1})$, $p = e + g + g^2 + \ldots + g^{p-1}$, i.e. $pg = p$, $g = e$. We have obtained a contradiction as $g \neq e$. Consequently, F has the characteristic p and Q is p-loop. Consequently, 1) $\Rightarrow$ 2).

Conversely, let the field F have a characteristic p and let Q be a finite p-loop. By Proposition 8.6 it will be centrally nilpotent loop. Let $H =< a >$ be a cyclic group of order p from the center $Z(Q)$ of Q. We will prove that the product

$$(e - a^{i^1})(e - a^{i^2}) \ldots (e - a^{i^m})$$

equals zero, if $m \geq p$. Indeed, if we use the identity $e - xy = (e - x) + (e - y) - (e - x)(e - y)$, then the last product is the sum of the factors of type $(e - a)^k$, $k \geq p$. Then

$$(e - a)^p = e - C_p^1 a + C_p^2 a^2 - \ldots \pm a^p.$$

As all binomial coefficients C_p^i divide by p, then they are zero in the field F. Consequently, $(\omega[H])^p = (0)$, where $\omega[H]$ means the augmentation ideal of the alternative loop algebra $F[H]$. Let μH means the ideal of the alternative loop algebra $F[Q]$, generated by the set $\{e - h | h \in H\}$. As the subloop H belongs to the center $Z(Q)$, then the equality $(\omega[H])^p = (0)$ entails the equality $(\mu H)^p = (0)$.

We will prove the nilpotency of augmentation ideal $\omega[Q]$ via induction on the order of the loop Q. As $H \subseteq Z(Q)$, then H is normal in Q and H induces the homomorphism $F[Q] \to F[Q/H]$. By item 4) of Proposition 4.4, we have $\omega[Q]/\mu H \cong \omega[Q/H]$. By inductive hypotheses, the augmentation ideal $\omega[Q/H]$ is nilpotent, for example, of index k. Then $(\omega[Q])^k \subseteq \mu H$ and $(\omega[Q])^{kp} \subseteq (\mu H)^p = (0)$. Consequently, the ideal $\omega[Q]$ is nilpotent. Hence 2) $\Rightarrow$ 1). Conversely, let us prove 1) $\Leftrightarrow$ 2).

The equivalence of 1) and 2) follows from Proposition 4.3. Now we suppose that 2) holds. Let $g_1, g_2, \ldots, g_k$ be all elements of Q. By item 5) of Lemma 6 ??? ωQ is a

finite sum of modules Fu_i, where $u_i = e - g_i$. The field F has a characteristic p and $g^{p^n} = e$ for some n. Then $u_i^{p^n} = (e - g_i)^{p^n} = 0$. Hence Fu_i satisfies the minimum condition for submodules. It easily follows from here that ωQ is Artinian, i.e the item 3) holds. Furthermore, it is known [6] that the Zhevlakov radical of an Artinian alternative algebra is nilpotent. By item 7) of Proposition 4.3, $J(F[Q]) = \omega[Q]$. Thus from 3) it follows 1). This completes the proof of Theorem 8.7. ∎

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