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NUMERICAL INVESTIGATIONS ON THE STABILITY OF THE PILOT-AIRPLANE SYSTEM USING FREQUENCY DOMAIN ANALYSIS

Adrian Toader[1], Ionel Iorga[2]

[1] *National Institute for Aerospace Research "Elie Carafoli", Bucharest, Romania*

[2] *University of Craiova, Craiova, Romania*

atoader@incas.ro, ioneliorga@yahoo.com

Abstract In the paper the absolute stability of a specific pilot-airplane system is analysed by using the Popov Criterion. The absolute stability Lurie problem defined by the presence in the system mathematical model of the actuator rate saturation is used. The case study is that of a longitudinal control chain of ADMIRE airplane model and the human operator is represented by a simple gain, without a specific delay.

Keywords: asymptotic absolute stability, Lurie problem, Popov Criterion, ADMIRE mathematical model, rate saturation.

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1. INTRODUCTION

The pilot induced oscillations phenomenon (shortly, PIO) is described in the literature as "sustained or uncontrollable oscillations resulting from the efforts of the pilot to control the airplane" [1], or, for example, as a "misadaptation between the pilot and the aircraft during some task in which tight closed loop control of the aircraft is required from the pilot" [12].

1.1. TYPES OF PIO

The following types of PIO are known [12]:

1. category I: linear oscillations of the pilot-vehicle system resulting from excessive lags introduced by filters, actuators, feel system and digital system time delays;
2. category II: quasi-linear oscillations which are mainly due to actuator rate limiting;
3. category III: severe life-threatening PIO, which are caused by nonlinearities and transitions in pilot or effective airplane dynamics.

If the reader wants to consult more about the PIO clasiffication a good reference is [12].

Fig. 1. YF-22A Pilot-Induced Oscillation [3]

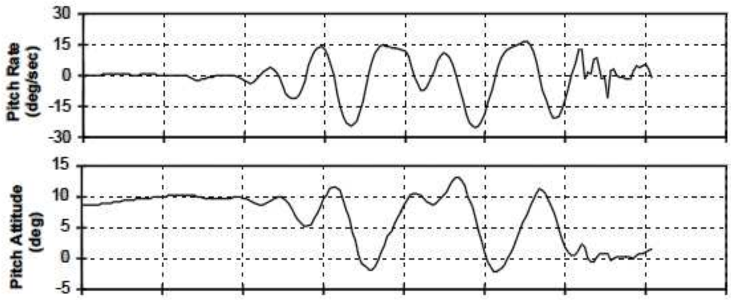

Fig. 2. YF-22A PIO time history [4]

1.2. AN EXAMPLE OF PIO: THE YF-22A PIO ACCIDENT

In [2] is described an example of longitudinal PIO which has started after the retraction of the landing gear and, also, happened nearly the same time as the after-burner was initiated in order to proceed another overflight above the runway. The oscillations which led to crash took about eleven seconds (their shape can be seen in Figure 1).
The accident happened in April 1992 at Edwards Air Force Base, in California and the test pilot survived. After four to five oscillations the aircraft impacted the runway. In Figure 2 the pitch rate and the pitch angle vs. time are shown.

In the YF-22A accident, the rate limit was involved, therefore it can be concluded that the PIO II phenomenon was present. See details in [11].

2. CLOSED LOOP CONTROL OF THE PILOTED SYSTEM

In Figure 3, the block diagram of the pilot-aircraft system, with rate limiter, is shown. The airplane dynamics is represented by the transfer function

$$G(s) = c^T(sI - A)^{-1}b \tag{1}$$

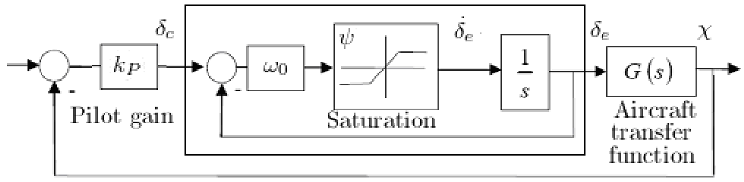

Fig. 3. Closed loop pilot-aircraft system with rate saturation of the flight control surface actuator

where χ is the output of the linear system, c^T, A, b are common notations from the state-space representation and s is the complex variable from the Laplace transform.

The actuator dynamics is modeled by a first order lag system with transfer function

$$\frac{\tilde{\delta}_e(s)}{\tilde{\delta}_c(s)} = \frac{1}{\tau s + 1} = \frac{\frac{1}{\tau}}{s + \frac{1}{\tau}} = \frac{\omega_0}{s + \omega_0} \tag{2}$$

where $\tau > 0$ is the actuator time constant, δ_c represents the input due to the stick and δ_e is the deflection of the longitudinal flight control surface. In other words,

$$\dot{\delta}_e + \omega_0 \delta_e = \omega_0 \delta_c$$

and

$$\dot{\delta}_e = \omega_0(\delta_c - \delta_e) \tag{3}$$

where

$$\delta_c(t) = \kappa_P \theta(t) \tag{4}$$

The rate saturation of the deflection of the longitudinal flight control surface, δ_e, is defined as:

$$\dot{\delta}_e = \begin{cases} |\dot{\delta}_e|, \text{ if } |\dot{\delta}_e| < V_L \\ V_L, \text{ if } \dot{\delta}_e \geq V_L \\ -V_L, \text{ if } \dot{\delta}_e \leq -V_L \end{cases} \tag{5}$$

where V_L is the rate limit value, herein $V_L = 1.57 \frac{rad}{s}$.

3. ABSOLUTE STABILITY ANALYSIS

Absolute stability refers to the global asymptotic stability of the zero equilibrium point of the general nonlinear system

$$\dot{x}(t) = A_\alpha x(t) - b_\alpha \psi(c_\alpha^{\mathrm{T}} x(t)) \tag{6}$$

having sector restricted nonlinearities [13]:

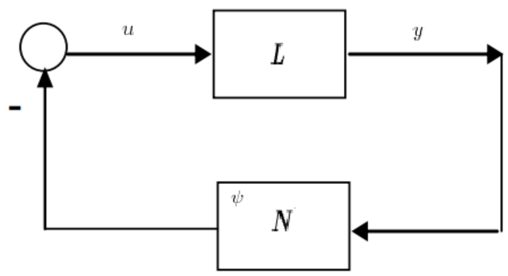

Fig. 4. Absolute stability feedback structure (Lurie problem)

$$0 \leq \underline{\psi} \leq \frac{\psi(y)}{y} \leq \bar{\psi} \leq \infty, \psi(0) = 0 \tag{7}$$

3.1. LURIE PROBLEM

Lurie problem (see [10], for details) refers to finding necessary and sufficient conditions for system (6)-(7) to be absolutely stable. The absolute stability feedback structure is shown in Figure 4, where L is a linear block (obtained by linearization of the original nonlinear system - in this paper the system (26) - around an equilibrium point) and N represents a non-linear block - modeled by the continuous function ψ (herein, $\psi(y)$ represents a saturation, like in the left part of Figure 5) which fulfills the general sector condition (7).
The continous functions ψ fulfilling the condition (7) belong to the class described by the right part of Figure 5.

An equivalence between rate limiter blocks (Figures 3 and 6) proof is made in the following. The starting point is the following relation

$$\tilde{\chi}(s) = G(s)\tilde{\delta}_e(s) \tag{8}$$

From the form of the Lurie problem the following output is considered

$$y_\alpha(t) = \omega_0 c^T x(t) + \omega_0 \delta_e(t) \tag{9}$$

where

$$\begin{cases} \dot{\delta}_e(t) = -\psi(y_\alpha(t)) \\ u(t) = -\psi(y_\alpha(t)) \end{cases} \tag{10}$$

The Laplace transform is applied

$$\begin{cases} s\tilde{x}(s) = A\tilde{x}(s) + b\tilde{\delta}_e(s) \\ s\tilde{\delta}_e(s) = \tilde{u}(s) \end{cases} \tag{11}$$

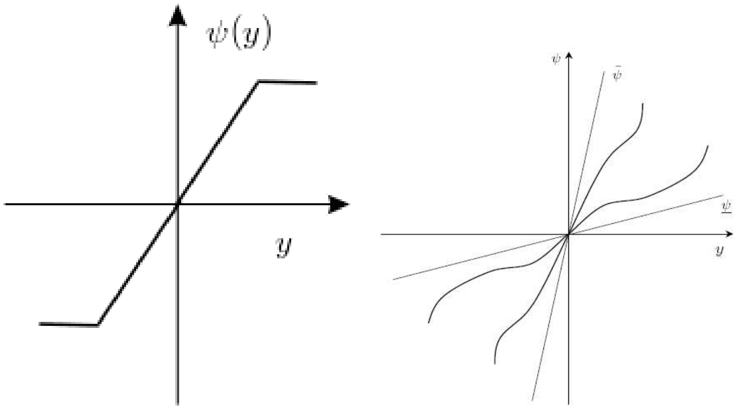

Fig. 5. Standard saturation nonlinearity. Sector restricted nonlinearity

$$\tilde{y}_\alpha(s) = \omega_0 c^T \tilde{x}(s) + \omega_0 \tilde{\delta}_e(s) \tag{12}$$

From (1) it follows that:

$$\begin{cases} \tilde{x}(s) = (sI - A)^{-1} b \tilde{\delta}_e(s) \\ \tilde{\delta}_e(s) = \dfrac{1}{s}\tilde{u}(s) \end{cases} \tag{13}$$

Therefore

$$\tilde{x}(s) = \frac{1}{s}(sI - A)^{-1} b \tilde{u}(s) \tag{14}$$

From (12), (13) and (14) follows

$$\tilde{y}_\alpha(s) = c^{\mathrm{T}}(sI - A)^{-1} b \frac{\omega_0}{s}\tilde{u}(s) + \frac{\omega_0}{s}\tilde{u}(s) \tag{15}$$

and, further on

$$G(s) = c^{\mathrm{T}}\,(sI - A)^{-1}\, b = \begin{pmatrix} 0 & 0 & \kappa_P \end{pmatrix} (sI - A)^{-1}\, b \tag{16}$$

from (15) and (16) the following relation is determined

$$\tilde{y}_\alpha(s) = G(s)\frac{\omega_0}{s}\tilde{u}(s) + \frac{\omega_0}{s}\tilde{u}(s) \tag{17}$$

which is equivalent to

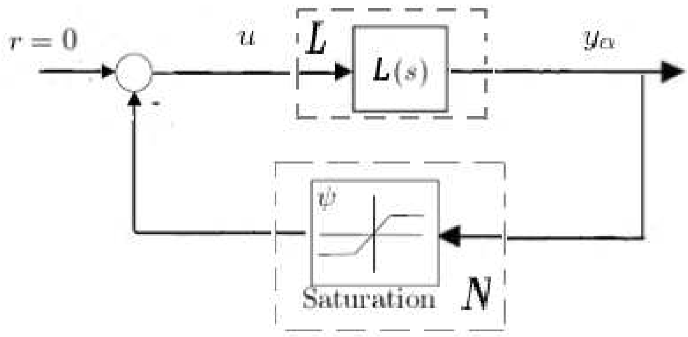

Fig. 6. Pilot-aircraft system with rate limiter written in terms of the Lurie problem

$$\begin{aligned}\tilde{y}_\alpha(s) &= L(s)\tilde{u}(s) = \omega_0\left(\mathrm{G}(s)\frac{1}{s} + \frac{1}{s}\right)\tilde{u}(s)\\ &= \omega_0\left[\begin{pmatrix} 0 & 0 & \kappa_P \end{pmatrix}(sI - A)^{-1}\, b\frac{1}{s} + \frac{1}{s}\right]\tilde{u}(s)\end{aligned} \tag{18}$$

The relation (18) shows that the transfer function is in the critical case of one simple pole in origin (the transfer function denominator - the characteristic polynomial - has all the roots in the open-left half plane, exception one which is zero).

When the coupled pilot-aircraft system, represented in Figure 3, is rewritten as a Lurie system, c^T, from (34), becomes $c^T = (0\ 0\ \kappa_P)$.

3.2. THE POPOV CRITERION - ASIMTOPTIC STABILITY WITH ONE POLE IN THE ORIGIN

The study of the absolute stability will be made by using the Popov criterion. First, the statement of the Popov Criterion [13] is given and then the criterion will be applied to a low-order ADMIRE aircraft mathematical model. Consider systems with one nonlinearity

$$\begin{cases} \dot{x}_\alpha(t) = A_\alpha x_\alpha(t) - b_\alpha \psi(y_\alpha(t)) \\ y_\alpha(t) = c_\alpha^T x_\alpha(t) \end{cases} \tag{19}$$

where A_α, b_α, c_α^T, x_α, y_α are the usual notations for the time-domain realisation of the $L(s)$ transfer function

$$A_\alpha = \begin{pmatrix} A & b \\ 0_{(1,\,n)} & 0 \end{pmatrix};\ b_\alpha = \begin{pmatrix} 0_{(n,\,1)} \\ 1 \end{pmatrix};\ c_\alpha^T = \omega_0(c^T\ 1) \tag{20}$$

($n \in N \setminus \{0, 1\}$) and ψ is a continous function having the property (7).

From [15] the following theorem is adapted (with $\underline{\psi} = 0$):

Theorem 3.1. *The trivial solution of the system (19) is asymptotic absolutely stable - under conditions (7)- if the following conditions are satisfied*

- *$det(sI_n - A) = 0$ has all the roots in the open left half plane and*

$$1 - c^T A^{-1} b > 0 \tag{21}$$

- *There exists a number $\xi \geq 0$, $\xi < \infty$ such that*

$$\frac{1}{\overline{\psi}} + Re[(1 + \imath\omega\xi)L(\imath\omega)] > 0 \tag{22}$$

for every real number ω.

Relation (22) can be written as:

$$\frac{1}{\overline{\psi}} + Re[(1 + \imath\omega\xi)(U(\omega) + \imath V(\omega))] > 0, \forall\omega \geq 0 \tag{23}$$

where

$$\begin{cases} U(\omega) = Re[L(\imath\omega)] \\ V(\omega) = Im[L(\imath\omega)] \end{cases} \tag{24}$$

which is equivalent with

$$\frac{1}{\overline{\psi}} + U(\omega) - \omega\xi V(\omega) > 0, \forall\omega \geq 0 \tag{25}$$

Remark 3.1. *In the previous relations ω is restricted to positive values because, from the general theory of functions of a complex variable with real coefficients, it is known that, for every function from the class mentioned, the real part is even and the imaginary part is odd (in conclusion, the above expression is always even).*

4. THE ADMIRE MODEL

The Generic Aerodata Model (GAM) is a theoretical model of a small fighter aircraft. This model has been developed by the Saab AB (the Swedish company that produces the JAS-39) and was used by the Swedish Defence Research Agency (FOI), in order to develop a mathematical model for a single-seated fighter with a delta canard configuration (ADMIRE). A detailed and explicit mathematical description of the two models is not given but informations can be obtained by consulting data tables. Suggestions regarding the distribution of the mass concerning the GAM theoretical model can be found in [5], also, a good reference for the ADMIRE is [6].

The original ADMIRE system has twelve states [6], but for the simplicity of the analysis a low order non-linear explicit differential model was used. This model has been described in [7], [8], [9].

From [7] the following nonlinear system is considered:

$$\begin{aligned}
&\dot{\alpha} = z_\alpha \alpha + q + \tfrac{g}{V_0}\cos(\theta) + z_{\delta_e}\delta_e \\
&\dot{q} = \bar{m}_\alpha \alpha + \bar{m}_q q - \tfrac{1}{a}\alpha q + \tfrac{g}{V_0}(m_{\dot{\alpha}}\cos\theta - \bar{a}\sin\theta) + \bar{m}_{\delta_e}\delta_e \\
&\dot{\theta} = q
\end{aligned} \tag{26}$$

The state system is $X = (\alpha, q, \theta)$, where α is the incidence angle, θ represents the pitch angle and q is the pitch rate. The input vector u one dimensional, $u = (\delta_e)$, δ_e represents the angle of the elevon.

Let consider equilibrium points of the above system. Using only the first two equations (from the third one results $\bar{q} = 0$), it follows that

$$\bar{\alpha} = -\frac{1}{z_\alpha}(\frac{g}{V_0}\cos\bar{\theta}_1 + z_{\delta_e}\bar{\delta}_e) \tag{27}$$

which, introduced in the second equation and with $\gamma = \tan(\frac{\bar{\theta}}{2})$ substitution, gives

$$\gamma^2(\bar{\delta}_e\varrho_3 - \varrho_1) + \gamma(2\varrho_2) + (\varrho_1 + \bar{\delta}_e\varrho_3) = 0 \tag{28}$$

where

$$\begin{cases} \varrho_1 = \frac{g}{V_0}(m_{\dot{\alpha}} - \frac{\bar{m}_\alpha}{z_\alpha}) \\ \varrho_2 = -\bar{a}\frac{g}{V_0} \\ \varrho_3 = m\bar{}_{\delta_e} - \frac{z_{\delta_e}}{z_\alpha}\bar{m}_\alpha \end{cases} \tag{29}$$

The positivity condition on the discriminant associated to the above equation equation gives

$$|\bar{\delta}_e| \leq \frac{\sqrt{\varrho_1^2 + \varrho_2^2}}{|\varrho|} \tag{30}$$

With a fixed value of $\bar{\delta}_e$, the roots of the equation (28) are:

$$\gamma_{1,\,2} = \frac{-\varrho_2 \pm \sqrt{\varrho_2^2 - (\bar{\delta}_e\varrho_3 - \varrho_1)(\varrho_1 + \bar{\delta}_e\varrho_3)}}{\bar{\delta}_e\varrho_3 - \varrho_1} \tag{31}$$

Therefore:

$$\bar{\theta} = 2tan^{-1}(\gamma) \tag{32}$$

4.0.1 The simplified linearized ADMIRE system. The linearized system associated to (26) has the form:

$$\begin{cases} \Delta\dot{x} = A\Delta x + b\Delta\delta_e \\ \Delta y = c^T\Delta x \end{cases} \tag{33}$$

where:

$$A = \begin{pmatrix} z_\alpha & 1 & -\frac{g}{V_0}\sin\bar{\theta} \\ \bar{m}_\alpha & \kappa_2 & -\frac{g}{V_0}\kappa_1 \\ 0 & 1 & 0 \end{pmatrix}; \Delta x = \begin{pmatrix} \Delta\alpha \\ q \\ \Delta\theta \end{pmatrix}; b = \begin{pmatrix} z_{\delta_e} \\ \bar{m}_{\delta_e} \\ 0 \end{pmatrix}; c^T = (0\;0\;1) \tag{34}$$

and

$$\begin{cases} \kappa_1 = m_{\dot{\alpha}}\sin\bar{\theta} + \bar{a}\cos\bar{\theta} \\ \kappa_2 = \bar{m}_q - \frac{1}{a}\bar{\alpha} \end{cases} \tag{35}$$

Let consider the notation:

$$\Delta\nu = \nu - \bar{\nu} \tag{36}$$

where $\bar{\nu}$ is the equilibrium point (trim value) of the state ν, $\nu \in \{\alpha, q, \theta\}$.
Therefore, the low-order transfer function of the ADMIRE aircraft is:

$$G(s) = \frac{\bar{m}_{\delta_e}s + \epsilon}{s^3 - s^2(\kappa_2 + z_\alpha) + \mu_1 s + \mu_2\frac{g}{V_0}} \tag{37}$$

where

$$\begin{cases} \epsilon = \bar{m}_\alpha z_{\delta_e} - \bar{m}_{\delta_e} z_\alpha \\ \mu_1 = z_\alpha\kappa_2 + \frac{g}{V_0}\kappa_1 - \bar{m}_\alpha \\ \mu_2 = \bar{m}_\alpha\sin\bar{\theta} - \kappa_1 z_\alpha \end{cases} \tag{38}$$

The open-loop transfer function in the case of rate saturation is denoted by $L(s)$ and from (18) we obtain

$$L(s) = \omega_0\frac{s\kappa_P\bar{m}_{\delta_e} + \kappa_P\epsilon}{P(s)} + \frac{\omega_0}{s} \tag{39}$$

where $P(s)$ is the characteristic polynomial

$$P(s) = s[s^3 - s^2(\kappa_2 + z_\alpha) + \mu_1 s + \mu_2\frac{g}{V_0}] \tag{40}$$

Using the Routh-Hurwitz criterion, the answer to the question if the transfer function (39) is stable (and, consequently, if a 'sector rotation', as described in [14], is needed) can be provided.

Consider the polynomial

$$D(s) = s^3 - s^2(\kappa_2 + z_\alpha) + \mu_1 s + \mu_2\frac{g}{V_0} \tag{41}$$

The Routh-Hurwitz condition (from [16]) is

$$-(\kappa_2 + z_\alpha)\mu_1 > \mu_2 \frac{g}{V_0} \tag{42}$$

The validation of condition (21), from Theorem (3.1), is given by the sign of

$$1 - \frac{\kappa_P V_0}{g(\kappa_1 z_\alpha - \bar{m}_\alpha)}(\bar{m}_\alpha z_{\delta_e} - z_\alpha \bar{m}_{\delta_e}) \tag{43}$$

4.1. STABILITY ANALYSIS IN THE CASE OF RATE LIMITER

If the following substition is made $s \to \imath\omega$ in (39), then

$$L(\imath\omega) = Re[L(\imath\omega)] + \imath Im[L(\imath\omega)] \tag{44}$$

where

$$\begin{cases} Re[L(\imath\omega)] = \omega_0\kappa_P \dfrac{\omega^2[\bar{m}_{\delta_e}(\kappa_2 + z_\alpha) + \epsilon] + \bar{m}_{\delta_e}\frac{g}{V_0}\mu_2 - \epsilon\mu_1}{[\omega^2(\kappa_2 + z_\alpha) + \frac{g}{V_0}\mu_2]^2 + \omega^2(\mu_1 - \omega^2)^2} \\ Im[L(\imath\omega)] = -\omega_0\omega[\kappa_P \dfrac{-\omega^4\bar{m}_{\delta_e} + \omega^2[\epsilon(\kappa_2 + z_\alpha) + \bar{m}_{\delta_e}\mu_1] + \epsilon\frac{g}{V_0}\mu_2}{\omega^2\{[\omega^2(\kappa_2 + z_\alpha) + \frac{g}{V_0}\mu_2]^2 + \omega^2(\mu_1 - \omega^2)^2\}} - \dfrac{1}{\omega^2}] \end{cases} \tag{45}$$

Based on (24) and (25), the following frequency domain inequality holds

$$\frac{-\omega_0\kappa_P\bar{m}_{\delta_e}\xi\omega^4 + \omega^2[\omega_0\kappa_P(\tau_1\xi + \tau_2)] + \tilde{\beta}_2\epsilon\omega_0\kappa_P\xi + \tau_3\omega_0\kappa_P}{(\omega^2\tilde{\beta}_1 + \tilde{\beta}_2)^2 + \omega^2(\mu_1 - \omega^2)^2} + \xi\omega_0 + 1 > 0 \tag{46}$$

where the following notations were introduced:

$$\tilde{\beta}_1 = \kappa_2 + z_\alpha; \tilde{\beta}_2 = \tfrac{g}{V_0}\mu_2; \tau_1 = \epsilon\tilde{\beta}_1 + \bar{m}_{\delta_e}\mu_1; \tau_2 = \bar{m}_{\delta_e}\tilde{\beta}_1 + \epsilon; \tau_3 = \bar{m}_{\delta_e}\tilde{\beta}_2 - \epsilon\mu_1 \tag{47}$$

With the notations:

$$\begin{cases} f(\xi, \omega) = -\bar{m}_{\delta_e}\xi\omega^4 + \omega^2(\tau_1\xi + \tau_2) + \tilde{\beta}_2\epsilon\xi + \tau_3 \\ g(\xi, \omega) = (\omega^2\tilde{\beta}_1 + \tilde{\beta}_2)^2 + \omega^2(\mu_1 - \omega^2)^2 \end{cases} \tag{48}$$

we get

$$\begin{aligned} h(\xi, \omega) &= \omega_0\kappa_P \frac{f(\omega)}{g(\omega)} + \xi\omega_0 + 1 \\ &= \omega_0\kappa_P \frac{-\bar{m}_{\delta_e}\xi\omega^4 + \omega^2(\tau_1\xi + \tau_2) + \tilde{\beta}_2\epsilon\xi + \tau_3}{\omega^6 + \omega^4\beta_4 + \omega^2\beta_2 + \beta_0} + \xi\omega_0 + 1 \end{aligned} \tag{49}$$

where

$$\beta_4 = \tilde{\beta}_1^2 - 2\mu_1; \beta_2 = \mu_1^2 + 2\tilde{\beta}_1\tilde{\beta}_2; \beta_0 = \tilde{\beta}_2^2 \tag{50}$$

Thus, the relation (46) can be written as

$$h(\xi, \omega) > 0, \forall \omega \geq 0 \tag{51}$$

Let now study the monotony of $h(\xi, \omega)$. First, the function behaviour is analyzed in the frontier points 0 and ∞:

$$\begin{cases} \lim_{\omega \to \infty} h(\xi, \omega) = \xi\omega_0 + 1 \\ \lim_{\omega \to 0} h(\xi, \omega) = \omega_0 \kappa_P \dfrac{\tilde{\beta}_2 \epsilon \xi + \tau_3}{\tilde{\beta}_2^2} + \xi\omega_0 + 1 \end{cases} \tag{52}$$

Because we have $\omega_0 > 0, \xi \geq 0$ (from (2) and Theorem 3.1) then the first relation of the above system is strictly positive.
In these conditions the sign of the second relation of the above system is given by the sign of

$$\tilde{\beta}_2 \epsilon \xi + \tau_3 \tag{53}$$

and, if the above relation is strictly positive, then

$$\kappa_P > -\frac{(\xi\omega_0 + 1)\tilde{\beta}_2^2}{\omega_0(\tilde{\beta}_2 \epsilon \xi + \tau_3)}$$

Finally, from positivity of κ_P, we denote that

$$\kappa_P \in (0, \infty) \tag{54}$$

On the other hand, if relation (53) is strictly negative, then

$$\kappa_P < -\frac{(\xi\omega_0 + 1)\tilde{\beta}_2^2}{\omega_0(\tilde{\beta}_2 \epsilon \xi + \tau_3)}$$

and, one obtains

$$\kappa_P \in (0, \ -\frac{(\xi\omega_0 + 1)\tilde{\beta}_2^2}{\omega_0(\tilde{\beta}_2 \epsilon \xi + \tau_3)}) \tag{55}$$

We notice that (54) and (55) are conditions obtained at limit (when $\omega \to 0$), thus we cannot say that they are valid for every ω but, however, if we want a maximum interval for κ_P we can assert the fact that the strict positive sign of (53) is in our advantage.
In what follows a discussion about restrictions imposed for ξ, in conditions in which

$\kappa_P > 0$ is fixed, is made. First, if we denote by $\upsilon = \omega^2$, we observe that the sign of (49) is given in relation with the sign of quantity

$$-\bar{m}_{\delta_e}\xi\upsilon^2 + (\tau_1\xi + \tau_2)\upsilon + \tilde{\beta}_2\epsilon\xi + \tau_3 \tag{56}$$

For the above relation the following discriminant is computed

$$\Delta = \xi^2\tau_1^2 + \xi(2\tau_1\tau_2 + 4\bar{m}_{\delta_e}\tilde{\beta}_2\epsilon) + \tau_2^2 + 4\bar{m}_{\delta_e}\tau_3 \tag{57}$$

We observe the fact that, for the positivity of (49), is sufficient that relation (56) to be strictly positive. The strict positivity of (56) is true if the above discriminant is strictly negative and if $-\bar{m}_{\delta_e}$ is strictly positive (the latter condition is obtained from the sign of (56) when $\upsilon \to \infty$), that is:

$$\begin{cases} \xi^2\tau_1^2 + \xi(2\tau_1\tau_2 + 4\bar{m}_{\delta_e}\tilde{\beta}_2\epsilon) + \tau_2^2 + 4\bar{m}_{\delta_e}\tau_3 < 0 \\ -\bar{m}_{\delta_e} > 0 \end{cases} \tag{58}$$

In order to make the evaluation of the above expression the following discriminant (relative to ξ) must be positive (because, otherwise, only complex values for ξ are obtained):

$$\Delta_\xi = 4^2\bar{m}_{\delta_e}(\tau_1\tau_2\tilde{\beta}_2\epsilon + \bar{m}_{\delta_e}\tilde{\beta}_2^2\epsilon^2 - \tau_1^2\tau_3) \tag{59}$$

Using the above result $\xi_{1,2}$ is determined:

$$\xi_{1,2} = -\frac{\tau_1\tau_2 + 2\bar{m}_{\delta_e}\tilde{\beta}_2\epsilon \pm 2\sqrt{\bar{m}_{\delta_e}(\tau_1\tau_2\tilde{\beta}_2\epsilon + m_{\delta_e}\tilde{\beta}_2^2\epsilon^2 - \tau_1^2\tau_3)}}{\tau_1^2} \tag{60}$$

In order for (51) to be true the following must hold

$$\xi \in (\xi_1, \xi_2) \tag{61}$$

because $\tau_2^2 + 4\bar{m}_{\delta_e} < 0$ (this is obtained from the fact that (59) is considered strictly positive and, obviously, from the strict positivity of τ_1^2).
The roots of the first derivative of $h(\xi, \omega)$ - with respect to ω (ξ is supposed fixed) - denoted by $h'(\omega)$, are studied in order to evaluate the sign and magnitude of the extremum points associated to $h(\xi, \omega)$.

When computing $h'(\omega)$ the intermediary relation is simplified with $\frac{1}{2\omega_0\kappa_P\omega}$:

$$\begin{aligned} h'(\omega) &= \frac{f'(\omega)g(\omega) - f(\omega)g'(\omega)}{g^2(\omega)} \\ &= \frac{\omega^8(\bar{m}_{\delta_e}\xi) + \omega^6\varpi_6 + \omega^4\varpi_4 + \omega^2\varpi_2 + \varpi_0}{(\omega^6 + \omega^4\beta_4 + \omega^2\beta_2 + \beta_0)^2} \end{aligned} \tag{62}$$

where

$$\begin{cases} \varpi_6 = -2(\tau_1\xi + \tau_2) \\ \varpi_4 = -[(\bar{m}_d e\beta_2 + \tau_1\beta_4 + 3\tilde{\beta}_2\epsilon)\xi + (\tau_2\beta_4 + 3\tau_3)] \\ \varpi_2 = -2[(\beta_4\tilde{\beta}_2\epsilon + \bar{m}_d e\beta_0)\xi + \beta_4\tau_3] \\ \varpi_0 = (\tau_1 - \beta_2\tilde{\beta}_2\epsilon)\xi + (\tau_2 - \beta_2\tau_3) \end{cases} \tag{63}$$

From (64), the following data are used:

$$\begin{cases} g = 9.81\frac{m}{s^2};\, V_0 = 84.5\frac{m}{s};\, a = -.2424;\, \bar{a} = 1.424 \\ z_\alpha = -.7986;\, z_{\delta_e} = -.2603;\, \bar{m}_{\delta_e} = -8.2668 \\ \bar{m}_\alpha = -6.5315;\, \bar{m}_q = -.6957;\, m_{\dot{\alpha}} = -.162 \end{cases} \tag{64}$$

These get (see (29))

$$\varrho_1 = -0.9683;\, \varrho_2 = -0.1653;\, \varrho_3 = -6.1379 \tag{65}$$

and, further on, the relations (65), (30), (32) and (27), yield:

$$\bar{\theta} = 0.055 \text{ rad};\, \bar{\alpha} = 0.197 \text{ rad};\, \bar{q} = 0 \text{ rad/s};\ \text{ with } \bar{\delta_e} = -0.159 \text{ rad} \tag{66}$$

The following values are obtained (from (35) and (38)) using the above equilibrium point:

$$\kappa_1 = 1.413;\, \kappa_2 = 0.117;\, \mu_1 = 6.6;\, \mu_2 = 0.77;\, \epsilon = -4.9 \tag{67}$$

$$\tilde{\beta}_1 = -0.6817;\, \tilde{\beta}_2 = 0.0894;\, \tau_1 = -51.2374;\, \tau_2 = 0.7336;\, \tau_3 = 31.6231 \tag{68}$$

From (67) and (68) we obtain that (53) is strictly positive and (54) is true (the maximum possible interval for κ_P is achieved).
Also, we are in the case in which $\tau_2^2 + 4\bar{m}_{\delta_e}\tau_3 < 0$ and, from (60) and (61), taking into account that $\xi \geq 0$, we obtain:

$$\xi \in [0, 0.642625) \tag{69}$$

Using the above values it follows that Routh-Hurwitz condition (42) is true (there is no need for a 'sector rotation') and, also, the condition (43) holds.

The coefficients of $h(\xi, \omega)$ are computed:

$$\beta_4 = -12.7396;\, \beta_2 = 43.4667;\, \beta_0 = 0.008 \tag{70}$$

In the numeric simulations, performed in the MatLab environment, the following values were employed:

$$\kappa_P = \frac{1}{2};\, \xi = 0.64;\, \omega_0 = 20\ \frac{rad}{s} \tag{71}$$

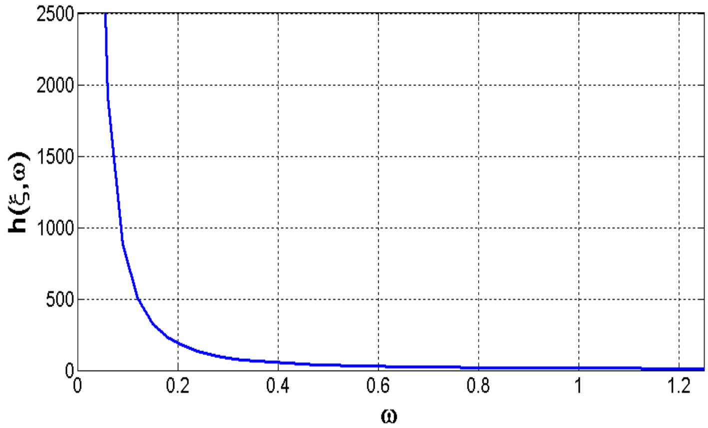

Fig. 7. Graphical representation of $h(\xi, \omega)$ function, for $\xi = 0.64$

With the usage of the previous values, the reader can see that the second relation of the system (52) is strictly positive.
The coefficients of (62) are given bellow

$$\varpi_6 = 70.265;\ \varpi_4 = -290;\ \varpi_2 = 797.74;\ \varpi_0 = -1395.5 \tag{72}$$

Taking into account the Remark 3.1, for table (1), only the strictly positive roots of (62) are used:

Table 1 $h(\xi, \omega)$ extremum points

	0	+	1.8795	+	2.6815	+	∞
$h'(\xi, \omega)$	$-1.3935 \cdot 10^3$	-	0	+	0	-	0
$h(\xi, \omega)$	$3.923 \cdot 10^4$	+	9.66088	+	42.77232	+	13.8

In the following figure, for the $\xi = 0.64$, $h(\xi, \omega)$ has the following extremum values are determined numerically

$$max[h(\xi, \omega)] = 3.923 \cdot 10^4;\ min[h(\xi, \omega)] = 9.66088 \tag{73}$$

5. CONCLUSIONS

From Table 1 or, equivalently, from relation (73), it becomes clear that $h(\xi, \omega)$ is strictly positive, thus implying that (25) is true. In conclusion, in the case of the pilot model acting like a simple gain, the simplified ADMIRE system, with one saturation nonlinearity given by the rate limiter, the asymptotic absolutely stability property is fulfilled as follows from the Popov Criterion, in the mentioned conditions.

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