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APPROXIMATE CONTROLLABILITY OF FRACTIONAL STOCHASTIC FUNCTIONAL EVOLUTION EQUATIONS DRIVEN BY A FRACTIONAL BROWNIAN MOTION

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Abstract In this paper, the approximate controllability result of a class of dynamic control systems described by nonlinear fractional stochastic functional differential equations in Hilbert space driven by a fractional Brownian motion with Hurst parameter $H > 1/2$ has been established and discussed by using the theory of fractional calculus, fixed point technique, stochastic analysis technique and methods adopted directly from deterministic control problems. As an application that illustrates the abstract results, an example is given.

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1. INTRODUCTION

The concept of controllability plays an important role in both the deterministic and the stochastic control theory. In recent years, significant progress has been made in the controllability of linear and nonlinear deterministic systems (Bashirov and Mahmudov, 1999; Klamka, 1991, 2000; Balachandran and Sakthivel, 2001). The controllability of nonlinear deterministic systems in a finite and infinite dimensional space by using different kinds of approaches have been considered in many publications (see [1, 2, 6] and the references therein). Moreover, the exact controllability enables to steer the system to arbitrary final state while approximate controllability means that the system can be steered to arbitrary small neighborhood of final state. Klamka [8] derived a set of sufficient conditions for the exact controllability of semilinear systems. Further, approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications. The approximate controllability of systems represented by nonlinear evolution equations has been investigated by several authors [9, 10], in which the authors effectively used

the fixed point approach. Fu and Mei [5] studied the approximate controllability of semilinear neutral functional differential systems with finite delay. The conditions are established with the help of semigroup theory and fixed point technique under the assumption that the linear part of the associated nonlinear system is approximately controllable.

Very recently, Sakthivel et al. [13] established a set of sufficient conditions for obtaining the approximate controllability of semilinear fractional differential systems in Hilbert spaces. Also, Sakthivel et al. in [14] investigated approximate controllability problem for nonlinear fractional stochastic systems driven by Wiener process, which are natural generalizations of the well known controllability concepts from the theory of infinite dimensional deterministic control systems. Specifically, they studied the approximate controllability of nonlinear fractional control systems under the assumption that the associated linear system is approximately controllable.

However, to the best of our knowledge, the approximate controllability problem for nonlinear fractional stochastic functional system driven by fractional Brownian motion in Hilbert spaces has not been investigated yet. Motivated by this consideration, in this paper we investigate the approximate controllability problem of a class of nonlinear fractional stochastic functional systems, we consider a mathematical model given by the following fractional functional equation with control:

$$\begin{aligned} {}^cD_t^q[x(t)-\varphi(t,x_t)] &= Ax(t)+Bu(t)+\phi(t,x_t)+\sigma_H(t)\frac{dB_Q^H(t)}{dt} \\ x(t) &= \psi(t), \quad t\in[-r,0], \end{aligned} \tag{1}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $(S(t))_{t\geq 0}$ in a Hilbert space U; $B_Q^H=\{B_Q^H(t), t\in[0,T]\}$ is a fBm with Hurst index $H\in(\frac{1}{2},1)$ defined in a complete probability space $(\Omega,\mathcal{F},\mathbf{P})$; $0<q<1$ and ${}^cD_t^q$ denotes the Caputo fractional derivative operator of order q. $x_t\in\mathcal{C}_r$ denote the function defined by $x_t(v)=x(t+v)$, $\forall v\in[-r,0]$, where $\mathcal{C}_r=C([-r,0],U)$ is the space of continuous functions f from $[-r,0]$ to U.

We will study the approximate controllability problem for nonlinear fractional control systems of the form (1) under the assumption that the associated linear system is approximately controllable.

The paper is organized as follows. In Section 2 we e will first revise some results concerning fractional calculus including pathwise stochastic integration with respect to fractional Brownian motion and some estimates for such integrals. Second, we provide some definitions, lemmas and notations necessary to establish our main results. In Section 3 we formulate and prove conditions for approximate controllability of the fractional stochastic functional dynamical control system (1) using the contraction mapping principle. As an application that illustrates the abstract results, an example is given.

2. PRELIMINARIES

In this section we introduce some notations, definitions, a technical lemmas and preliminary fact which are used in what follows.
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \in [0,T]), \mathbf{P})$ be a complete probability space with a filtration satisfying the standard conditions.

Definition 2.1. *The fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a Gaussian process $B_t^H = \{B_t^H, \mathcal{F}_t, t \in [0,T]\}$, having the properties $B_0^H = 0$, $\mathbf{E}B_t^H = 0$ and $\mathbf{E}B_t^H B_s^H = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$.*

Let $T > 0$ and denote by Υ the linear space of $\mathcal{R}$-valued step functions on $[0,T]$, that is, $\phi \in \Upsilon$ if

$$\phi(t) = \sum_{i=1}^{n-1} z_i \chi_{[t_i, t_{i+1})}(t),$$

where $t \in [0,T]$, $x_i \in \mathbf{R}$ and $0 = t_1 < t_2 < \cdot < t_n = T$. For $\phi \in \Upsilon$ its Wiener integral with respect to B^H is

$$\int_0^T \phi(s) dB^H(s) = \sum_{i=1}^{n-1} z_i \Big(B^H(t_{i+1}) - B^H(t_i) \Big).$$

Let $\mathcal{H}$ be the Hilbert space defined as the closure of Υ with respect to the scalar product $\langle \chi_{[0,t]}, \chi_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s)$. Then the mapping

$$\phi = \sum_{i=1}^{n-1} z_i \chi_{[t_i, t_{i+1})} \mapsto \int_0^T \phi(s) dB^H(s)$$

is an isometry between Υ and the linear space $span\{B^H(t), t \in [0,T]\}$, which can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fBm $\overline{span}^{L^2(\Omega)}\{B^H(t), t \in [0,T]\}$ (see [12]). The image of an element $\phi \in \mathcal{H}$ by this isometry is called the Wiener integral of ϕ with respect to B^H.

Let us now consider the Kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

where $c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right)^{\frac{1}{2}}$, where β denoting the Beta function, and $t > s$. It is not difficult to see that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(\frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}.$$

Let $\mathcal{K}_H : \Upsilon \mapsto L^2([0,T])$ be the linear operator given by

$$\mathcal{K}_H\phi(s)(s) = \int_s^t \phi(t)\frac{\partial K_H}{\partial t}(t,s)dt.$$

Then $(\mathcal{K}_H\chi_{[0,t]})(s) = K_H(t,s)\chi_{[0,t]}(s)$ and $\mathcal{K}_H$ is an isometry between Υ and $L^2([0,T])$ that can be extended to $\mathcal{H}$.

Denoting $L^2_{\mathcal{H}}([0,T]) = \{\phi \in \mathcal{H}, \mathcal{K}_H\phi \in L^2([0,T])\}$, since $H > 1/2$, we have

$$L^{1/H}([0,T]) \subset L^2_{\mathcal{H}}([0,T]). \tag{2}$$

Moreover the following result hold:

Lemma 2.1 ([12]). *For $\phi \in L^{1/H}([0,T])$,*

$$H(2H-1)\int_0^T\int_0^T |\phi(r)||\phi(u)||r-u|^{2H-2}drdu \le c_H\|\phi\|^2_{L^{1/H}([0,T])}.$$

Let us now consider two separable Hilbert spaces $(U, |\cdot|_U, <.,.>_U)$ and $(V, |\cdot|_V, <.,.>_V)$. Let $L(V,U)$ denote the space of all bounded linear operator from V to U and $Q \in L(V,V)$ be a non-negative self adjoint operator. Denote by $L^0_Q(V,U)$ the space of all $\xi \in L(V,U)$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. the norm is given by

$$|\xi|^2_{L^0_Q(V,U)} = \left|\xi Q^{\frac{1}{2}}\right|^2_{HS} = tr(\xi Q\xi^*).$$

Then ξ is called a Q-Hilbert-Schmidt operator from V to U.

Let $\{B^H_n(t)\}_{n\in\mathbf{N}}$ be a sequence of two-side one-dimensional fBm mutually independent on the complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, $\{e_n\}_{n\in\mathbf{N}}$ be a complete orthonormal basis in V. Define the V-valued stochastic process $B^H_Q(t)$ by

$$B^H_Q(t) = \sum_{n=1}^{\infty} B^H_n(t)Q^{\frac{1}{2}e_n}, \quad t \ge 0.$$

If Q is a non-negative self-adjoint trace class operator, then this series converges in the space V, that is, it holds that $B^H_Q(t) \in L^2(\Omega, V)$. Then, we say that $B^H_Q(t)$ is a V-valued Q-cylindrical fBm with covariance operator Q.

Let $\psi : [0,T] \to L^0_Q(V,U)$ such that

$$\sum_{n=1}^{\infty} \|\mathcal{K}_H(\psi Q^{\frac{1}{2}})e_n\|_{L^2([0,T],U)} < \infty. \tag{3}$$

Definition 2.2. *Let $\psi : [0,T] \to L^0_Q(V,U)$ satisfy (4). Then, its stochastic integral with respect to the fBm B^H_Q is defined for $t \ge 0$ as*

$$\int_0^t \psi(s)dB^H_Q(s) := \sum_{n=1}^{\infty}\int_0^t \psi(s)Q^{\frac{1}{2}}e_n dB^H_n(s) = \sum_{n=1}^{\infty}\int_0^t \left(\mathcal{K}_H(\psi Q^{\frac{1}{2}}e_n)\right)(s)dW(s),$$

where W is a Wiener process.

Notice that if

$$\sum_{n=1}^{\infty} \|\psi Q^{\frac{1}{2}} e_n\|_{L^{1/H}([0,T],U)} < \infty, \tag{4}$$

then in particular (4) holds, which follows immediately from (3).

The following lemma is proved in [12] and obtained as a simple application of Lemma 1.

Lemma 2.2 ([12])**.** *For any $\psi : [0,T] \to L_Q^0(V,U)$ such that (5) holds, and for any $\alpha, \beta \in [0,T]$ with $\alpha > \beta$,*

$$\mathbf{E}\Big| \int_\beta^\alpha \psi(s) dB_Q^H(s) \Big|_U^2 \leq cH(2H-1)(\alpha-\beta)^{2H-1} \sum_{n=1}^{\infty} \int_\beta^\alpha |\psi Q^{\frac{1}{2}} e_n|_U^2 ds,$$

where $c = c(H)$. If in addition

$$\sum_{n=1}^{\infty} |\psi Q^{\frac{1}{2}} e_n|_U \ \text{ is uniformly convergent for } \ t \in [0,T], \tag{5}$$

then

$$\mathbf{E}\Big| \int_\beta^\alpha \psi(s) dB_Q^H(s) \Big|_U^2 \leq cH(2H-1)(\alpha-\beta)^{2H-1} \int_\beta^\alpha |\psi(s)|_{L_Q^0(V,U)}^2 ds. \tag{6}$$

Now, we recall the following known definitions on the fractional integral and derivative

Definition 2.3 ([12])**.** *Let $f \in L^1(0,T)$ and $\alpha > 0$. The fractional Riemann-Liouville integral of f of order α is defined for almost all $t \in (0,T)$ by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(\alpha) = \int_0^\infty \theta^{\alpha-1} e^{-\theta} d\theta$ is the Euler function.

Definition 2.4. *Riemann-Liouville derivative of order α with lower limit zero for a function $f : [0,\infty) \to \mathbf{R}$ can be written as*

$${}^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, n-1 < \alpha < n. \tag{7}$$

Definition 2.5. *The Caputo derivative of order α for a function $f : [0,\infty) \to \mathbf{R}$ can be written as*

$${}^c D^\alpha f(t) = {}^L D^\alpha \Big(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^k(0) \Big), \quad t > 0, n-1 < \alpha < n. \tag{8}$$

If $f(t) \in C^n[0,\infty)$, then

$$ {}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s)ds = I^{n-\alpha} f^n(s), \quad t > 0, n-1 < \alpha < n $$

Now, we denote by $\mathcal{C}(0,T;L^2(\Omega;U)) = \mathcal{C}(0,T;L^2(\Omega,\mathcal{F},\mathbf{P};U))$ the Banach space of all continuous functions from $[0,T]$ into $L^2(\Omega;U)$ equipped with the sup norm.

Let us consider a fixed real number $r \geq 0$. If $x \in \mathcal{C}(-r,T;L^2(\Omega;U))$ for each $t \in [0,T]$ we denote by $x_t \in \mathcal{C}(-r,0;L^2(\Omega;U))$ the function defined by $x_t(v) = x(t+v)$, for $v \in [-r,0]$.

Consider the fractional functional equation with control of the form

$$ \begin{cases} {}^cD_t^q[x(t) - \varphi(t,x_t)] &= Ax(t) + Bu(t) + \phi(t,x_t) + \sigma_H(t)\dfrac{dB_Q^H(t)}{dt} \\ x(t) &= \psi(t), \quad t \in [-r,0], \end{cases} \tag{9} $$

where $B_Q^H(t)$ is the fractional Brownian motion which was introduced above, the initial data $\psi \in \mathcal{C}(-r,0;L^2(\Omega;U))$ and $A : Dom(A) \subset U \to U$ is the infinitesimal generator of a strongly continuous semigroup $S(.)$ on U. Here, for $0 < q < 1$, ${}^cD_t^q$ denote the Caputo fractional derivative operator of order q, control function $u(.)$ is given in $L^2([0,T],\tilde{U})$, a Banach space of admissible control functions with $\tilde{U}$ is a Hilbert space and $B \in L(\tilde{U},U)$. Further $\varphi,\phi : [0,T] \times \mathcal{C}(-r,0;U) \to U$ and $\sigma_H : [0,T] \to L_Q^0(V,U)$ are appropriate functions.

Definition 2.6. *A U-valued process $x(t)$ is called a mild solution of (1) if $x \in \mathcal{C}(-r,T;L^2(\Omega;U))$, $x(t) = \psi(t)$ for $t \in [-r,0]$, and, for $t \in [0,T]$, satisfies*

$$ \begin{aligned} & x(t) = \\ & S(t)\Big[\psi(0) - \varphi(0,x_0)\Big] + \varphi(t,x_t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s)\Big[Bu(s) + \phi(s,x_s)\Big]ds \\ + \quad & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s)\varphi(s,x_s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S(t-s)\sigma_H(s)dB_Q^H(s). \end{aligned} \tag{10} $$

We will make use of the following assumptions on data of the problem:

($\mathbf{H}_1$) The semigroup $(S(t))_{t\geq 0}$ is a bounded linear operator on U and satisfies for $t \geq 0$

$$ |S(t)x|_U \leq Me^{\lambda t}|x|, \qquad M \geq 1, \lambda \in \mathbf{R} \text{ and } x \in U. $$

($\mathbf{H}_2$) The functions ϕ,φ satisfy the following Lipschitz condition: there exist constants $c_1, c_2 > 0$ for $x,y \in U$ and $t \geq 0$ such that

$$ \begin{aligned} |\phi(t,x) - \phi(t,y)|_U^2 &\leq c_1|x-y|_U^2, \\ |\varphi(t,x) - \varphi(t,y)|_U^2 &\leq c_2|x-y|_U^2. \end{aligned} $$

($\mathbf{H}_3$) The functions ϕ, φ are continuous and satisfy the usual linear growth condition i.e., there exist constants $c_3, c_4 > 0$ for $x, y \in U$ and $t \geq 0$ such that

$$\begin{array}{rcl} |\phi(t,x)|_U^2 & \leq & c_3(1+|x|_U^2), \\ |\varphi(t,x)|_U^2 & \leq & c_4(1+|x|_U^2). \end{array}$$

($\mathbf{H}_4$) The function σ_H satisfies the following conditions: for the complete orthonormal basis $\{e_n\}_{n\in\mathbf{N}}$ in V, we have

$$\sum_{n=1}^{\infty} \|\sigma_H Q^{1/2} e_n\|_{L^2([0,T];U)} < \infty.$$

$$\sum_{n=1}^{\infty} |\sigma_H(t,x(t))Q^{1/2}e_n|_U \text{ is uniformly convergent for } t \in [0,T].$$

Note that, by assumption ($\mathbf{H}_4$), for every $t \in [0,T]$, $\displaystyle\int_0^t |\sigma_H(s)|^2_{L^0_Q(V,U)} ds < \infty$.

($\mathbf{H}_5$) The linear stochastic system is approximately controllable on $[0,T]$.

Let B^*, $S^*(.)$ be respectively the operator adjoint of B and $S(.)$. Define the controllability Grammian operator by $\Theta_0^t = \int_0^t (t-s)^{2(q-1)} S(t-s)BB^*S^*(t-s)ds$. Then, for each $0 \leq s \leq t$, the operator $\theta\Big(\theta I + \Theta_0^t\Big)^{-1} \to 0$ in the strong operator topology as $\theta \to 0^+$.

We consider the corresponding linear fractional deterministic control system to (9)

$$\left\{ \begin{array}{rcl} {}^cD_t^q x(t) & = & Ax(t) + (Bu)(t) \quad t \in [0,T], \\ x(t) & = & \psi(t), \quad t \in [-r,0], \end{array} \right. \tag{11}$$

Proposition 2.1. *The deterministic system (11) is approximately controllable on* $[0,T]$ *iff the operator* $\theta\Big(\theta I + \Theta_0^t\Big)^{-1} \to 0$ *as* $\theta \to 0^+$.

We note that the approximate controllability for linear fractional deterministic control system (11) is a natural generalization of approximate controllability of linear first order control system (see [11], Theorem 2).

Definition 2.7. *System* (9) *is approximately controllable on* $[0,T]$ *if* $\overline{\mathcal{R}(T)} = L^2(\Omega, \mathcal{F}, \mathbf{P}; U)$, *where* $\mathcal{R}(t) = \{x(t) = x(t,u) : u \in L^2([0,T], \tilde{U})\}$.

The following lemma is required to define the control function. The reader can refer to [10] for the proof.

Lemma 2.3. *For all adapted, U-valued process* $z_T \in L^2(\Omega; U)$, *there exists* $f \in L^2\Big(\Omega; L^2([0,T]; L^0_Q)\Big)$ *such that* $z_T = \mathbf{E}z_T + \displaystyle\int_0^T f(s)dB_Q^H(s)$.

Let $\theta > 0$ and $z_T \in L^2(\Omega; U)$. Define the control function in the following form:

$$
\begin{aligned}
& u^\theta(t,x) = \\
& = B^*(T-t)^{q-1}S^*(T-t)\Big[(\theta I + \Theta_0^T)^{-1}\Big(\mathbf{E}z_T - S(T)[\psi(0) - \varphi(0,x_0)] - \varphi(T,x(T))\Big) \\
+ \quad & \int_0^T (\theta I + \Theta_0^T)^{-1} f(s) dB_Q^H(s) - \frac{1}{\Gamma(q)}\int_0^T (\theta I + \Theta_0^T)^{-1}(T-s)^{q-1}S(T-s)\varphi(s,x(s))ds \\
- \quad & \frac{1}{\Gamma(q)}\int_0^T (\theta I + \Theta_0^T)^{-1}(T-s)^{q-1}S(T-s)\phi(s,x(s))ds \\
- \quad & \frac{1}{\Gamma(q)}\int_0^T (\theta I + \Theta_0^T)^{-1}(T-s)^{q-1}S(T-s)\sigma_H(s)dB_Q^H(s)\Big].
\end{aligned}
$$

Now for our convenience, let us assume that the function f satisfies the condition $(\mathbf{H}_4)$. Set $c_5 = \max\{|f(s)|_U^2 : 0 \le s < t \le T\}$.

Lemma 2.4. *There exists a positive real constant N such that for all $x \in \mathcal{C}(-r,T;U)$, we have*

$$
\mathbf{E}|u^\theta(t,x)|^2 \le \frac{N}{\theta^2}\Big(1 + \int_0^t \mathbf{E}|x(s)|_U^2 ds\Big). \tag{12}
$$

Proof. Let $x \in \mathcal{C}(-r,T;U)$ and $T > 0$ be fixed. We have

$$
\begin{aligned}
& \mathbf{E}|u^\theta(t,x)|^2 \le \\
\le \quad & 6\mathbf{E}\Big|B^*(T-t)^{q-1}S^*(T-t)(\theta I + \Theta_0^T)^{-1}(\mathbf{E}z_T - S(T)[\psi(0) - \varphi(0,x_0)])\Big|^2 \\
+ \quad & 6\mathbf{E}\Big|B^*(T-t)^{q-1}S^*(T-t)(\theta I + \Theta_0^T)^{-1}\varphi(T,x(T))\Big|^2 \\
+ \quad & 6\mathbf{E}\Big|B^*(T-t)^{q-1}S^*(T-t)\int_0^t(\theta I + \Theta_0^T)^{-1}f(s)dB_Q^H(s)\Big|^2 \\
+ \quad & 6\mathbf{E}\Big|B^*(T-t)^{q-1}S^*(T-t)\frac{1}{\Gamma(q)}\int_0^t(\theta I + \Theta_0^T)^{-1}(T-s)^{q-1}S(T-s)\varphi(s,x(s))ds\Big|^2 \\
+ \quad & 6\mathbf{E}\Big|B^*(T-t)^{q-1}S^*(T-t)\frac{1}{\Gamma(q)}\int_0^t(\theta I + \Theta_0^T)^{-1}(T-s)^{q-1}S(T-s)\phi(s,x(s))ds\Big|^2 \\
+ \quad & 6\mathbf{E}\Big|B^*(T-t)^{q-1}S^*(T-t)\frac{1}{\Gamma(q)}\int_0^t(\theta I + \Theta_0^T)^{-1}(T-s)^{q-1}S(T-s)\sigma_H(s)dB_Q^H(s)\Big|^2.
\end{aligned}
$$

From the Hölders inequality, the assumption on the data and Lemma 2.4., we have

$$
\begin{aligned}
\mathbf{E}|u^\theta(t,x)|^2 \quad \le \quad & \frac{6}{\theta^2}|B|^2T^{2q-2}\Big(\frac{M^2e^{2\lambda t}}{q-1}\Big)^2\Big[\mathbf{E}|z_T|^2 - \mathbf{E}|\psi(0) - \varphi(0,x_0)|^2\Big] \\
+ \quad & \frac{6}{\theta^2}|B|^2T^{2q-2}\Big(\frac{Me^{\lambda t}}{q-1}\Big)^2\Big[\mathbf{E}|\varphi(T,x(T))|^2 - c_6H(2H-1)T^{2H}\Big] \\
+ \quad & \frac{6}{\theta^2}|B|^2T^{4q-3}\Big(\frac{M^2e^{2\lambda t}}{(q-1)\Gamma(q)}\Big)^2 c_7\int_0^t(1 + \mathbf{E}|x(s)|_U^2)ds \\
+ \quad & \frac{6}{\theta^2}|B|^2T^{4q-3}\Big(\frac{M^2e^{2\lambda t}}{(q-1)\Gamma(q)}\Big)^2 cH(2H-1)T^{2H-1}\int_0^t|\sigma_H(s)|^2_{L_Q^0(V,U)}ds,
\end{aligned}
$$

where $c_6 = cc_5$ and $c_7 = c_3 + c_4$.
Remark that condition $(\mathbf{H}_4)$ ensures the existence of a positive constant c_8 such that

$$6|B|^2 T^{4q-3}\left(\frac{M^2 e^{2\lambda t}}{(q-1)\Gamma(q)}\right)^2 cH(2H-1)T^{2H-1}\int_0^t |\sigma_H(s)|^2_{L^0_Q(V,U)} ds \le c_8, \quad \text{for all} \quad t \ge 0.$$

Thus it follows from the above inequalities and linear growth condition that there exists $N > 0$ such that

$$\mathbf{E}|u^\theta(t,x)|^2 \le \frac{N}{\theta^2}\left(1 + \int_0^t \mathbf{E}|x(v)|^2_U)dv\right).$$

∎

3. CONTROLLABILITY RESULTS

In this section, we formulate and prove conditions for approximate controllability of the fractional functional stochastic dynamical control system (9) using the contraction mapping principle. In particular, we establish approximate controllability of nonlinear fractional functional stochastic control system (9) under the assumptions that the corresponding linear system is approximately controllable.

We first define the operator $\mathcal{P}_\theta : \mathcal{C}(0,T;L^2(\Omega,U)) \to \mathcal{C}(0,T;L^2(\Omega,U))$, $\theta > 0$ by

$$\begin{aligned}
&\mathcal{P}_\theta x(t) = \\
&S(t)\Big[\psi(0) - \varphi(0,x_0)\Big] + \varphi(t,x_t) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} S(t-s)\Big[Bu^\theta(s,x) + \phi(s,x_s)\Big]ds \\
+ \quad & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} S(t-s)\varphi(s,x_s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} S(t-s)\sigma_H(s)dB^H_Q(s).
\end{aligned} \tag{13}$$

Lemma 3.1. *For any $x \in \mathcal{C}(0,T;L^2(\Omega,U))$, $(\mathcal{P}_\theta x)(.)$ is continuous on [0,T] in L^2 sense.*

Proof. Let $x \in \mathcal{C}(0,T;L^2(\Omega,U))$ be fixed and $0 \le t_1 < t_2 \le T$. Then from Eq. (13) we have

$$\begin{aligned}
&\mathbf{E}|(\mathcal{P}_\theta x)(t_2) - (\mathcal{P}_\theta x)(t_1)|^2 \le \\
&\le 6\Big[\mathbf{E}|(S(t_2) - S(t_1))[\psi(0) - \varphi(0,x_0)]|^2 + \mathbf{E}|\varphi(t_2,x_{t_2}) - \varphi(t_1,x_{t_1})|^2\Big] \\
+ \quad & 6\Big[\sum_{i=1}^4 \mathbf{K}|\Sigma_i^x(t_2) - \Sigma_i^x(t_1)|^2\Big].
\end{aligned}$$

From the strong continuity of $S(.)$, the first term on the R.H.S goes to zero as $t_2 - t_1 \to 0$ [5]. The Lipschitz condition on φ implies that the second term goes to zero as

$t_2 - t_1 \to 0$.

Next, it follows from Hölder's inequality and assumptions on the theorem that

$$\begin{aligned}
&\mathbf{K}|\Sigma_1^x(t_2) - \Sigma_1^x(t_1)|^2 \leq \\
&3\mathbf{K}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}[S(t_2-s)-S(t_1-s)]\phi(s,x_s)ds\right|^2 \\
+\ &3\mathbf{K}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]S(t_2-s)\phi(s,x_s)ds\right|^2 \\
+\ &3\mathbf{K}\left|\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\phi(s,x_s)ds\right|^2 \\
\leq\ &3\frac{t_1^{2q-1}}{(2q-1)\Gamma^2(q)}\int_0^{t_1}\mathbf{K}\Big|[S(t_2-s)-S(t_1-s)]\phi(s,x_s)\Big|^2 ds \\
+\ &3\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2\left(\int_0^{t_1}\Big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\Big]^2 ds\right)\left(\int_0^{t_1}\mathbf{E}|\phi(s,x_s)|^2 ds\right) \\
+\ &3\frac{(t_2-t_1)^{2q-1}}{2q-1}\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2\int_{t_1}^{t_2}\mathbf{E}|\phi(s,x_s)|^2 ds.
\end{aligned}$$

Further, we obtain

$$\begin{aligned}
&\mathbf{E}|\Sigma_2^x(t_2) - \Sigma_2^x(t_1)|^2 \leq \\
&\leq 3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}[S(t_2-s)-S(t_1-s)]Bu^\alpha(s,x)ds\right|^2 \\
+\ &3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]S(t_2-s)Bu^\alpha(s,x)ds\right|^2 \\
+\ &3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}Bu^\alpha(s,x)ds\right|^2 \\
\leq\ &3\frac{t_1^{2q-1}}{(2q-1)\Gamma^2(q)}\int_0^{t_1}\mathbf{E}\Big|[S(t_2-s)-S(t_1-s)]Bu^\alpha(s,x)\Big|^2 ds \\
+\ &3\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2|B|^2\left(\int_0^{t_1}\Big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\Big]^2 ds\right)\left(\int_0^{t_1}\mathbf{E}|u^\alpha(s,x)|^2 ds\right) \\
+\ &3\frac{(t_2-t_1)^{2q-1}}{2q-1}\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2|B|^2\int_{t_1}^{t_2}\mathbf{E}|u^\alpha(s,x)|^2 ds.
\end{aligned}$$

$$\begin{aligned}
&\mathbf{E}|\Sigma_3^x(t_2) - \Sigma_3^x(t_1)|^2 \leq \\
&\leq 3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}[S(t_2-s)-S(t_1-s)]\varphi(s,x_s)ds\right|^2 \\
+\ &3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]S(t_2-s)\varphi(s,x_s)ds\right|^2 \\
+\ &3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\varphi(s,x_s)ds\right|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq 3\frac{t_1^{2q-1}}{(2q-1)\Gamma^2(q)}\int_0^{t_1}\mathbf{E}\Big|[S(t_2-s)-S(t_1-s)]\varphi(s,x_s)\Big|^2ds\\
&+ 3\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2\left(\int_0^{t_1}\Big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\Big]^2ds\right)\left(\int_0^{t_1}\mathbf{E}|\varphi(s,x_s)|^2ds\right)\\
&+ 3\frac{(t_2-t_1)^{2q-1}}{2q-1}\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2\int_{t_1}^{t_2}\mathbf{E}|\varphi(s,x_s)|^2ds.
\end{aligned}
$$

Similarly, using Lemma 2.4 and assumptions on the theorem we get

$$
\begin{aligned}
&\mathbf{E}|\Sigma_4^x(t_2)-\Sigma_4^x(t_1)|^2\leq\\
&\leq 3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}[S(t_2-s)-S(t_1-s)]\sigma_H(s)dB_Q^H(s)\right|^2\\
&+ 3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]S(t_2-s)\sigma_H(s)dB_Q^H(s)\right|^2\\
&+ 3\mathbf{E}\left|\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\sigma_H(s)dB_Q^H(s)\right|^2\\
&\leq \frac{3cH(2H-1)T^{2H-1}t_1^{2q-1}}{(2q-1)\Gamma^2(q)}\int_0^{t_1}\mathbf{E}\Big|[S(t_2-s)-S(t_1-s)]\sigma_H(s)\Big|^2ds\\
&+ \frac{3}{\Gamma^2(q)}cH(2H-1)T^{2H-1}\left(\int_0^{t_1}\Big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\Big]^2ds\right)\\
&\times \left(\int_0^{t_1}\mathbf{E}|S(t_2-s)\sigma_H(s)|^2ds\right)\\
&+ \frac{3cH(2H-1)T^{2H-1}(t_2-t_1)^{2q-1}}{2q-1}\left(\frac{Me^{\lambda t}}{\Gamma(q)}\right)^2\int_{t_1}^{t_2}\mathbf{E}|S(t_2-s)\sigma_H(s)|^2ds.
\end{aligned}
$$

Hence using the strong continuity $S(t)$ and Lebesgue's dominated convergence theorem, we conclude that the right-hand side of the above inequalities tends to zero as $t_2-t_1\to 0$. Thus we conclude $\mathcal{P}_\theta(x)(t)$ is continuous from the right in $[0,T)$. A similar argument shows that it is also continuous from the left in $(0,T]$. This completes the proof of this lemma. ∎

Theorem 3.1. *Assume assumptions* $(\mathbf{H}_1)$-$(\mathbf{H}_4)$ *are satisfied. Then the system* (9) *has a mild solution on* $[0,T]$.

Proof. We prove the existence of a fixed point of the operator $\mathcal{P}_\theta$ by using the contraction mapping principle. Let $x\in\mathcal{C}(0,T;L^2(\Omega,U))$. From (13) we obtain

$$
\mathbf{E}\Big|\mathcal{P}_\theta x\Big|_C^2\leq 6\left[\sup_{0\leq t\leq T}\mathbf{E}|S(t)[\psi(0)-\varphi(0,x_0)]|^2+\sup_{0\leq t\leq T}\mathbf{E}|\varphi(t,x_t)|^2+\sup_{0\leq t\leq T}\sum_{i=1}^4\mathbf{E}|\Sigma_i^x(t)|^2\right]. \tag{14}
$$

Using assumptions $(\mathbf{H}_1)$-$(\mathbf{H}_4)$ and Lemma 2.12., we get

$$\sup_{0\le t\le T} \mathbf{E}|S(t)[\psi(0)-\varphi(0,x_0)]|^2 \le M^2 e^{2\lambda T}\Big[|\psi(0)|^2+|\varphi(0,x_0)|^2\Big] \tag{15}$$

and

$$\begin{aligned}\sum_{i=1}^{4}\mathbf{E}|\Sigma_i^x(t)|^2 &\le 3\left(\frac{Me^{\lambda T}}{\Gamma(q)}\right)^2\frac{T^{2q-1}}{2q-1}c_7\Big(1+|x|_C^2\Big)\\ &+ 3\left(\frac{Me^{\lambda T}}{\Gamma(q)}\right)^2\frac{T^{2q}}{2q-1}|B|^2\frac{N}{\theta^2}\Big(1+|x|_C^2\Big)+c_9,\end{aligned} \tag{16}$$

where c_9 is a positive constant such that

$$3cH(2H-1)T^{2H-1}\frac{T^{2q-1}}{2q-1}\left(\frac{Me^{\lambda T}}{\Gamma(q)}\right)^2\int_0^t|\sigma_H(s)|^2_{L_Q(V,U)}ds \le c_9.$$

Inequalities (15) and (16) together imply that $\mathbf{E}\Big|\mathcal{P}_\theta x\Big|_C^2 < \infty$. By Lemma 3.1., $\mathcal{P}_\theta x \in \mathcal{C}(0,T;L^2(\Omega,U))$. Thus for each $\theta>0$, the operator $\mathcal{P}_\theta$ maps $\mathcal{C}(0,T;L^2(\Omega,U))$ into itself.

Now, we are going to use the Banach fixed point theorem to prove that $\mathcal{P}_\theta$ has a unique fixed point in $\mathcal{C}(0,T;L^2(\Omega,U))$. We claim that $\mathcal{P}_\theta$ is a contraction on $\mathcal{C}(0,T;L^2(\Omega,U))$. For $x,y\in\mathcal{C}(0,T;L^2(\Omega,U))$ we have

$$\begin{aligned}&\mathbf{E}\Big|(\mathcal{P}_\theta)(x)-\mathcal{P}_\theta)(y)\Big|_C^2\\ \le\ & 4\mathbf{E}\sum_{i=1}^{4}|\Sigma_i^x(t)-\Sigma_i^y(t)|^2\\ \le\ & 4c_9+4\left(\frac{Me^{\lambda T}}{\Gamma(q)}\right)^2\left[\frac{T^{2q-1}}{2q-1}c_1+\frac{N}{\theta^2}|B|^2\frac{T^{2q-1}}{2q-1}T^2+\frac{T^{2q-1}}{2q-1}c_2\right]\int_0^t\mathbf{E}|x(s)-y(s)|^2ds\\ =\ & 4c_9+4\left(\frac{Me^{\lambda T}}{\Gamma(q)}\right)^2\left[\frac{T^{2q-1}}{2q-1}c_{10}+\frac{N}{\theta^2}|B|^2\frac{T^{2q+1}}{2q-1}\right]\int_0^t\mathbf{E}|x(s)-y(s)|^2ds.\end{aligned}$$

It results that

$$\begin{aligned}&\sup_{0\le t\le T}\mathbf{E}\Big|(\mathcal{P}_\theta)(x)-\mathcal{P}_\theta)(y)\Big|_C^2\\ \le\ & \left[4c_9+4\left(\frac{Me^{\lambda T}}{\Gamma(q)}\right)^2\left(\frac{c_{10}}{2q-1}+\frac{N|B|^2}{\theta^2(2q-1)}\right)\right]T^{2q+2}\sup_{0\le t\le T}\mathbf{E}|x(t)-y(t)|_C^2.\end{aligned} \tag{17}$$

Therefore we conclude that $\mathcal{P}_\theta$ is a contraction mapping on $\mathcal{C}(0,T;L^2(\Omega,U))$. Then the mapping $\mathcal{P}_\theta$ has a unique fixed point $x(.)\in\mathcal{C}(0,T;L^2(\Omega,U))$, which is a mild solution of (9). ∎

Theorem 3.2. *Assume that the assumptions* $(\mathbf{H}_1)$-$(\mathbf{H}_5)$ *hold. If the function* ϕ *and* φ *are uniformly bounded and* $\{S(t);t\ge 0\}$ *is compact, then the system* (9) *is approximately controllable on* $[0,T]$.

Proof. Let x_θ be a fixed point of $\mathbf{P}_\theta$. By the stochastic Fubini theorem, it can be easily seen that

$$\begin{aligned} x_\theta(T) &\leq z_T - \theta(\theta I + \Theta)^{-1}\Big(\mathbf{E}z_T - S(T)[\psi(0) - \varphi(0, x_0)] - \varphi(t, (x_\theta)_t)\Big) \\ &+ \frac{\theta}{\Gamma(q)} \int_0^T (\theta I + \Theta_s^T)^{-1}(T-s)^{q-1} S(T-s)[\phi(s, x_\theta(s)) + \varphi(s, x_\theta(s))]ds \\ &+ \frac{\theta}{\Gamma(q)} \int_0^T (\theta I + \Theta_s^T)^{-1}[(T-s)^{q-1} S(T-s)\sigma_H(s) - \Gamma(q) f(s)]dB_Q^H(s). \end{aligned}$$

It follows from the assumption on ϕ and φ that there exists $c_{11} > 0$ such that

$$|\phi(s, x_\theta(s))|^2 + |\varphi(s, x_\theta(s))|^2 \leq c_{11} \tag{18}$$

Then there is a subsequence still denoted by $\{\phi(s, x_\theta(s)), \varphi(s, x_\theta(s))\}$ which converges to weakly to, say, $\{\phi(s), \varphi(s)\}$ in U.
We have

$$\begin{aligned} &\mathbf{E}\Big|x_\theta(T) - z_T\Big|^2 \\ \leq\ & 7\mathbf{E}\Big|\theta(\theta I + \Theta_0^T)^{-1}\Big(\mathbf{E}z_T - S(T)[\psi(0) - \varphi(0, x_0)] - \varphi(t, (x_\theta)_t)\Big)\Big|^2 \\ +\ & 7c_6 H(2H-1)T^{2H-1}\Big(\int_0^T |\theta(\theta I + \Theta_s^T)^{-1}|^2_{L_Q^0} ds\Big) \\ +\ & 7\mathbf{E}\Big(\int_0^T (T-s)^{q-1}|\theta(\theta I + \Theta_s^T)^{-1}|\Big|S(T-s)(\phi(s, x_\theta(s)) - \phi(s)\Big|ds\Big)^2 \\ +\ & 7\mathbf{E}\Big(\int_0^T (T-s)^{q-1}|\theta(\theta I + \Theta_s^T)^{-1} S(T-s)\phi(s)|ds\Big)^2 \\ +\ & 7\mathbf{E}\Big(\int_0^T (T-s)^{q-1}|\theta(\theta I + \Theta_s^T)^{-1}|\Big|S(T-s)(\varphi(s, x_\theta(s)) - \varphi(s)\Big|ds\Big)^2 \\ +\ & 7\mathbf{E}\Big(\int_0^T (T-s)^{q-1}|\theta(\theta I + \Theta_s^T)^{-1} S(T-s)\varphi(s)|ds\Big)^2 \\ +\ & 7cH(2H-1)T^{2H-1}\Big(\int_0^T |\theta(\theta I + \Theta_s^T)^{-1} S(T-s)\sigma_H(s)|^2_{L_Q^0} ds\Big). \end{aligned}$$

By assumption ($\mathbf{H}_5$), for all $0 \leq s \leq T$ the operator $\theta(\theta I + \Theta_s^T)^{-1} \to 0$ strongly as $\theta \to 0^+$ and moreover $|\theta(\theta I + \Theta_s^T)^{-1}| \leq 1$. Finally, by the Lebesgue dominated convergence theorem and the compactness of $S(.)$ we get $\mathbf{E}\Big|x_\theta(T) - z_T\Big|^2 \to 0$ as $\theta \to 0^+$ which implies the approximate controllability of system (9). ∎

4. AN EXAMPLE

As a specific application of the theoretical result established in the preceding Theorem, we can consider the following example.

Let $V = L^2(0,\pi)$ and $e_n = \sqrt{\frac{2}{\pi}}\sin(nx)$, $n = 1,2,\ldots$. Then $\{e_n\}_n$ is a complete orthonormal basis in V. Let $U = L^2(0,\pi)$ and $A = \frac{\partial^2}{\partial z^2}$ with domain $D(A) = H_0^1(0,\pi) \cap H^2(0,\pi)$. Then, it is well-known that $Av = -\sum_{n=1}^{\infty} n^2\langle v, e_n\rangle e_n$ for any $v \in U$, and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t) : U \to U$, where $S(t)v = \sum_{n=1}^{\infty}\exp(-n^2t)\langle v, e_n\rangle e_n$. In order to define the operator $Q : V \to V$, we choose a sequence $\{\eta_n\}_{n\geq 1} \subset \mathbf{R}^+$ and set $Qe_n = \eta_n e_n$, and assume that $tr(Q) = \sum_{n=1}^{\infty}\sqrt{\eta_n} < \infty$. Define the process $B_Q^H(s)$ by $B_Q^H(t) = \sum_{n=1}^{\infty}\sqrt{\eta_n}B_n^H(t)e_n$, where $H \in (1/2, 1)$ and $\{B_n^H\}_{n\geq 0}$ is a sequence of mutually independent two-sided one-dimensional fractional Brownian motions.

We consider the following fractional stochastic control system of the form

$$\begin{cases} {}^cD_t^q[x(t,z) - \varphi(t, x_t(z))] &= \dfrac{\partial^2 x(t,z)}{\partial z^2} + y(t,z) + \phi(t, x_t(z)) + \sigma_H(t)\dfrac{dB_Q^H(t)}{dt} \\ x(t,0) &= x(t,\pi) = 0, \quad t \geq 0 \\ x(t,z) &= \psi(t,z); \quad t \in [-r, 0], \quad z \in [0,\pi], \end{cases} \tag{19}$$

where $0 < q < 1$, $r \in (0,1)$ and $T > 0$. We define $x(t)(z) = x(t,z)$, $\phi(t, x_t)(z) = \phi(t, x_t(z))$ and the bounded linear operator $B : \tilde{U} \to U$ by $Bu(t)(z) = y(t,z)$, $0 \leq z \leq \pi$, $u \in \tilde{U}$. On the other hand, it can be easily seen that the deterministic linear fractional control system corresponding to (18) is approximately controllable on $[0,\pi]$ (see [5]). Therefore, with the above choices, the system (18) may be written in the abstract form (9) and all conditions of Theorem 3.3 are satisfied. Thus, by its conclusion, the fractional stochastic control system (18) is approximately controllable on $[0,\pi]$.

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