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BRANCHING EQUATIONS IN THE ROOT-SUBSPACES AND POTENTIALITY CONDITIONS FOR THEM, FOR ANDRONOV-HOPF BIFURCATION. II.

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Abstract As the prolongation of the article [1] on the base of the relevant stationary [2;3] and non-stationary [1] bifurcation results the conditions are established for Lyapunov-Schmidt branching equations and branching equations in the root-subspaces at Poincaré-Andronov-Hopf bifurcation would be of potential type, with invariant potentials under group symmetry conditions.

Keywords: Banach spaces; nonlinear differential equation; small parameter; Lyapunov-Schmidt method; Lyapunov-Schmidt branching equation and branching equation in the root-subspaces; potentiality condition; group symmetry; potential invariance.

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1. INTRODUCTION

Since this article is the direct prolongation of the previous one [1], the contained there short presentation of the articles [2,3] basic results are omitted here. In many applications of bifurcation theory [4-6] often the following situation arises when the original nonlinear problem has not the variational structure, while the relevant Lyapunov-Schmidt branching equation (BEq) and BEq in the root-subspaces (BEqR) turn out to be potential. In the articles [2,3] for such situation in stationary problems of branching theory sufficient conditions for the potentiality of the equivalent to bifurcation problem BEq and also in [7] for the potential type BEq are established. In the article [1] such conditions are obtained for BEqRs of dynamic branching (Poincaré-Andronov-Hopf (P-A-H) bifurcation). Here sufficient conditions are established for the corresponding Lyapunov-Schmidt BEq and BEqR in dynamic branching theory to be systems of potential type. Everywhere below the terminology and notations of the works [1-7] are used.

2. BEQ AND BEQR CONSTRUCTION

As in the article [1], in real Banach spaces E_1 and E_2, for differential equation with sufficiently smooth by ε operators

$$A(\varepsilon)\frac{dx}{dt} = B(\varepsilon)x - R(x,\varepsilon),\ R(0,\varepsilon) = 0,\ R_x(0,\varepsilon) = 0,\ A_0 = A(0),\ B_0 = B(0), \quad (1)$$

P-A-H bifurcation is considered under assumption that A_0-spectrum $\sigma_{A_0}(B_0)$ of densely defined closed Fredholmian operator B_0 is decomposed on two parts: $\sigma^-_{A_0}(B_0)$ lying strictly in the left half-plane and $\sigma^0_{A_0}(B_0)$ consisting of the eigenvalues $\pm i\alpha$ of the multiplicity n with eigenelements $u_j^{(1)} = u_j = u_{1j} \pm iu_{2j}$, and eigenelements $v_j^{(1)} = v_j = v_{1j} \pm iv_{2j}$ of the conjugate operator $A_0^\star : D_{A_0^\star} \to E_1^\star$, $B_0^\star : D_{B_0^\star} \to E_1^\star$, i.e. $(B_0 - i\alpha A_0)u_j = 0$, $(B_0 + i\alpha A_0)\overline{u}_j = 0$, $(B_0^\star + i\alpha A_0^\star)v_j = 0$, $(B_0^\star - i\alpha A_0^\star)\overline{v}_j = 0$, $j = \overline{1,n}$.

H. Poincaré substitution $t = \frac{\tau}{\alpha+\mu}$, $x(t) = y(\tau)$, $\mu = \mu(\varepsilon)$ reduces the problem of $\frac{2\pi}{\alpha+\mu}$-periodic solutions construction to the determination of 2π -periodic solutions of the equation

$$\begin{aligned} \mathcal{B}y = \mu A(\varepsilon)\frac{dy}{d\tau} + \alpha(A(\varepsilon) - A_0)\frac{dy}{d\tau} - (B(\varepsilon) - B_0)y + R(y,\varepsilon) \equiv \\ \equiv \mu\mathcal{C}(\varepsilon)y + \mathcal{R}(y,\varepsilon), \quad R_y(0,\varepsilon) = 0, \end{aligned} \quad (2)$$

$$\mathcal{B}y = (\mathcal{B}y)(\tau) \equiv B_0 y(\tau) - \alpha A_0\frac{dy}{d\tau}, \quad \mathcal{C}(\varepsilon)y = (\mathcal{C}(\varepsilon)y)(\tau) \equiv A(\varepsilon)\frac{dy}{d\tau},$$

where the supposed Fredholmian operator $\mathcal{B}$ and the operators in (2) are mapping the space Y of 2π-periodic continuously differentiable functions τ with values in $\mathcal{E}_1 = E_1 + iE_1$ in the space $\mathcal{E}_2 = E_2 + iE_2$ with duality between $Y, Y^\star$ $(Z, Z^\star)$ determined by the functionals

$$\ll y, f \gg = \frac{1}{2\pi}\int_0^{2\pi}\langle y(\tau), f(\tau)\rangle d\tau, y \in Y, f \in Y^\star (y \in Z, f \in Z^\star), \quad (3)$$

(in (3) $\langle\cdot,\cdot\rangle$ represents the duality between $\mathcal{E}_1, \mathcal{E}_1^\star$, $(\mathcal{E}_2, \mathcal{E}_2^\star)$). Then the zero-subspaces of the operators $\mathcal{B}$ and $\mathcal{B}^\star$ are $2n$-dimensional:

$$\mathcal{N}(\mathcal{B}) = span\left\{\varphi_j^{(1)} = \varphi_j,\ \ \varphi_j(\tau) = u_j e^{i\tau}; \overline{\varphi}_j\right\}_1^n,$$

$$\mathcal{N}(\mathcal{B}^\star) = span\left\{\psi_j^{(1)} = \psi_j,\ \ \psi_j(\tau) = v_j e^{i\tau}; \overline{\psi}_j\right\}_1^n.$$

Introduce the systems $\{\gamma_s^{(1)}\}_1^n \in Y^\star$ and $\{z_s^{(1)}\}_1^n \in Z^\star$ biorthogonal in the sense (3) to $\{\varphi_k^{(1)}\}_1^n \in \mathcal{N}(\mathcal{B})$ and $\{\psi_k^{(1)}\}_1^n \in \mathcal{N}(\mathcal{B}^\star)$ respectively. As such systems can be chosen the $A_0^\star$- and A_0-images of the last elements of the complete $A_0^\star$- and A_0-Jordan sets of the elements $\{\psi_k^{(1)}\}_1^n$ and $\{\varphi_k^{(1)}\}_1^n$ respectively which are always existed ($\pm i\alpha$ are the isolated eigenvalues) and are determined by the formulae for the generalized Jordan chains [8,9]

$$(B_0 - i\alpha A_0)u_j^{(k)} = A_0 u_j^{(k-1)}, (B_0 + i\alpha A_0)\overline{u}_j^{(k)} = -A_0\overline{u}_j^{(k-1)};$$

$$(B_0^\star + i\alpha A_0^\star)v_j^{(k)} = -A_0^\star v_j^{(k-1)}, (B_0^\star - i\alpha A_0^\star)\overline{v}_j^{(k)} = A_0^\star \overline{v}_j^{(k-1)},$$

$$\text{z}_j^{(k)} = A_0 u_j^{(p_j+1-k)}, \vartheta_j^{(k)} = A_0^\star v_j^{(p_j+1-k)}, k = \overline{1, p_j}, j = \overline{1, n}$$

with the biorthogonality conditions

$$\langle u_j^{(k)}, \vartheta_s^{(l)}\rangle = \delta_{js}\delta_{kl}, \langle \text{z}_j^{(k)}, v_s^{(l)}\rangle = \delta_{js}\delta_{kl}$$

and respectively

$$\mathcal{B}\varphi_j^{(k)} = A_0\varphi_j^{(k-1)}, \mathcal{B}\overline{\varphi}_j^{(k)} = -A_0\overline{\varphi}_j^{(k-1)},$$

$$\mathcal{B}^\star\psi_j^{(k)} = \left(B_0^\star + \alpha A_0^\star \frac{d}{d\tau}\right)\psi_j^{(k)} = -A_0^\star\psi_j^{(k-1)},$$

$$\mathcal{B}_0^\star\overline{\psi}_j^{(k)} = \left(B_0^\star + \alpha A_0^\star \frac{d}{d\tau}\right)\overline{\psi}_j^{(k)} = A_0^\star\overline{\psi}_j^{(k-1)},$$

where

$$\varphi_j^{(k)} = u_j^{(k)}e^{i\tau}, \overline{\varphi}_j^{(k)} = \overline{u}_j^{(k)}e^{-i\tau}, \psi_j^{(k)} = v_j^{(k)}e^{i\tau}, \overline{\psi}_j^{(k)} = \overline{v}_j^{(k)}e^{-i\tau}$$

$$z_j^{(k)} = \text{z}_j^{(k)}e^{i\tau}, \gamma_s^{(l)} = \vartheta_s^{(l)}e^{i\tau}, k(l) = 1, p_j(p_s), j, s = \overline{1, n}$$

with the biorthogonality conditions

$$\ll \varphi_j^{(k)}, \gamma_s^{(l)} \gg = \delta_{js}\delta_{kl}, \ll z_j^{(k)}, \psi_s^{(l)} \gg = \delta_{js}\delta_{kl}, k(l) = \overline{1, p_j(p_s)}, j, s = \overline{1, n} \tag{4}$$

$K = p_1 + p_2 + ... + p_n$ is the root-number.

Introduce the following notations, available further for the writing of the projectors: $\Phi = (\varphi_1^{(1)}, ..., \varphi_1^{(p_1)}, \varphi_n^{(1)}, ..., \varphi_n^{(p_n)})$. The vectors γ, Ψ and Z are defined analogously.

Lemma 2.1. [10, 11] *Biorthogonality conditions (4) allow to introduce the projectors*

$$\mathbf{P} = \sum_{j=1}^{n}\sum_{k=1}^{p_i} \ll \cdot, \gamma_j^{(k)} \gg \varphi_j^{(k)} = \ll \cdot, \gamma \gg \Phi, \quad \overline{\mathbf{P}} = \ll \cdot, \overline{\gamma} \gg \Phi, \quad \mathbb{P} = \mathbf{P} + \overline{\mathbf{P}},$$

$$\mathbf{Q} = \sum_{j=1}^{n}\sum_{k=1}^{p_i} \ll \cdot, \psi_j^{(k)} \gg z_j^{(k)} = \ll \cdot, \psi \gg z, \overline{\mathbf{Q}} = \ll \cdot, \overline{\psi} \gg \overline{z}, \quad \mathbb{Q} = \mathbf{Q} + \overline{\mathbf{Q}},$$

generating expansions of the spaces Y and Z in direct sums $Y = Y^{2K}\dot{+}Y^{\infty-2K}$, $Z = Z_{2K}\dot{+}Z_{\infty-2K}$, $Y^{2K} = span\{\varphi_j^{(k)}, \overline{\varphi_j^{(k)}}\}_{j=\overline{1,n},k=\overline{1,p_j}}$ is the root-subspace of A-adjoint elements of the operator $\mathcal{B}$,$Z_{2K} = span\{z_j^k, \overline{z_j^k}\}_{j=\overline{1,n},k=\overline{1,p_j}}$. The operators $\mathcal{B}$ and A_0 are intertwined by the projectors $\mathbf{P}$ and $\mathbf{Q}$, $\overline{\mathbf{P}}$ and $\overline{\mathbf{Q}}$, $\mathcal{B}\mathbf{P}u = \mathbf{Q}\mathcal{B}u$ on $\mathbf{D}_{\mathcal{B}}$, $\mathcal{B}\Phi = \mathfrak{A}_0 Z$, $\mathcal{B}^\star\Psi = \mathfrak{A}\gamma$, $\mathfrak{A} = diag\{B_1,...,B_n\}$ is cell-diagonal matrix, B_i is $p_i\times p_i$-matrix with units along secondary subdiagonal and zeros on other places; $A_0\mathbf{P} = \mathbf{Q}A_0$, $\mathcal{C}_0\mathbf{P} = \mathbf{Q}\mathcal{C}_0$ on D_{A_0}, $\mathcal{C}_0 = \mathcal{C}(0)$, $A_0\Phi = \mathfrak{A}_1 Z$, $A_0^\star\Psi = \mathfrak{A}_1\gamma$, $\mathfrak{A}_1 = diag\{B^1,...,B^n\}$ is cell-diagonal matrix, B^i is $p_i\times p_i$-matrix with units along secondary diagonal and zeros on other places. Operators A_0 and $\mathcal{B}$ act in invariant pairs of subspaces Y^{2K}, Z_{2K} and $Y^{\infty-2K}$, $Z_{\infty-2K}$ and $\mathcal{B}: Y^{\infty-2K}\bigcap D_{\mathcal{B}} \to Z_{\infty-2K}$, $A_0 : Y^{2K} \to Z_{2K}$ are isomorphisms.

Remark 2.1. [9, 10, 4]. *Because of the invariance property of the root-number K under perturbation we can work with A_0-adjoint elements of the operator $\mathcal{B}$, the more so parameter μ enters linearly in the equation (2).*

Consider now the Lyapunov-Schmidt BEqR construction [4,11]. The usage of the E.Schmidt regularizator [4]

$$\widetilde{\mathcal{B}} = \mathcal{B} + \sum_{s=1}^{n}[\ll \cdot, \gamma_i^{(1)} \gg z_i^{(1)} + \ll \cdot, \overline{\gamma}_i^{(1)} \gg \overline{z}_i^{(1)}], \quad \widetilde{\mathcal{B}}^{-1} = \Gamma$$

allows to rewrite the equation (2) in the form of the system

$$\begin{aligned}&\widetilde{\mathcal{B}}y = \mu\mathcal{C}(\varepsilon)y + \mathcal{R}(y,\varepsilon) + \sum_{i=1}^{n}(\xi_{i1}z_i^{(1)} + \overline{\xi}_{i1}\overline{z}_i^{(1)}),\\ &\xi_{s\sigma} = \ll y, \gamma_s^{(\sigma)} \gg, \quad \overline{\xi}_{s\sigma} = \ll y, \overline{\gamma}_s^{(\sigma)} \gg, \quad \sigma = \overline{1,p_s}, s = \overline{1,n}\end{aligned} \tag{5}$$

the unique solution of the first equation of which is sought in the form

$$y = u + \xi\cdot\Phi + \overline{\xi}\cdot\overline{\Phi} = u + v(\xi,\overline{\xi},\mu,\varepsilon), \xi = \xi(\mu(\varepsilon),\varepsilon), \overline{\xi} = \overline{\xi}(\mu(\varepsilon),\varepsilon). \tag{6}$$

Then the first equation of the system (5) gives

$$u = -(I-\mu\Gamma\mathcal{C}_0)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{p_i}(\xi_{ij}\varphi_i^{(j)} + \overline{\xi}_{ij}\overline{\varphi}_i^{(j)}) + \mu(I-\mu\Gamma\mathcal{C}_0)^{-1}\Gamma\mathcal{C}_0(\xi\cdot\Phi + \overline{\xi}\cdot\overline{\Phi}) +$$

$$+\mu(I-\mu\Gamma\mathcal{C}_0)^{-1}\Gamma(\mathcal{C}(\varepsilon)-\mathcal{C}_0)u + \mu(I-\mu\Gamma\mathcal{C}_0)^{-1}\Gamma(\mathcal{C}(\varepsilon)-\mathcal{C}_0)(\xi\cdot\Phi + \overline{\xi}\cdot\overline{\Phi}) +$$

$$+\mu(I-\mu\Gamma\mathcal{C}_0)^{-1}\Gamma\mathcal{R}(u+v,\varepsilon) =$$

$$-\sum_{i=1}^{n}\sum_{j=2}^{p_i}(\xi_{ij}\varphi_i^{(j)} + \overline{\xi_{ij}}\overline{\varphi_i^{(j)}}) + \mu\Gamma\mathcal{C}_0(I-\mu\Gamma\mathcal{C}_0)^{-1}\sum_{i=1}^{n}(\xi_{i1}\varphi_i^{(1)} + \overline{\xi}_{i1}\overline{\varphi}_i^{(1)}) +$$

$$+\mu\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}(\mathcal{C}(\varepsilon)-\mathcal{C}_0)u + \mu\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}(\mathcal{C}(\varepsilon)-\mathcal{C}_0)(\xi\cdot\Phi + \overline{\xi}\cdot\overline{\Phi}) +$$

$$+\mu\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathcal{R}(u+v,\varepsilon).$$

Taking into account the relations $\mu\Gamma\mathcal{C}_0\varphi_i^{(1)} = i\mu\varphi_i^{(2)}$, $\mu^2(\Gamma\mathcal{C}_0)^2\varphi_i^{(1)} = (i\mu)^2\varphi_i^{(3)}$,..., $\mu^{p_i-1}(\Gamma\mathcal{C}_0)^{p_i-1}\varphi_i^{(1)} = (i\mu)^{p_i-1}\varphi_i^{(p_i)}$, $\mu^{p_i}(\Gamma\mathcal{C}_0)^{p_i}\varphi_i^{(1)} = (i\mu)^{p_i}\varphi_i^{(1)}$, and generally, $\mu^l(\Gamma\mathcal{C}_0)^l\varphi_i^{(1)} = (i\mu)^l\varphi_i^{(l+1-[\frac{l+1}{p_i}]p_i)}$ according to the formulae $\Gamma^\star\gamma_s^{(1)} = \psi_s^{(1)}$, $\Gamma^\star\gamma_s^{(\sigma)} = \psi_s^{(p_s+2-\sigma)}$ [10,11] from the second equalities of the system (5) E. Schmidt BEqR follows

$$\begin{aligned}
&t_{s1}(\xi,\overline{\xi},\mu,\varepsilon) \equiv -\ll u,\gamma_s^{(1)} \gg = -\tfrac{(i\mu)^{p_s}}{1-(i\mu)^{p_s}}\xi_{s1} - \mu \ll (I-\mu\mathcal{C}_0)\Gamma)^{-1}(\mathcal{C}(\varepsilon)-\\
&-\mathcal{C}_0)(u+v),\psi_s^{(1)} \gg - \ll (I-\mu\mathcal{C}_0\Gamma)^{-1}\mathcal{R}(u+v,\varepsilon),\psi_s^{(p_s+2-\sigma)} \gg = 0,\\
&t_{s\sigma}(\xi,\overline{\xi},\mu,\varepsilon) \equiv -\ll u,\gamma_s^{(\sigma)} \gg = \xi_{s\sigma} - \tfrac{(i\mu)^{\sigma-1}}{1-(i\mu)^{p_s}}\xi_{s1} - \mu \ll (I-\mu\mathcal{C}_0\Gamma)^{-1}(\mathcal{C}(\varepsilon)-\\
&-\mathcal{C}_0)(u+v),\psi_s^{(p_s+2-\sigma)} \gg - \ll (I-\mu\mathcal{C}_0\Gamma)^{-1}\mathcal{R}(u+v,\varepsilon),\psi_s^{(p_s+2-\sigma)} \gg = 0,\\
&\sigma = \overline{2,p_s},\ s = \overline{1,n}.
\end{aligned} \tag{7}$$

For the BEq construction, write the equation (2) in the form of the system

$$\begin{aligned}
&\widetilde{\mathcal{B}}y = \mu\mathcal{C}(\varepsilon)y + \mathcal{R}(y,\varepsilon) + \sum_{i=1}^{n}(\xi_{i1}z_i^{(1)} + \overline{\xi}_{i1}\overline{z}_i^{(1)}),\\
&\xi_s = \ll y,\gamma_s^{(1)} \gg, \quad \overline{\xi}_s = \ll y,\overline{\gamma}_s^{(1)} \gg .
\end{aligned} \tag{8}$$

The unique solution of the first equation (8)

$$y = \Gamma(\mu\mathcal{C}(\varepsilon)y) + \Gamma\mathcal{R}(y,\varepsilon) + \sum_{j=1}^{n}(\xi_j\varphi_j + \overline{\xi}_j\overline{\varphi}_j) \tag{9}$$

find in the form

$$y = \sum_{j=1}^{n}(\xi_j\varphi_j + \overline{\xi}_j\overline{\varphi}_j) + u(\xi,\overline{\xi},\mu,\varepsilon) \tag{10}$$

Then the second equalities (8) by using the relations $\Gamma z_j^{(1)} = \varphi_j^{(1)}$, $\Gamma\overline{z}_j^{(1)} = \overline{\varphi}_j^{(1)}$, $\Gamma^\star\gamma_j^{(1)} = \psi_j^{(1)}$, $\Gamma^\star\overline{\gamma}_j^{(1)} = \overline{\psi}_j^{(1)}$, $Y = Y^{2n} \dotplus Y^{\infty-2n}$, $Z = Z_{2n} \dotplus Z_{\infty-2n}$, $Y^{2n} = \mathcal{N}(\mathcal{B})$, $Z_{2n} = span\{z_s,\overline{z}_s\}_{s=1}^n$, $Y^{2n} = (P_n+\overline{P}_n)Y = \mathbf{P}_nY$, $P_n = \sum_{j=1}^{n} \ll \cdot,\gamma_j \gg \overline{\varphi}_j$, $\overline{P}_n = \sum_{j=1}^{n} \ll \cdot,\overline{\gamma}_j^{(1)} \gg \overline{\varphi}_j^{(1)}$, $Z_{2n} = (Q_n+\overline{Q}_n) = \mathbf{Q}_nZ$, $Q_n = \sum_{j=1}^{n} \ll \cdot,\psi_j^{(1)} \gg z_j^{(1)}$, $\overline{Q}_n = \sum_{j=1}^{n} \ll \cdot,\overline{\psi}_j^{(1)} \gg \overline{z}_j^{(1)}$ give E. Schmidt BEq

$$\begin{aligned}
&t_s(\xi,\overline{\xi},\mu,\varepsilon) \equiv -\ll u,\gamma_s^{(1)} \gg = -\tfrac{(i\mu)^{p_s}}{1-(i\mu)^{p_s}}\xi_s - \mu \ll (I-\mu\mathcal{C}_0\Gamma)^{-1}(\mathcal{C}(\varepsilon)-\\
&-\mathcal{C}_0)(u+v),\psi_s^{(1)} \gg - \ll (I-\mu\mathcal{C}_0\Gamma)^{-1}\mathcal{R}(u+v,\varepsilon),\psi_s^{(1)} \gg = 0,\\
&t_s(\xi,\overline{\xi},\mu,\varepsilon) \equiv -\ll u,\overline{\gamma}_s^{(1)} \gg = 0,\ s = \overline{1,n}.
\end{aligned} \tag{11}$$

3. CONDITIONS OF BEQ AND BEQR POTENTIALITY TYPES $A(B)$

Definition 3.1. [7] *BEq* $t(\xi,\bar{\xi},\mu,\varepsilon)=0$, $\xi=(\xi_1,\xi_2,...,\xi_n)$ *(BEqR* $\mathbf{t}(\xi,\bar{\xi},\mu,\varepsilon)=0$, $\xi=(\xi_{11},\xi_{12},\xi_{1p_1},...,\xi_{n1},\xi_{n2},...,\xi_{np_n},)$*) for dynamic branching problem of branching theory is called BEq of potential type A or BEq of potential type B if* $t(\xi,\bar{\xi},\mu,\varepsilon)=d\cdot grad_{\xi,\bar{\xi}}U(\xi,\bar{\xi},\mu,\varepsilon)\sim(t_1,\bar{t}_1,\dots,t_n,\bar{t}_n)^T=$
$=d\cdot\left(\frac{\partial U}{\partial\bar{\xi}_1},\frac{\partial U}{\partial\xi_1},\dots,\frac{\partial U}{\partial\bar{\xi}_n},\frac{\partial U}{\partial\xi_n}\right)$ *or* $t(\xi,\bar{\xi},\mu,\varepsilon)=grad_{\xi,\bar{\xi}}U(\xi,\bar{\xi},\mu,\varepsilon)\cdot d\sim$
$\sim(t_1,\bar{t}_1,\dots,t_n,\bar{t}_n)=\left(\frac{\partial U}{\partial\bar{\xi}_1},\frac{\partial U}{\partial\xi_1},\dots,\frac{\partial U}{\partial\bar{\xi}_n},\frac{\partial U}{\partial\xi_n}\right)\cdot d$ *with an invertible matrix d (respectively* $\mathbf{t}(\xi,\bar{\xi},\mu,\varepsilon)=d\cdot grad_{\xi,\bar{\xi}}U(\xi,\bar{\xi},\mu,\varepsilon)\sim$
$\sim(t_{11},\bar{t}_{11},\dots,t_{1p_1},\bar{t}_{1p_1},\dots,t_{n1},\bar{t}_{n1},\dots,t_{np_n},\bar{t}_{np_n})^T=$
$=d\cdot\left(\frac{\partial U}{\partial\bar{\xi}_{11}},\frac{\partial U}{\partial\xi_{11}},\dots,\frac{\partial U}{\partial\bar{\xi}_{1p_1}},\frac{\partial U}{\partial\xi_{1p_1}},\dots,\frac{\partial U}{\partial\bar{\xi}_{n1}},\frac{\partial U}{\partial\xi_{n1}},\dots,\frac{\partial U}{\partial\bar{\xi}_{np_n}},\frac{\partial U}{\partial\xi_{np_n}}\right)$ *or* $\mathbf{t}(\xi,\bar{\xi},\mu,\varepsilon)=grad_{\xi,\bar{\xi}}U(\xi,\bar{\xi},\mu,\varepsilon)\cdot d\sim$
$\sim(t_{11},\bar{t}_{11},\dots,t_{1p_1},\bar{t}_{1p_1},\dots,t_{n1},\bar{t}_{n1},\dots,t_{np_n},\bar{t}_{np_n})=$
$=\left(\frac{\partial U}{\partial\bar{\xi}_{11}},\frac{\partial U}{\partial\xi_{11}},\dots,\frac{\partial U}{\partial\bar{\xi}_{1p_1}},\frac{\partial U}{\partial\xi_{1p_1}},\dots,\frac{\partial U}{\partial\bar{\xi}_{n1}},\frac{\partial U}{\partial\xi_{n1}},\dots,\frac{\partial U}{\partial\bar{\xi}_{np_n}},\frac{\partial U}{\partial\xi_{np_n}}\right)\cdot d$*).*

Remark 3.1. *Note here that potentiality conditions for BEq and BEqR of potentiality type* $A(B)$ *in stationary branching are obtained and proved in [7] and respectively in our communication to Int.Conf. [12].*

In the development of the article [1] results here similarly to n.2 and n.4 of [1] sufficient potentiality conditions are established for BEq (11) and BEqR (7) would be of potential type $A(B)$. Since the notions of the operators symmetrizability [2, 3] here are introduced for the equation (2) in the spaces Y, Z, elements of which are complex-valued functions in the definition 3.1 and analogs of n.2 , 4 assertions of [1] in the notion of matrices symmetricity the complex conjugation must be used as this is accepted in [1]. As there it is used for the proofs of operators $\mathcal{B}$, $\mathcal{C}(\varepsilon)$ and $\mathcal{R}_y$ symmetrizability. The finite-dimensional symmetrizators

$$J_n=\sum_{j=1}^{n}(\ll\cdot,\psi_j\gg\gamma_j+\ll\cdot,\overline{\psi_j}\gg\overline{\gamma_j})$$

for BEq and

$$J_n=\sum_{j=1}^{n}\sum_{k=1}^{p_j}[\ll\cdot,\psi_j^{(k)}\gg\gamma_j^{(k)}+\ll\cdot,\overline{\psi}_j^{(k)}\gg\overline{\gamma}_j^{(k)}]$$

for BEqR are used.

A.

Lemma 3.1. . *For the BEq (11) would be of potential type $A(B)$ it is sufficient the symmetricity of the matrices* $[\frac{\partial(d^{-1}\cdot t)}{\partial\xi\partial\bar{\xi}}]$, $[\frac{\partial(t\cdot d^{-1})}{\partial\xi\partial\bar{\xi}}]$ *,i.e.*

$$\begin{aligned}
&\sum_{s=1}^{n}\left(d_{2p-1,2s-1}\frac{\partial t_s}{\partial\xi_q}+d_{2p-1,2s}\frac{\partial\bar{t}_s}{\partial\xi_q}\right)=\overline{\sum_{s=1}^{n}\left(d_{2q-1,2s-1}\frac{\partial t_s}{\partial\xi_p}+d_{2q-1,2s}\frac{\partial\bar{t}_s}{\partial\xi_p}\right)},\\
&\sum_{s=1}^{n}\left(d_{2p,2s-1}\frac{\partial t_s}{\partial\bar{\xi}_q}+d_{2p,2s}\frac{\partial\bar{t}_s}{\partial\bar{\xi}_q}\right)=\overline{\sum_{s=1}^{n}\left(d_{2q,2s-1}\frac{\partial t_s}{\partial\bar{\xi}_p}+d_{2q,2s}\frac{\partial\bar{t}_s}{\partial\bar{\xi}_p}\right)},\\
&\sum_{s=1}^{n}\left(d_{2p-1,2s-1}\frac{\partial t_s}{\partial\bar{\xi}_q}+d_{2p-1,2s}\frac{\partial\bar{t}_s}{\partial\bar{\xi}_q}\right)=\overline{\sum_{s=1}^{n}\left(d_{2q,2s-1}\frac{\partial t_s}{\partial\xi_p}+d_{2q,2s}\frac{\partial\bar{t}_s}{\partial\xi_p}\right)},
\end{aligned}\tag{12}$$

for the type A and

$$\begin{aligned}
&\sum_{s=1}^{n}\left(\frac{\partial t_s}{\partial\xi_q}d_{2s-1,2p-1}+\frac{\partial\bar{t}_s}{\partial\xi_q}d_{2s,2p-1}\right)=\overline{\sum_{s=1}^{n}\left(\frac{\partial t_s}{\partial\xi_p}d_{2s-1,2q-1}+\frac{\partial\bar{t}_s}{\partial\xi_p}d_{2s,2q-1}\right)},\\
&\sum_{s=1}^{n}\left(\frac{\partial t_s}{\partial\bar{\xi}_q}d_{2s-1,2p}+\frac{\partial\bar{t}_s}{\partial\bar{\xi}_q}d_{2s,2p}\right)=\overline{\sum_{s=1}^{n}\left(\frac{\partial t_s}{\partial\bar{\xi}_p}d_{2s-1,2q}+\frac{\partial\bar{t}_s}{\partial\bar{\xi}_p}d_{2s,2q}\right)},\\
&\sum_{s=1}^{n}\left(\frac{\partial t_s}{\partial\bar{\xi}_q}d_{2s-1,2p-1}+\frac{\partial\bar{t}_s}{\partial\bar{\xi}_q}d_{2s,2p-1}\right)=\overline{\sum_{s=1}^{n}\left(\frac{\partial t_s}{\partial\xi_p}d_{2s-1,2q}+\frac{\partial\bar{t}_s}{\partial\xi_p}d_{2s,2q}\right)}
\end{aligned}\tag{13}$$

for the type B.

The proof follows from the definition 3.1 at the usage of designation

$$d^{-1}=\begin{pmatrix} d_{2k-1,2s-1} & d_{2k-1,2s}\\ d_{2k,2s-1} & d_{2k,2s}\end{pmatrix}_{k,s=\overline{1,n}}.$$

The finding of solutions to (9) in the form (10) with the subsequent differentiation leads to relations

$$\frac{\partial y}{\partial\xi_s}=[\mu\Gamma\mathcal{C}(\varepsilon)+\Gamma\mathcal{R}_y]\frac{\partial y}{\partial\xi_s}+\varphi_s\Rightarrow\frac{\partial y}{\partial\xi_s}=[I-\Gamma(\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)]^{-1}\varphi_s,$$

$$\frac{\partial y}{\partial\xi_s}=\varphi_s+\frac{\partial u}{\partial\xi_s}\Rightarrow\varphi_s+\frac{\partial u}{\partial\xi_s}=\varphi_s+\Gamma(\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)[I-\Gamma(\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)]^{-1}\varphi_s\Rightarrow$$

$$\frac{\partial t_s}{\partial\xi_q}=-\ll\frac{\partial u}{\partial\xi_q},\gamma_s^{(1)}\gg=-\ll\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)[I-\Gamma(\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)]^{-1}\varphi_q,\psi_s\gg$$

and analogously $\frac{\partial\overline{t_s}}{\partial\xi_q}=-\ll\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)[I-\Gamma(\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)]^{-1}\varphi_q,\overline{\psi_s}\gg$,

$$\frac{\partial t_s}{\partial\overline{\xi_q}}=-\ll\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)[I-\Gamma(\mu\mathcal{C}(\varepsilon)+\mathcal{R}_y)]^{-1}\overline{\varphi_q},\psi_s\gg,$$

$$\frac{\partial \overline{t_s}}{\partial \overline{\xi_q}} = - \ll \mu \mathcal{C}(\varepsilon) + \mathcal{R}_y)[I - \Gamma(\mu \mathcal{C}(\varepsilon) + \mathcal{R}_y)]^{-1}\overline{\varphi_q}, \overline{\psi_s} \gg .$$

Corolar 3.1. *When $d = I$ the usual potentiality conditions for BEq from [1] follow:* $\frac{\partial t_k}{\partial \xi_s} = \overline{\frac{\partial \overline{t_s}}{\partial \overline{\xi_k}}}$, $\frac{\partial \overline{t_k}}{\partial \xi_s} = \overline{\frac{\partial \overline{t_s}}{\partial \xi_k}}$ *and* $\frac{\partial t_k}{\partial \overline{\xi_s}} = \overline{\frac{\partial t_s}{\partial \overline{\xi_k}}}$, $k, s = \overline{1, n}$.

Lemma 3.2. *Let the operator $\mathcal{B} : Y \supset D_{\mathcal{B}} \to Z$ be J–symmetrizable on $D = D_{\mathcal{B}}$ and the operator $J : Z \to Y^\star$ satisfies the requirements:*

1°. $\forall y \in Y^{\infty-2n} \Rightarrow J^\star y \in Z^\star_{\infty-2n} = \{f \in Z^\star| \ll z_s, f \gg = 0, \ll \overline{z}_s, f \gg = 0,$ $s = \overline{1, n}\}$*;*

2°.*The matrix* $\ll (\varphi, \overline{\varphi}), J(z, \overline{z}) \gg$ *is symmetric, i.e.* $\ll \varphi_s, Jz_k \gg = \overline{\ll \overline{\varphi}_k, J\overline{z}_s \gg}$, $\ll \overline{\varphi_s}, J\overline{z_k} \gg = \overline{\ll \overline{\varphi_k}, J\overline{z_s} \gg}$, $\ll \overline{\varphi_s}, Jz_k \gg = \overline{\ll \varphi_k, J\overline{z_s} \gg}$.

Then the operator $\Gamma = \widetilde{\mathcal{B}}^{-1}$ is J^* -symmetrizable on Z.

Now the following analog of the Theorem 4.1 [1] is true.

Theorem 3.1. *Let there exists a linear operator $J\colon Z \to Y^\star$, such that* $J^\star \varphi_p = \sum_{s=1}^{n} (\overline{d}_{2p-1,2s-1}\psi_s + \overline{d}_{2p-1,2s}\overline{\psi_s})$, $J^\star \overline{\varphi_p} = \sum_{s=1}^{n} (\overline{d}_{2p,2s-1}\psi_s + \overline{d}_{2p,2s}\overline{\psi_s})$ *(resp.* $J^\star \varphi_p = \sum_{s=1}^{n} (\overline{d}_{2s-1,2p-1}\psi_s + \overline{d}_{2s,2p}\overline{\psi_s})$, $J^\star \overline{\varphi_p} = \sum_{s=1}^{n} (\overline{d}_{2s-1,2p}\psi_s + \overline{d}_{2s,2p}\overline{\psi_s})$ *and the following requirements are realized:*

1°. *Operator $\mathcal{B}$ is J-symmetrizable on D;*

2°. *Operator $\mathcal{C}(\varepsilon)$ and operators $B(\varepsilon) - B_0)$, $\mathcal{R}(y, \varepsilon)$ for any (y, ε) in some neighborhood of the point $(0, 0)$ are J-symmetrizable on D;*

3°. *For any $y \in Y^{\infty-2n} \bigcap D$ follows that $J^\star y \in Z^\star_{\infty-2n}$.*

Then the BEq (10) is the system of potential type A (resp. B).

The proof follows from the analogs of assertions n.3 [1] and lemmas 3.1, 3.2.

Corollary 3.1. *When $d = I$, Theorem 3.1 coincides with Theorem 4.1 of [1].*

Remark 3.2. *In applications the matrix d often turns out to be diagonal.*

B. For the simplicity of presentation further potential BEqRs are considered.

Lemma 3.3. *For the BEqR (7) potentiality it is sufficient the symmetricity of the matrix*

$$\mathbf{D} = \mathbf{D}(\frac{\mathbf{t}, \bar{\mathbf{t}}}{\xi, \bar{\xi}}) = \frac{D(t_{11}, \overline{t_{11}}, \ldots, t_{1p_1}, \overline{t_{1p_1}} \ldots, t_{n1}, \overline{t_{n1}} \ldots, t_{np_n}, \overline{t_{np_n}})}{D(\xi_{11}, \overline{\xi_{11}}, \ldots, \xi_{1p_1}, \overline{\xi_{1p_1}}, \ldots, \xi_{n1}, \overline{\xi_{n1}}, \ldots, \xi_{np_n}, \overline{\xi_{np_n}})},$$

i.e. the equalities relations

$$\frac{\partial t_{kl}}{\partial \xi_{s\sigma}} = \overline{\frac{\partial t_{s\sigma}}{\partial \xi_{kl}}}, \quad \frac{\partial \bar{t}_{kl}}{\partial \bar{\xi}_{s\sigma}} = \overline{\frac{\partial \bar{t}_{s\sigma}}{\partial \bar{\xi}_{kl}}}, \quad \frac{\partial t_{kl}}{\partial \bar{\xi}_{s\sigma}} = \overline{\frac{\partial \bar{t}_{s\sigma}}{\partial \xi_{kl}}}. \tag{14}$$

Remark 3.3. *From (14) the reality of diagonal elements* **D** *follows (when* $k = s$, $\sigma = l$*).*

For the shortening of computation in (6) the following designation will be used
$u = -\sum_{i=1}^{n}\sum_{j=2}^{p_i}(\xi_{ij}\varphi_i^{(j)} + \overline{\xi_{ij}\varphi_i^{(j)}}) + \mu\Gamma\mathcal{C}_0(I-\mu\Gamma\mathcal{C}_0)^{-1}\sum_{i=1}^{n}(\xi_{i1}\varphi_i^{(1)} + \overline{\xi_{i1}}\overline{\varphi_i^{(1)}}))+$
$+\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}(u+v,\mu,\varepsilon)$, i.e.

$$\mathbf{R}(y,\mu,\varepsilon) = \mu(\mathcal{C}(\varepsilon)-\mathcal{C}_0)y + \mathcal{R}(y,\varepsilon), \quad v = \xi\cdot\Phi + \overline{\xi}\cdot\overline{\Phi}.$$

For the verification (14) the computation of the derivatives $\frac{\partial u}{\partial \xi_{sk}}$, $\frac{\partial u}{\partial \overline{\xi}_{sk}}$ is required:

$$\frac{\partial u}{\partial \xi_{s1}} = \mu\Gamma\mathcal{C}_0(I-\mu\Gamma\mathcal{C}_0)^{-1}\varphi_s^{(1)} + \Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y(\frac{\partial u}{\partial \xi_{s1}} + \varphi_s^{(1)}) \Rightarrow$$

$$\Rightarrow \frac{\partial u}{\partial \xi_{s1}} = [I-\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y]^{-1}\{\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y\varphi_s^{(1)} + \Gamma(I-\mu\mathcal{C}_0)\Gamma)^{-1}\mu\mathcal{C}_0\varphi_s^{(1)}\} =$$

$$= \Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}[I-\Gamma(I-\mu\mathcal{C}_0)\Gamma)^{-1}\mathbf{R}_y]^{-1}(\mathbf{R}_y + \mu\mathcal{C}_0)\varphi_s^{(1)},$$

$$\frac{\partial u}{\partial \xi_{sk}} = -\varphi_s^{(k)} + \Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y(\frac{\partial u}{\partial \xi_{sk}} + \varphi_s^{(k)}),$$

whence it follows $\frac{\partial u}{\partial \xi_{sk}} = -\varphi_s^{(k)}$ when $k > 1$. Now (7) and (14) give the relations of the following type

$$\begin{aligned} &\ll (I-\mu\mathcal{C}_0\Gamma)^{-1}[I-\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y]^{-1}(\mathbf{R}_y+\mu\mathcal{C}_0)\varphi_s^{(1)},\psi_k^{(1)} \gg= \\ &=\ll (I-\mu\mathcal{C}_0\Gamma)^{-1}[I-\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y]^{-1}(\mathbf{R}_y+\mu\mathcal{C}_0)\varphi_k^{(1)},\psi_s^{(1)} \gg, \end{aligned} \tag{15}$$

$$\ll (I-\mu\mathcal{C}_0\Gamma)^{-1}[I-\Gamma(I-\mu\mathcal{C}_0\Gamma)^{-1}\mathbf{R}_y]^{-1}(\mathbf{R}_y+\mu\mathcal{C}_0)\varphi_k^{(1)},\psi_s^{(p_s+2-\sigma)} \gg= 0, \tag{16}$$

for $\sigma \geq 2$, since $\frac{\partial t_{kl}}{\partial \xi_{s\sigma}} = -\ll \varphi_s^{(\sigma)},\gamma_k^{(l)} \gg= -\delta_{sk}\delta_{\sigma l}$.

Corolar 3.2. *Formulae (15) and (16) mean that BEqR potentiality is equivalent to BEq potentiality.*

4. SYMMETRY IN P-A-H BIFURCATION PROBLEM WITH POTENTIAL BRANCHING EQUATIONS

SH(2) – symmetry. As the prolongation of the article [1] results, return now to model example of P-A-H bifurcation with $SH(2)$-symmetry generated by the following zero-subspace of the linearized operator with relevant pure imaginary eigenvalues

$$N = N(\mathcal{B}) = span\{\varphi_1 = (chx + ishx)e^{it}, \overline{\varphi_1}, \varphi_2 = (chx - ishx)e^{it}, \overline{\varphi_2}\},$$

with the following matrix representation in the definition of group invariance $\mathcal{B}_g t(\xi,\bar{\xi}) = t(\mathcal{A}_g\xi, \overline{\mathcal{A}_g\xi})$ correspondingly to the basis N

$$\mathcal{B}(\alpha_0,\alpha) = \mathcal{A}(\alpha_0,\alpha) = \\ = \begin{pmatrix} e^{i\alpha_0}ch\alpha & 0 & -ie^{i\alpha_0}sh\alpha & 0 \\ 0 & e^{-i\alpha_0}ch\alpha & 0 & ie^{-i\alpha_0}sh\alpha \\ ie^{i\alpha_0}sh\alpha & 0 & e^{i\alpha_0}ch\alpha & 0 \\ 0 & -ie^{-i\alpha_0}sh\alpha & 0 & e^{-i\alpha_0}ch\alpha \end{pmatrix}$$

Here the branching equation potentiality of the types A and B take place simultaneously. In the articles [13,14] at the usage of group analysis methods [15] the general form of C^1-BEq was constructed on allowed group symmetry (see also [1]).

$$\begin{aligned} t_1 &= \frac{1}{\bar{\xi}_1^2+\bar{\xi}_2^2}\left[\bar{\xi}_1 F_1(I_1(\xi), I_2(\xi)) + \bar{\xi}_2 F_2(I_1(\xi), I_2(\xi))\right] \\ I_1 &= \sqrt{|\xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2|},\ I_2 = \sqrt{|\xi_1\bar{\xi}_2 + \bar{\xi}_1\xi_2|}, \\ t_2 &= \frac{1}{\bar{\xi}_1^2+\bar{\xi}_2^2}\left[\bar{\xi}_1 F_2(I_1(\xi), I_2(\xi)) - \bar{\xi}_2 F_1(I_1(\xi), I_2(\xi))\right] \end{aligned} \tag{17}$$

where the functions F_1,F_2 are real-valued.

The definition 3.1 of potential type BEq (here simultaneously A and B) with $d = diag(1,1,-1,-1)$ means the symmetricity of the matrix $\mathbf{D}$, i.e. the realization of the equalities (12) or (13).

$$\begin{pmatrix} \frac{\partial t_1}{\partial \xi_1} & \frac{\partial t_1}{\partial \bar{\xi}_1} & \frac{\partial t_1}{\partial \xi_2} & \frac{\partial t_1}{\partial \bar{\xi}_2} \\ \frac{\partial \bar{t}_1}{\partial \xi_1} & \frac{\partial \bar{t}_1}{\partial \bar{\xi}_1} & \frac{\partial \bar{t}_1}{\partial \xi_2} & \frac{\partial \bar{t}_1}{\partial \bar{\xi}_2} \\ -\frac{\partial t_2}{\partial \xi_1} & -\frac{\partial t_2}{\partial \bar{\xi}_1} & -\frac{\partial t_2}{\partial \xi_2} & -\frac{\partial t_2}{\partial \bar{\xi}_2} \\ -\frac{\partial \bar{t}_2}{\partial \xi_1} & -\frac{\partial \bar{t}_2}{\partial \bar{\xi}_1} & -\frac{\partial \bar{t}_2}{\partial \xi_2} & -\frac{\partial \bar{t}_2}{\partial \bar{\xi}_2} \end{pmatrix} \Rightarrow \begin{matrix} \frac{\partial t_1}{\partial \xi_1} = \overline{\frac{\partial \bar{t}_1}{\partial \bar{\xi}_1}}, & \frac{\partial t_2}{\partial \xi_2} = \overline{\frac{\partial \bar{t}_2}{\partial \bar{\xi}_2}} \Rightarrow a) \\ \frac{\partial t_1}{\partial \bar{\xi}_1} = \frac{\partial \bar{t}_1}{\partial \xi_1}, & \frac{\partial t_2}{\partial \bar{\xi}_2} = \frac{\partial \bar{t}_2}{\partial \xi_2} \Rightarrow b) \\ \frac{\partial t_1}{\partial \xi_2} = -\overline{\frac{\partial t_2}{\partial \xi_1}}, & \frac{\partial \bar{t}_1}{\partial \bar{\xi}_2} = -\overline{\frac{\partial \bar{t}_2}{\partial \bar{\xi}_1}} \Rightarrow c) \\ \frac{\partial t_1}{\partial \bar{\xi}_2} = -\overline{\frac{\partial \bar{t}_2}{\partial \xi_1}}, & \frac{\partial \bar{t}_1}{\partial \xi_2} = -\overline{\frac{\partial t_2}{\partial \bar{\xi}_1}} \Rightarrow d) \end{matrix} \tag{18}$$

Here a) is the reality of diagonal elements, b) the partial potentiality, c) the symmetry along secondary subdiagonals, d) symmetry along secondary diagonal. Conditions (18) lead to the following relations, where the symbols $F_{k,1}$, $F_{k,2}$ mean the derivatives of F_k on the relevant invariants I_1, I_2:

$$\begin{aligned}\frac{\partial t_1}{\partial \xi_1} &= \frac{1}{\bar{\xi}_1^2+\bar{\xi}_2^2}\left[\bar{\xi}_1 F_{1,1}\frac{\bar{\xi}_1}{2I_1}+\bar{\xi}_1 F_{1,2}\frac{\bar{\xi}_2}{2I_2}+\bar{\xi}_2 F_{2,1}\frac{\bar{\xi}_1}{2I_1}+\bar{\xi}_2 F_{2,2}\frac{\bar{\xi}_2}{2I_2}\right],\\ \frac{\partial \bar{t}_1}{\partial \bar{\xi}_1} &= \frac{1}{\xi_1^2+\xi_2^2}\left[\xi_1 F_{1,1}\frac{\xi_1}{2I_1}+\bar{\xi}_1 F_{1,2}\frac{\xi_2}{2I_2}+\xi_2 F_{2,1}\frac{\xi_1}{2I_1}+\xi_2 F_{2,2}\frac{\xi_2}{2I_2}\right],\end{aligned} \tag{19}$$

the second equality of a) and also the equalities b) are verified analogously,

$$\begin{aligned}&\frac{\partial t_1}{\partial \xi_2} = -\overline{\frac{\partial t_2}{\partial \xi_1}} \Rightarrow (-F_{1,1})[\xi_1\xi_2(\bar{\xi}_1^2+\bar{\xi}_2^2)+\bar{\xi}_1\bar{\xi}_2(\xi_1^2+\xi_2^2)]I_2+F_{2,2}[\xi_1\xi_2(\bar{\xi}_1^2+\bar{\xi}_2^2)+\\ &+\bar{\xi}_1\bar{\xi}_2(\xi_1^2+\xi_2^2)]I_1+F_{1,2}[\bar{\xi}_1^2(\xi_1^2+\xi_2^2)-\xi_2^2(\bar{\xi}_1^2+\bar{\xi}_2^2)]I_1+\\ &+F_{2,1}[\xi_1^2(\bar{\xi}_1^2+\bar{\xi}_2^2)-\bar{\xi}_2^2(\xi_1^2+\xi_2^2)]I_2=0,\end{aligned} \tag{20}$$

$$\begin{aligned}&\frac{\partial \bar{t}_1}{\partial \bar{\xi}_2} = -\overline{\frac{\partial \bar{t}_2}{\partial \bar{\xi}_1}} \Rightarrow (-F_{1,1})[\xi_1\xi_2(\bar{\xi}_1^2+\bar{\xi}_2^2)+\bar{\xi}_1\bar{\xi}_2(\xi_1^2+\xi_2^2)]I_2+F_{2,2}[\xi_1\xi_2(\bar{\xi}_1^2+\bar{\xi}_2^2)+\\ &+\bar{\xi}_1\bar{\xi}_2(\xi_1^2+\xi_2^2)]I_1+F_{1,2}[\xi_1^2(\bar{\xi}_1^2+\bar{\xi}_2^2)-\bar{\xi}_2^2(\xi_1^2+\xi_2^2)]I_1+\\ &+F_{2,1}[\bar{\xi}_1^2(\xi_1^2+\xi_2^2)-\xi_2^2(\bar{\xi}_1^2+\bar{\xi}_2^2)]I_2=0,\end{aligned} \tag{21}$$

$$\frac{\partial t_1}{\partial \bar{\xi}_2} = -\overline{\frac{\partial \bar{t}_2}{\partial \xi_1}} \Rightarrow -F_{1,1}I_2^3+F_{2,2}I_2^2I_1+F_{2,1}I_2I_1^2+F_{1,2}I_1^3=0, \tag{22}$$

$$\frac{\partial \bar{t}_1}{\partial \bar{\xi}_2} = -\overline{\frac{\partial \bar{t}_2}{\partial \xi_1}},\ \frac{\partial \bar{t}_1}{\partial \xi_2} = -\overline{\frac{\partial t_2}{\partial \bar{\xi}_1}} \Rightarrow -F_{1,1}I_2^3+F_{2,2}I_2^2I_1+F_{2,1}I_2I_1^2+F_{1,2}I_1^3=0. \tag{23}$$

However the relations (20) and (21) are differed from (22) only by the cofactor $|\xi_1|^2 + |\xi_2|^2$. Consequently the following assertion is proved

Theorem 4.1. *C^1-BEq of potential type for P-A-H bifurcation with the symmetry $SH(2)$ on spatial variables has the form (17), where the functions F_1 and F_2 satisfy the differential equation (22).*

Corollary 4.1. *The potential is determined by the following formula [16]*

$$\sum_{k=1}^{n}[\int_0^1 t_k(\tau\xi_1,\tau\xi_2,\mu,\varepsilon)\bar{\xi}_k d\tau+\int_0^1 t_k(\tau\xi_1,\tau\xi_2,\mu,\varepsilon)\bar{\xi}_k d\tau]=U(\xi,\bar{\xi},\mu,\varepsilon).$$

In the case of analytic BEq as invariants $I_1 = \xi_1\overline{\xi_1} - \xi_2\overline{\xi_2}$ and $I_2 = \xi_1\overline{\xi_2} + \overline{\xi_1}\xi_2$ are chosen. As before the conditions a) and b) (18) are verified directly, while the conditions c) and d) (18) give

$$\begin{aligned}&[-F_{1,1}+F_{2,2}](\xi_1\bar{\xi}_2+\bar{\xi}_1\xi_2)+[F_{1,2}+F_{2,1}](\xi_1\bar{\xi}_1-\xi_2\bar{\xi}_2)=\\ &=[-F_{1,1}+F_{2,2}]I_2+[F_{1,2}+F_{2,1}]I_1=0.\end{aligned} \tag{24}$$

Theorem 4.2. *Analytic BEq of potential type for P-A-H bifurcation allowing the symmetry SH(2) has the form (17), where the functions $F_1(I_1(\xi), I_2(\xi))$ and $F_2(I_1(\xi), I_2(\xi))$ satisfy the differential equation (24) with invariants $I_1(\xi) = \xi_1\overline{\xi_1} - \xi_2\overline{\xi_2}$ and $I_2(\xi) = \xi_1\overline{\xi_2} + \overline{\xi_1}\xi_2$.*

Remark 4.1. *Accepted in [12,13] potentiality conditions of BEq mean the symmetry of the matrix $\dfrac{D(t_1, \bar{t}_1, -t_2, -\bar{t}_2)}{D(\bar{\xi}_1, \xi_1, \bar{\xi}_2, \xi_2)}$, where only its coincidence with transposed one was taken into account. Therefore in [1] we could not construct the finite dimensional symmetrizator (symmetryzing operator) and could not prove Theorem 4.1 for C^1-BEq. However, for analytic case (in [12] and [13] and correspondingly in [1]), it is said erroneously that C^1-BEq is considered) the results of [12,13] about the general form of analytic BEq and its potential turn out to be valid.*

Remark 4.2. *As in the previous our article, we assume in the future to investigate dynamic bifurcation problems with symmetries $SO(2)$ and $SH(2)$ at high order degeneration of the linearized operator.*

5. EXISTENCE OF BIFURCATION POINT

Similarly to the articles [1,3], by using the approach of Section 3, the existence theorem of P-A-H bifurcation can be proved.

Lemma 5.1. *Let be $A(\varepsilon) \equiv A_0$ and $\mu\mathcal{C}_0 y + (B(\varepsilon) - B_0)y + \mathcal{R}(y, \varepsilon) = \mu\mathcal{C}_0 y + \mathbf{R}(y, \varepsilon)$, $\mathbf{R}(0, \varepsilon) \equiv 0$. Then at the realization of the Th.3.1 conditions, potential $U(\xi, \tilde{\xi}, \mu, \varepsilon)$ of the potential type A(B) BEq is generated by the symmetric matrices $d^{-1} \cdot \ll \rho(0, \mu(\varepsilon), \varepsilon)(\varphi; \overline{\varphi}), (\psi; \overline{\psi}) \gg$ for the case A and $\ll \rho(0, \mu(\varepsilon), \varepsilon)(\varphi; \overline{\varphi}), (\psi; \overline{\psi}) \gg \cdot d^{-1}$ for the case B with relevant square form on $\xi, \overline{\xi}$ and residual term $\omega(\xi, \overline{\xi}, \mu(\varepsilon), \varepsilon)$, $\|\omega\| = o(\sqrt{|\xi|^2 + |\overline{\xi}|^2})$ as $\xi \to 0$. Components of the symmetric matrices are continuous functions in some neighborhood of the point $\mu = 0$, $\varepsilon = 0$, the function ω is continuous in the same neighborhood together with partial derivatives on $\xi, \overline{\xi}$ up to second order. The symmetricity of the matrix in the main part of potential is understanding in the sense of (12) for the case A ((13) for the case B), i.e. (here the symbol $[\ldots]^\star$ means the complex conjugation to the expression $[\ldots]$)*

$$\sum_{s=1}^{n}[d_{2p-1,2s-1} \ll \rho(0, \mu(\varepsilon), \varepsilon)\varphi_q^{(1)}, \psi_s^{(1)} \gg + d_{2p-1,2s} \ll \rho(0, \mu(\varepsilon), \varepsilon)\varphi_q^{(1)}, \overline{\psi}_s^{(1)} \gg] =$$

$$= \sum_{s=1}^{n}[d_{2q-1,2s-1} \ll \rho(0, \mu(\varepsilon), \varepsilon)\varphi_p^{(1)}, \psi_s^{(1)} \gg + d_{2q-1,2s} \ll \rho(0, \mu(\varepsilon), \varepsilon)\varphi_p^{(1)}, \overline{\psi}_s^{(1)} \gg]^\star,$$

$$\sum_{s=1}^{n}[d_{2p,2s-1} \ll \rho(0, \mu(\varepsilon), \varepsilon)\overline{\varphi}_q^{(1)}, \psi_s^{(1)} \gg + d_{2p,2s} \ll \rho(0, \mu(\varepsilon), \varepsilon)\overline{\varphi_q^{(1)}}, \overline{\psi_s^{(1)}} \gg] =$$

$$= \sum_{s=1}^{n}[d_{2q,2s-1} \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_p^{(1)}}, \psi_s^{(1)} \gg +d_{2q,2s} \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_p^{(1)}}, \overline{\psi_s^{(1)}} \gg]^\star,$$

$$\sum_{s=1}^{n}[d_{2p-1,2s-1} \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi}_q^{(1)}, \psi_s^{(1)} \gg +d_{2p-1,2s} \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_q^{(1)}}, \overline{\psi_s^{(1)}} \gg] =$$

$$= \sum_{s=1}^{n}[d_{2q,2s-1} \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \psi_s^{(1)} \gg +d_{2q,2s} \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \overline{\psi}_s^{(1)} \gg]^\star,$$

for the case A and

$$\sum_{s=1}^{n}[\ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_q^{(1)}, \psi_s^{(1)} \gg d_{2s-1,2p-1}+ \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_q^{(1)}, \overline{\psi}_s^{(1)} \gg d_{2s,2p-1}] =$$

$$= \sum_{s=1}^{n}[\ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \psi_s^{(1)} \gg d_{2s-1,2q-1}+ \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \overline{\psi}_s^{(1)} \gg d_{2s,2q-1}]^\star,$$

$$\sum_{s=1}^{n}[\ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_q^{(1)}}, \psi_s^{(1)} \gg d_{2s-1,2p}+ \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_q^{(1)}}, \overline{\psi_s^{(1)}} \gg d_{2s,2p}] =$$

$$= \sum_{s=1}^{n}[\ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_p^{(1)}}, \psi_s^{(1)} \gg d_{2s-1,2q}+ \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_p^{(1)}}, \overline{\psi}_s^{(1)} \gg d_{2s,2q}]^\star,$$

$$\sum_{s=1}^{n}[\ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_q^{(1)}}, \psi_s^{(1)} \gg d_{2s-1,2p-1}+ \ll \rho(0,\mu(\varepsilon),\varepsilon)\overline{\varphi_q^{(1)}}, \overline{\psi_s^{(1)}} \gg d_{2s,2p-1}] =$$

$$\sum_{s=1}^{n}[\ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \psi_s^{(1)} \gg d_{2s-1,2q}+ \ll \rho(0,\mu(\varepsilon),\varepsilon)\varphi_p^{(1)}, \overline{\psi_s^{(1)}} \gg d_{2s,2q}]^\star,$$

for the case B.

Here $\rho(0,\mu(\varepsilon),\varepsilon) = (\mu\mathcal{C}(\varepsilon) + \mathcal{R}_y)[I - \Gamma(\mu\mathcal{C}(\varepsilon) + \mathcal{R}_y)]^{-1}$, in accordance with [1] and sections 2,3. The proof follows from the definition 3.1 and Lemma 3.1.

Similarly to the article [3] introduce the condition suitable also for ε belonging to some normed space Λ: α) let in some neighborhood of $\varepsilon = 0$ there exists the set S, containing the point $\varepsilon = 0$, which is continuum presented in the form $S = \overline{S}_+ \bigcup \overline{S}_-$, $0 \in \partial S_+ \bigcap \partial S_-$. Let be

$$det[d^{-1} \ll \rho(0,\mu(\varepsilon),\varepsilon), (\varphi,\overline{\varphi}), (\psi,\overline{\psi} \gg)]_{\varepsilon\in S_+\bigcup S_-} \neq 0,$$

(resp. $det[\ll \rho(0,\mu(\varepsilon),\varepsilon), (\varphi,\overline{\varphi}), (\psi,\overline{\psi} \gg) \cdot d^{-1}]_{\varepsilon\in S_+\bigcup S_-} \neq 0$) and the matrix $[d^{-1}\cdot \ll \rho(0,\mu(\varepsilon),\varepsilon), (\varphi,\overline{\varphi}), (\psi,\overline{\psi} \gg)]$ in case A (resp.the matrix $[\ll \rho(0,\mu(\varepsilon),\varepsilon), (\varphi,\overline{\varphi}), (\psi,\overline{\psi} \gg) \cdot d^{-1}]$ for the case B) has at $\varepsilon \in S_-$ ($\varepsilon \in S_+$) precisely ν_1 negative eigenvalues (ν_2 negative eigenvalues).

Lemma 5.2. [3]. *Let the condition α) with $\nu_1 \neq \nu_2$ be realized. Then for any $\delta > 0$ there exists $\varepsilon^\star$ in a neighborhood $|\varepsilon| < \delta$ such that the function $U(\xi, \overline{\xi}, \mu(\varepsilon^\star), \varepsilon^\star)$ has in it a stationary point $\xi^\star \neq 0$.*

The proof follows from homotopic invariance of Conley-Morse index [17,Th.1.4,p.67].

Theorem 5.1. *Let the branching equation of the problem (1), under Lemma 5.1 conditions, be potential type A(or B) and the condition α) be fulfilled with $\nu_1 \neq \nu_2$. Then $\varepsilon = 0 \in S$ is the bifurcation point.*

Remark 5.1. *The results [2,3] can be found in the more available collective monograph [6].*

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