EXISTENCE OF POSITIVE SOLUTIONS FOR A HIGHER-ORDER MULTI-POINT BOUNDARY VALUE PROBLEM

Rodica Luca, Ciprian Deliu

“Gh. Asachi” Technical University, Department of Mathematics, Iași, Romania
rlucatudor@yahoo.com, cipriandeliu@gmail.com

Abstract We study the existence and nonexistence of positive solutions for a system of nonlinear higher-order ordinary differential equations with multi-point boundary conditions.

Keywords: Higher-order differential system, multi-point boundary conditions, positive solutions.

2010 MSC: 34B10, 34B18.

1. INTRODUCTION

We consider the system of nonlinear higher-order ordinary differential equations

\[
\begin{align*}
(S) & : \begin{cases} 
u^{(n)}(t) + \alpha(t)f(v(t)) = 0, & t \in (0, T) , \\
v^{(m)}(t) + \beta(t)g(u(t)) = 0, & t \in (0, T),
\end{cases} \\
(BC) & : \begin{cases} 
u(0) = \sum_{i=1}^{p-2} a_i u(\xi_i) + a_0 , & \nu'(0) = \cdots = \nu^{(n-2)}(0) = 0 , & \nu(T) = 0 , \\
u(0) = \sum_{i=1}^{q-2} b_i v(\eta_i) + b_0 , & \nu'(0) = \cdots = \nu^{(m-2)}(0) = 0 , & \nu(T) = 0 ,
\end{cases}
\end{align*}
\]

with the multi-point boundary conditions

where \( n, m, p, q \in \mathbb{N} , n \geq 2 , m \geq 2 , p \geq 3 , q \geq 3 , 0 < \xi_1 < \cdots < \xi_{p-2} < T \) and \( 0 < \eta_1 < \cdots < \eta_{q-2} < T \).

By using the Schauder fixed point theorem, we shall prove the existence of positive solutions of problem \((S) - (BC)\). By a positive solution of \((S) - (BC)\) we mean a pair of functions \((u, v) \in C^n([0, T]) \times C^m([0, T])\) satisfying \((S)\) and \((BC)\) with \(u(t) > 0, v(t) > 0\) for all \(t \in [0, T]\). We shall also give sufficient conditions for the nonexistence of positive solutions for this problem.
The system \((S)\) with the boundary conditions

\[
(BC_1) \quad \begin{cases}
u(0) = ν'(0) = \cdots = ν^{(m-2)}(0) = 0, & ν(T) = \sum_{i=1}^{p-2} b_i ν(ξ_i) + b_0, \\
u(0) = ν'(0) = \cdots = ν^{(m-2)}(0) = 0, & ν(T) = \sum_{i=1}^{q-2} a_i ν(ξ_i) + a_0,
\end{cases}
\]

has been investigated in [7]. In [26], the authors used the fixed point index theory to prove the existence of positive solutions for the system \((S)\) where \(f\) and \(g\) are dependent of \(u\) and \(v\), and the boundary conditions \((BC_1)\) with \(a_0 = b_0 = 0\) and \(\frac{1}{2} \leq ξ_1 < \cdots < ξ_{p-2} < 1, \frac{1}{2} \leq η_1 < \cdots < η_{q-2} < 1, T = 1\). For multi-point boundary value problems for nonlinear higher-order ordinary differential equations we mention the papers [1], [19].

Multi-point boundary value problems for systems of ordinary differential equations which involve positive eigenvalues were studied in recent years by J. Henderson, R. Luca, S. K. Ntouyas and I. K. Purnaras, by using the Guo-Krasnosel’skii fixed point theorem. Namely, in [2], the authors give sufficient conditions for \(λ, μ, f\) and \(g\) such that the system of differential equations

\[
(S_1) \quad \begin{cases}
u^{(m)}(t) + λf(t)ν(t) + ν(t) = 0, & t \in (0, T), \\
u^{(m)}(t) + μg(t)ν(t) = 0, & t \in (0, T),
\end{cases}
\]

with the boundary conditions \((BC_1)\) with \(a_0 = b_0 = 0\) has positive solutions. The system \((S_1)\) with \(f_1(u, v) = f(v), g_1(u, v) = g(u)\) and \(n = m\) (denoted by \((\bar{S}_1)\)) with the boundary conditions \((BC_1)\) with \(a_0 = b_0 = 0\), where \(n = m, p = q, a_i = b_i, ξ_i = η_i\) for \(i = 1, \ldots, p - 2\), has been studied in [22]. In [9], the authors studied the system \((\bar{S}_1)\) with \(T = 1\) and the boundary conditions \(u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, u(1) = au(η), ν(0) = ν'(0) = \cdots = ν^{(n-2)}(0) = 0, v(1) = av(η)\), where \(0 < η < 1\) and \(0 < aη^{n-1} < 1\).

The systems \((S)\) and \((S_1)\) with \(n = m = 2\) subject to various boundary conditions were studied in [3], [4], [10], [11], [13], [14], [23]. Some discrete versions of these nonlinear second-order boundary value problems have been investigated in [5], [6], [12], [15], [21], [24].

Our results obtained in this paper were inspired by the paper [20], where the authors studied the existence and nonexistence of positive solutions for the \(m\)-point boundary value problem on time scales

\[
\begin{cases}
u^{N}(t) + a(t)f(u(t)) = 0, & t \in (0, T), \\
βu(0) - γu^{Δ}(0) = 0, & μ(T) - \sum_{i=1}^{m-2} a_i u(ξ_i) = b, \quad m \geq 3, \quad b > 0,
\end{cases}
\]

where \((0, T)\) denotes a time scale interval.
Multi-point boundary value problems for ordinary differential equations or finite difference equations have applications in a variety of different areas of applied mathematics and physics. For example, the vibrations of a guy wire of a uniform cross-section and composed of \( N \) parts of different densities can be set up as a multi-point boundary value problem (see [25]); also many problems in the theory of elastic stability can be handled as multi-point problems (see [27]). The study of multi-point boundary value problems for second order differential equations was initiated by Il’in and Moiseev (see [16], [17]). Since then such multi-point boundary value problems (continuous or discrete cases) have been studied by many authors, by using different methods, such as fixed point theorems in cones, the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder and coincidence degree theory.

In Section 2, we shall present some auxiliary results which investigate a boundary value problem for a \( n \)-th order differential equation (problem (1)–(2) below), and in Section 3, we shall give our main results.

2. AUXILIARY RESULTS

In this section, we shall present some auxiliary results from [18] related to the following \( n \)-th order differential equation with \( p \)-point boundary conditions

\[
\begin{align*}
  u^{(n)}(t) + y(t) &= 0, \quad t \in (0, T), \\
  u(0) &= \sum_{i=1}^{p-2} a_i u(\xi_i), \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(T) = 0.
\end{align*}
\]

We shall present these results for the interval \([0, T]\) of the \( t \)-variable. Their proofs are similar to those from [18] where \( T = 1 \).

**Lemma 2.1.** If \( d = T^{n-1} - \sum_{i=1}^{p-2} a_i(T^{n-1} - \xi_i^{n-1}) \neq 0, \quad 0 < \xi_1 < \cdots < \xi_{p-2} < T \) and \( y \in C([0, T]) \), then the solution of (1)-(2) is given by

\[
\begin{align*}
  u(t) &= - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s) \, ds + \frac{d}{d} \sum_{i=1}^{p-2} a_i \int_0^t \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s) \, ds \\
  & \quad + \left( 1 - \sum_{i=1}^{p-2} a_i \int_0^t \frac{(T-s)^{n-1}}{(n-1)!} y(s) \, ds \right) + \frac{1}{d} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} y(s) \, ds \\
  & \quad - \frac{T^{n-1}}{d} \sum_{i=1}^{p-2} a_i \int_0^t \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s) \, ds.
\end{align*}
\]

**Lemma 2.2.** Under the assumptions of Lemma 2.1, the Green’s function for the boundary value problem (1)-(2) is

\[
G_1(t, s) = g_1(t, s) + \frac{T^{n-1} - \xi_i^{n-1}}{d} \sum_{i=1}^{p-2} a_i g_1(\xi_i, s),
\]
where
\[ g_1(t, s) = \frac{1}{(n-1)!T^{n-1}} \left\{ \begin{array}{ll} t^{n-1}(T - s)^{n-1} - T^{n-1}(t - s)^{n-1}, & 0 \leq s \leq t \leq T, \\ t^{n-1}(T - s)^{n-1}, & 0 \leq t \leq s \leq T. \end{array} \right. \]

Using the above Green’s function the solution of problem (1)-(2) is expressed as
\[ u(t) = \int_0^T G_1(t, s)y(s) ds. \]

**Lemma 2.3.** The function \( g_1 \) has the properties
a) \( g_1 \) is a continuous function on \([0, T] \times [0, T]\) and \( g_1(t, s) \geq 0 \) for all \((t, s) \in [0, T] \times [0, T] \); 
b) \( g_1(t, s) \leq g_1(\theta_1(s), s) \), for all \((t, s) \in [0, T] \times [0, T] \); 
c) For any \( c \in \left(0, \frac{T}{n-1}\right) \),
\[ \min_{\theta_1 \in [c, T-c]} g_1(t, s) \geq \frac{T^{n-1}}{n-1} g_1(\theta_1(s), s), \quad \text{for all} \ s \in [0, T], \]

where \( \theta_1(s) = s \) if \( n = 2 \) and \( \theta_1(s) = \left\{ \begin{array}{ll} s & s \in (0, T], \\ 1 - \left(1 - \frac{s}{T}\right)^{\frac{n}{n-2}} & s = 0, \end{array} \right. \] if \( n \geq 3 \).

In the case \( n \geq 3 \), we choose the values of \( \theta_1 \) in \( s = 0 \) and \( s = T \) such that \( \theta_1 \) be a continuous function on \([0, T] \) (see also [8]).

**Lemma 2.4.** Assume that \( a_i \geq 0 \) for all \( i = 1, \ldots, p-2, \ 0 < \xi_1 < \cdots < \xi_{p-2} < T \) and \( d > 0 \). Then the Green’s function \( G_1 \) of problem (1)-(2) has the properties
a) \( G_1 \) is a continuous function on \([0, T] \times [0, T] \) and \( G_1(t, s) \geq 0 \) for all \((t, s) \in [0, T] \times [0, T] \); 
b) \( G_1(t, s) \leq J_1(s) \) for all \((t, s) \in [0, T] \times [0, T] \) and for any \( c \in (0, T/2) \) we have
\[ \min_{\theta_1 \in [c, T-c]} G_1(t, s) \geq \frac{T^{n-1}}{n-1} J_1(s) \quad \text{for all} \ s \in [0, T], \]

where \( J_1(s) = g_1(\theta_1(s), s) + \frac{T^{n-2}}{d} \sum_{i=1}^{p-2} a_i g_1(\xi_i, s), \quad \forall s \in [0, T]. \)

**Lemma 2.5.** If \( a_i \geq 0 \) for all \( i = 1, \ldots, p-2, \ 0 < \xi_1 < \cdots < \xi_{p-2} < T, \ d > 0, \ y \in C([0, T]) \) and \( y(t) \geq 0 \) for all \( t \in [0, T] \), then the solution of problem (1)-(2) satisfies \( u(t) \geq 0 \) for all \( t \in [0, T] \).

**Lemma 2.6.** Assume that \( a_i \geq 0 \) for all \( i = 1, \ldots, p-2, \ 0 < \xi_1 < \cdots < \xi_{p-2} < T, \ d > 0, \ y \in C([0, T]) \) and \( y(t) \geq 0 \) for all \( t \in [0, T] \). Then the solution of problem (1)-(2) satisfies
\[ \min_{\theta_1 \in [c, T-c]} u(t) \geq \frac{T^{n-1}}{n-1} \max_{t \in [0, T]} u(t'). \]

We can also formulate similar results as Lemma 2.1 - Lemma 2.6 above for the boundary value problem
\[ v^{(m)}(t) + h(t) = 0, \quad t \in (0, T), \] (3)
Existence of positive solutions for a higher-order multi-point boundary value problem

\[ v(0) = \sum_{i=1}^{q-2} b_i v(\eta_i), \quad v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(T) = 0, \]

where \( 0 < \eta_1 < \cdots < \eta_{q-2} < T, b_i \geq 0 \) for all \( i = 1, \ldots, q-2 \) and \( h \in C([0,T]) \). If \( e = T^{m-1} \sum_{i=1}^{q-2} b_i \left( T^{m-1} - \eta_i^{m-1} \right) \neq 0 \), we denote by \( G_2 \) the Green’s function associated to problem (3)-(4) and defined in a similar manner as \( G_1 \). We also denote by \( g_2, \theta_2 \) and \( J_2 \) the corresponding functions for (3)-(4) defined in a similar manner as \( g_1, \theta_1 \) and \( J_1 \), respectively.

3. MAIN RESULTS

We present the assumptions that we shall use in the sequel:

(H1) \( 0 < \xi_1 < \cdots < \xi_{p-2} < T, 0 < \eta_1 < \cdots < \eta_{q-2} < T, a_i \geq 0, i = 1, \ldots, p-2, \)
\( b_i \geq 0, \quad i = 1, \ldots, q-2, \quad d = T^{n-1} - \sum_{i=1}^{p-2} a_i (T^{n-1} - \xi_i^{n-1}) > 0, \)
\( e = T^{m-1} - \sum_{i=1}^{q-2} b_i (T^{m-1} - \eta_i^{m-1}) > 0. \)

(H2) The functions \( \alpha, \beta : [0, T] \to [0, \infty) \) are continuous and for any \( c \in (0, T/2) \) there exist \( t_0, \tilde{t}_0 \in [c, T - c] \) such that \( \alpha(t_0) > 0, \beta(\tilde{t}_0) > 0. \)

(H3) The functions \( f, g : [0, \infty) \to [0, \infty) \) are continuous and there exists \( r_0 > 0 \) such that \( f(u) < \frac{r_0}{L}, g(u) < \frac{r_0}{L} \) for all \( u \in [0, r_0] \), where

\[ L = \max \left\{ \int_0^T J_1(s) \alpha(s) \, ds, \int_0^T J_2(s) \beta(s) \, ds \right\}. \]

(H4) The functions \( f, g : [0, \infty) \to [0, \infty) \) are continuous and satisfy the conditions \( \lim_{u \to \infty} \frac{f(u)}{u} = \infty, \lim_{u \to \infty} \frac{g(u)}{u} = \infty. \)

First, we present an existence result for the positive solutions of \((S) - (BC)\).

**Theorem 3.1.** Assume that the assumptions (H1), (H2) and (H3) hold. Then the problem \((S) - (BC)\) has at least one positive solution for \( a_0 > 0 \) and \( b_0 > 0 \) sufficiently small.

**Proof.** We consider the problems

\[
\begin{cases}
    h^{(n)}(t) = 0, & t \in (0, T), \\
    h(0) = \sum_{i=1}^{p-2} a_i h(\xi_i) + 1, & h'(0) = \cdots = h^{(n-2)}(0) = 0, & h(T) = 0,
\end{cases}
\]

\[ (5) \]

\[
\begin{cases}
    w^{(m)}(t) = 0, & t \in (0, T), \\
    w(0) = \sum_{i=1}^{q-2} b_i w(\eta_i) + 1, & w'(0) = \cdots = w^{(m-2)}(0) = 0, & w(T) = 0.
\end{cases}
\]

\[ (6) \]
The above problems (5) and (6) have the solutions
\[ h(t) = \frac{T^{n-1} - t^{n-1}}{d}, \quad w(t) = \frac{T^{m-1} - t^{m-1}}{e}, \quad t \in [0, T]. \] (7)

We define the functions \( x(t) \) and \( y(t) \), \( t \in [0, T] \) by
\[ x(t) = u(t) - a_0 h(t), \quad y(t) = v(t) - b_0 w(t), \quad t \in [0, T], \]
where \((u, v)\) is a solution of \((S) - (BC)\). Then \((S) - (BC)\) can be equivalently written as
\[
\begin{cases}
  x^{(n)}(t) + \alpha(t)f(y(t) + b_0 w(t)) = 0, & t \in (0, T), \\
  y^{(m)}(t) + \beta(t)g(x(t) + a_0 h(t)) = 0, & t \in (0, T),
\end{cases}
\] (8)
with the boundary conditions
\[
\begin{cases}
  x(0) = \sum_{i=1}^{d-2} a_i x(\xi_i), \quad x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(T) = 0, \\
  y(0) = \sum_{i=1}^{d-2} b_i y(\eta_i), \quad y'(0) = \cdots = y^{(m-2)}(0), \quad y(T) = 0.
\end{cases}
\] (9)

Using the Green’s functions given in Section 2, a pair \((x, y)\) is a solution of the problem (8)-(9) if and only if \((x, y)\) is a solution for the nonlinear integral equations
\[
\begin{cases}
  x(t) = \int_0^T G_1(t, s) \alpha(s) f \left( \int_0^T G_2(s, \tau) \beta(\tau) g(x(\tau) + a_0 h(\tau)) d\tau + b_0 w(s) \right) ds, \\
  y(t) = \int_0^T G_2(t, s) \beta(s) g(x(s) + a_0 h(s)) ds, \quad 0 \leq t \leq T,
\end{cases}
\] (10)
where \( h(t) \) and \( w(t) \) for \( t \in [0, T] \) are given by (7).

We consider the Banach space \( X = C([0, T]) \) with the supremum norm \( \| \cdot \| \) and define the set
\[ K = \{ x \in C([0, T]), \quad 0 \leq x(t) \leq r_0, \quad \forall t \in [0, T] \} \subset X. \]

We also define the operator \( \mathcal{A} : K \to X \) by
\[ \mathcal{A}(x)(t) = \int_0^T G_1(t, s) \alpha(s) f \left( \int_0^T G_2(s, \tau) \beta(\tau) g(x(\tau) + a_0 h(\tau)) d\tau + b_0 w(s) \right) ds, \quad 0 \leq t \leq T, \quad x \in K. \]

For sufficiently small \( a_0 > 0 \) and \( b_0 > 0 \), by \((H3)\), we deduce
\[ f(y(t) + b_0 w(t)) \leq \frac{r_0}{L}, \quad g(x(t) + a_0 h(t)) \leq \frac{r_0}{L}, \quad \forall t \in [0, T], \quad \forall x, y \in K. \]
Then, by using Lemma 2.3, we obtain $A(x)(t) \geq 0$ for all $t \in [0, T]$ and $x \in K$. By Lemma 2.4, for all $x \in K$, we have

$$
\int_0^T G_2(s, \tau)\beta(\tau)g(x(\tau) + a_0h(\tau))\,d\tau \leq \int_0^T J_2(\tau)\beta(\tau)g(x(\tau) + a_0h(\tau))\,d\tau
$$

and

$$
A(x)(t) \leq \int_0^T J_1(s)\alpha(s)f\left(\int_0^T G_2(s, \tau)\beta(\tau)g(x(\tau) + a_0h(\tau))\,d\tau + b_0w(s)\right)\,ds
$$

and

$$
\frac{r_0}{L} \int_0^T J_2(\tau)\beta(\tau)\,d\tau \leq r_0, \quad \forall \ s \in [0, T],
$$

Therefore $A(K) \subset K$.

Using standard arguments, we deduce that $A$ is completely continuous ($A$ is compact, that is for any bounded set $B \subset K$, $A(B) \subset K$ is relatively compact by Arzèla-Ascoli theorem, and $A$ is continuous). By the Schauder fixed point theorem, we conclude that $A$ has a fixed point $x \in K$. This element together with $y$ given by

$$
y(t) = \int_0^T G_2(t, s)\beta(s)g(x(s) + a_0h(s))\,ds, \quad t \in [0, T]
$$

represents a solution for (8)-(9). This shows that our problem $(S) - (BC)$ has a positive solution $u = x + a_0h, \ v = y + b_0w$ for sufficiently small positive $a_0$ and $b_0$.

In what follows, we present sufficient conditions for the nonexistence of the positive solutions of $(S) - (BC)$.

**Theorem 3.2.** Let the assumptions (H1), (H2) and (H4) be satisfied. Then the problem $(S) - (BC)$ has no positive solution for $a_0$ and $b_0$ sufficiently large.

**Proof.** We suppose that $(u, v)$ is a positive solution of $(S) - (BC)$. Then $x = u - a_0h, \ y = v - b_0w$ is a solution for (8)-(9), where $h$ and $w$ are the solutions of problems (5) and (6) (given by (7)). By Lemma 2.5, we have $x(t) \geq 0, \ y(t) \geq 0$ for all $t \in [0, T]$, and by (H2) we deduce that $\|x\| > 0, \|y\| > 0$. Using Lemma 2.6, for $c \in (0, T/2)$, we also have

$$
\inf_{t \in [c, T-c]} x(t) \geq \frac{c^{n-1}}{T^{n-1}}\|x\| \quad \text{and} \quad \inf_{t \in [c, T-c]} y(t) \geq \frac{c^{m-1}}{T^{m-1}}\|y\|.
$$

Using now (7), we deduce that $\inf_{t \in [c, T-c]} h(t) = [T^{n-1} - (T - c)^{n-1}]/d$. Therefore

$$
\inf_{t \in [c, T-c]} h(t) = \frac{T^{n-1} - (T - c)^{n-1}}{T^{n-1}}\|h\| \geq \frac{c^{n-1}}{T^{n-1}}\|h\|.
$$

In a similar manner we obtain $\inf_{t \in [c, T-c]} w(t) \geq c^{m-1}/T^{m-1}\|w\|$. 


Therefore, we obtain
\[
\inf_{t \in [c, T - c]} (x(t) + a_0h(t)) \geq \inf_{t \in [c, T - c]} x(t) + a_0 \inf_{t \in [c, T - c]} h(t) \\
\geq \frac{e^{\alpha t}}{T^{n-1}} (||x|| + a_0||h||) \geq \frac{e^{\alpha t}}{T^{n-1}} ||x + a_0h||,
\]
\[
\inf_{t \in [c, T - c]} (y(t) + b_0w(t)) \geq \inf_{t \in [c, T - c]} y(t) + b_0 \inf_{t \in [c, T - c]} w(t) \\
\geq \frac{e^{\beta t}}{T^{m-1}} (||y|| + b_0||w||) \geq \frac{e^{\beta t}}{T^{m-1}} ||y + b_0w||.
\]

We now consider
\[
R = \frac{T^{n+m-2}}{e^{(n+m-2)c}} \left( \min \left\{ \int_c^{T-c} J_1(s)\alpha(s) \, ds, \int_c^{T-c} J_2(s)\beta(s) \, ds \right\} \right)^{-1} > 0.
\]

By \((H4)\), for \(R\) defined above, we deduce that there exists \(M > 0\) such that \(f(u) > 2Ru, g(u) > 2Ru\) for all \(u \geq M\).

We consider \(a_0 > 0\) and \(b_0 > 0\) sufficiently large such that
\[
\inf_{t \in [c, T - c]} (x(t) + a_0h(t)) \geq M \quad \text{and} \quad \inf_{t \in [c, T - c]} (y(t) + b_0w(t)) \geq M.
\]

By using Lemma 2.4 and the above considerations, we have
\[
y(c) = \int_0^T G_2(c, s)\beta(s)g(x(s) + a_0h(s)) \, ds \geq \\
\geq \int_c^{T-c} G_2(c, s)\beta(s)g(x(s) + a_0h(s)) \, ds \geq \\
\geq \frac{e^{\alpha c}}{T^{n-1}} \int_c^{T-c} J_2(s)\beta(s)g(x(s) + a_0h(s)) \, ds \geq \\
\geq \frac{2Re^{\alpha c}}{T^{n-1}} \int_c^{T-c} J_2(s)\beta(s)(x(s) + a_0h(s)) \, ds \geq \\
\geq \frac{2Re^{\alpha c}}{T^{n-1}} \inf_{t \in [c, T - c]} (x(t) + a_0h(t)) \int_c^{T-c} J_2(s)\beta(s) \, ds \geq \\
\geq \frac{2Re^{\alpha c}}{T^{n-1}} ||x + a_0h|| \int_c^{T-c} J_2(s)\beta(s) \, ds \geq 2||x + a_0h|| \geq 2||x||.
\]

Therefore, we obtain
\[
||x|| \leq y(c)/2 \leq ||y||/2. \tag{11}
\]
In a similar manner, we deduce
\[ x(c) = \int_{T-n}^{T} G_1(c, s) \alpha(s) f(y(s) + b_0 w(s)) \, ds \geq \]
\[ \geq \frac{c^{n-1}}{T-n-1} \int_{T-c}^{T-c} J_1(s) \alpha(s) f(y(s) + b_0 w(s)) \, ds \geq \]
\[ \geq \frac{2Rc^{n-1}}{T-n-1} \inf_{\tau \in [c, T-c]} (y(\tau) + b_0 w(\tau)) \int_{c}^{T-c} J_1(s) \alpha(s) \, ds \geq \]
\[ \geq \frac{2Rc^{n+m-2}}{T-n+m-2} \int_{c}^{T-c} J_1(s) \alpha(s) \, ds \geq 2[|y| + b_0 w|] \geq 2[|y|]. \]

So, we obtain
\[ ||y|| \leq x(c)/2 \leq ||x||/2. \] (12)

By (11) and (12), we obtain \( ||x|| \leq ||y||/2 \leq ||x||/4 \), which is a contradiction, because \( ||x|| > 0 \). Then, for \( d_0 \) and \( b_0 \) sufficiently large, our problem \((S) - (BC)\) has no positive solution. \hfill \Box

Acknowledgement. This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0557, Romania.

References