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EXISTENCE OF POSITIVE SOLUTIONS FOR A HIGHER-ORDER MULTI-POINT BOUNDARY VALUE PROBLEM

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Abstract We study the existence and nonexistence of positive solutions for a system of nonlinear higher-order ordinary differential equations with multi-point boundary conditions.

Keywords: Higher-order differential system, multi-point boundary conditions, positive solutions.

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1. INTRODUCTION

We consider the system of nonlinear higher-order ordinary differential equations

$$(S) \qquad \begin{cases} u^{(n)}(t) + \alpha(t) f(v(t)) = 0, & t \in (0, T), \\ v^{(m)}(t) + \beta(t) g(u(t)) = 0, & t \in (0, T), \end{cases}$$

with the multi-point boundary conditions

$$(BC) \qquad \begin{cases} u(0) = \displaystyle\sum_{i=1}^{p-2} a_i u(\xi_i) + a_0, \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(T) = 0, \\ v(0) = \displaystyle\sum_{i=1}^{q-2} b_i v(\eta_i) + b_0, \quad v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(T) = 0, \end{cases}$$

where n, m, p, $q \in \mathbb{N}$, $n \geq 2$, $m \geq 2$, $p \geq 3$, $q \geq 3$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$ and $0 < \eta_1 < \cdots < \eta_{q-2} < T$.

By using the Schauder fixed point theorem, we shall prove the existence of positive solutions of problem $(S) - (BC)$. By a positive solution of $(S) - (BC)$ we mean a pair of functions $(u, v) \in C^n([0, T]) \times C^m([0, T])$ satisfying (S) and (BC) with $u(t) > 0$, $v(t) > 0$ for all $t \in [0, T)$. We shall also give sufficient conditions for the nonexistence of positive solutions for this problem.

The system (S) with the boundary conditions

$$(BC_1)\qquad \begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(T) = \displaystyle\sum_{i=1}^{p-2} a_i u(\xi_i) + a_0, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(T) = \displaystyle\sum_{i=1}^{q-2} b_i v(\eta_i) + b_0, \end{cases}$$

has been investigated in [7]. In [26], the authors used the fixed point index theory to prove the existence of positive solutions for the system (S) where f and g are dependent of u and v, and the boundary conditions (BC_1) with $a_0 = b_0 = 0$ and $\frac{1}{2} \le \xi_1 < \cdots < \xi_{p-2} < 1$, $\frac{1}{2} \le \eta_1 < \cdots < \eta_{q-2} < 1$, $T = 1$. For multi-point boundary value problems for nonlinear higher-order ordinary differential equations we mention the papers [1], [19].

Multi-point boundary value problems for systems of ordinary differential equations which involve positive eigenvalues were studied in recent years by J. Henderson, R. Luca, S. K. Ntouyas and I. K. Purnaras, by using the Guo-Krasnosel'skii fixed point theorem. Namely, in [2], the authors give sufficient conditions for λ, μ, f and g such that the system of differential equations

$$(S_1)\qquad \begin{cases} u^{(n)}(t) + \lambda\alpha(t) f_1(u(t), v(t)) = 0, \quad t \in (0, T), \\ v^{(m)}(t) + \mu\beta(t) g_1(u(t), v(t)) = 0, \quad t \in (0, T), \end{cases}$$

with the boundary conditions (BC_1) with $a_0 = b_0 = 0$ has positive solutions. The system (S_1) with $f_1(u, v) = \widetilde{f}(v)$, $g_1(u, v) = \widetilde{g}(u)$ and $n = m$ (denoted by $(\widetilde{S}_1)$) with the boundary conditions (BC_1) with $a_0 = b_0 = 0$, where $n = m$, $p = q$, $a_i = b_i$, $\xi_i = \eta_i$ for $i = 1, ..., p-2$, has been studied in [22]. In [9], the authors studied the system $(\widetilde{S}_1)$ with $T = 1$ and the boundary conditions $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, $u(1) = \alpha u(\eta)$, $v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0$, $v(1) = \alpha v(\eta)$, where $0 < \eta < 1$ and $0 < \alpha\eta^{n-1} < 1$.

The systems (S) and (S_1) with $n = m = 2$ subject to various boundary conditions were studied in [3], [4], [10], [11], [13], [14], [23]. Some discrete versions of these nonlinear second-order boundary value problems have been investigated in [5], [6], [12], [15], [21], [24].

Our results obtained in this paper were inspired by the paper [20], where the authors studied the existence and nonexistence of positive solutions for the m-point boundary value problem on time scales

$$\begin{cases} u^{\Delta\nabla}(t) + a(t) f(u(t)) = 0, \quad t \in (0, \widetilde{T}), \\ \beta u(0) - \gamma u^{\Delta}(0) = 0, \quad u(\widetilde{T}) - \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i) = b, \quad m \ge 3, \quad b > 0, \end{cases}$$

where $(0, \widetilde{T})$ denotes a time scale interval.

Multi-point boundary value problems for ordinary differential equations or finite difference equations have applications in a variety of different areas of applied mathematics and physics. For example the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary value problem (see [25]); also many problems in the theory of elastic stability can be handled as multi-point problems (see [27]). The study of multi-point boundary value problems for second order differential equations was initiated by Il'in and Moiseev (see [16], [17]). Since then such multi-point boundary value problems (continuous or discrete cases) have been studied by many authors, by using different methods, such as fixed point theorems in cones, the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder and coincidence degree theory.

In Section 2, we shall present some auxiliary results which investigate a boundary value problem for a n-th order differential equation (problem (1) – (2) below), and in Section 3, we shall give our main results.

2. AUXILIARY RESULTS

In this section, we shall present some auxiliary results from [18] related to the following n-th order differential equation with p-point boundary conditions

$$u^{(n)}(t) + y(t) = 0, \quad t \in (0, T), \tag{1}$$

$$u(0) = \sum_{i=1}^{p-2} a_i u(\xi_i), \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(T) = 0. \tag{2}$$

We shall present these results for the interval $[0, T]$ of the t-variable. Their proofs are similar to those from [18] where $T = 1$.

Lemma 2.1. *If* $d = T^{n-1} - \sum_{i=1}^{p-2} a_i (T^{n-1} - \xi_i^{n-1}) \neq 0$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$ *and* $y \in C([0, T])$, *then the solution of* (1)-(2) *is given by*

$$\begin{aligned} u(t) = &- \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\, ds + \frac{t^{n-1}}{d} \left[\sum_{i=1}^{p-2} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s)\, ds \right. \\ &+ \left. \left(1 - \sum_{i=1}^{p-2} a_i\right) \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} y(s)\, ds \right] + \frac{1}{d} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} y(s)\, ds \\ &- \frac{T^{n-1}}{d} \sum_{i=1}^{p-2} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s)\, ds. \end{aligned}$$

Lemma 2.2. *Under the assumptions of Lemma 2.1, the Green's function for the boundary value problem* (1)-(2) *is*

$$G_1(t, s) = g_1(t, s) + \frac{T^{n-1} - t^{n-1}}{d} \sum_{i=1}^{p-2} a_i g_1(\xi_i, s),$$

where

$$g_1(t,s) = \frac{1}{(n-1)!T^{n-1}} \begin{cases} t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1}, & 0 \le s \le t \le T, \\ t^{n-1}(T-s)^{n-1}, & 0 \le t \le s \le T. \end{cases}$$

Using the above Green's function the solution of problem (1)-(2) is expressed as $u(t) = \int_0^T G_1(t,s)y(s)\,ds$.

Lemma 2.3. *The function g_1 has the properties*

a) g_1 is a continuous function on $[0,T]\times[0,T]$ and $g_1(t,s) \ge 0$ for all $(t,s) \in [0,T]\times[0,T]$;

b) $g_1(t,s) \le g_1(\theta_1(s),s)$, for all $(t,s) \in [0,T]\times[0,T]$;

c) For any $c \in \left(0, \frac{T}{2}\right)$,

$$\min_{t\in[c,T-c]} g_1(t,s) \ge \frac{c^{n-1}}{T^{n-1}} g_1(\theta_1(s),s), \text{ for all } s \in [0,T],$$

where $\theta_1(s) = s$ *if* $n = 2$ *and* $\theta_1(s) = \begin{cases} \dfrac{s}{1-\left(1-\frac{s}{T}\right)^{\frac{n-1}{n-2}}}, & s \in (0,T], \\ \frac{T(n-2)}{n-1}, & s = 0, \end{cases}$ *if* $n \ge 3$.

In the case $n \ge 3$, we choose the values of θ_1 in $s = 0$ and $s = T$ such that θ_1 be a continuous function on $[0,T]$ (see also [8]).

Lemma 2.4. *Assume that $a_i \ge 0$ for all $i = 1,\dots,p-2$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$ and $d > 0$. Then the Green's function G_1 of problem* (1)-(2) *has the properties*

a) G_1 is a continuous function on $[0,T]\times[0,T]$ and $G_1(t,s) \ge 0$ for all $(t,s) \in [0,T]\times[0,T]$;

b) $G_1(t,s) \le J_1(s)$ for all $(t,s) \in [0,T]\times[0,T]$ and for any $c \in (0,T/2)$ we have

$$\min_{t\in[c,T-c]} G_1(t,s) \ge \frac{c^{n-1}}{T^{n-1}} J_1(s) \text{ for all } s \in [0,T],$$

where $J_1(s) = g_1(\theta_1(s),s) + \dfrac{T^{n-1}}{d}\displaystyle\sum_{i=1}^{p-2} a_i g_1(\xi_i,s), \quad \forall\, s \in [0,T]$.

Lemma 2.5. *If $a_i \ge 0$ for all $i = 1,\dots,p-2$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $d > 0$, $y \in C([0,T])$ and $y(t) \ge 0$ for all $t \in [0,T]$, then the solution of problem* (1)-(2) *satisfies $u(t) \ge 0$ for all $t \in [0,T]$.*

Lemma 2.6. *Assume that $a_i \ge 0$ for all $i = 1,\dots,p-2$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $d > 0$, $y \in C([0,T])$ and $y(t) \ge 0$ for all $t \in [0,T]$. Then the solution of problem* (1)-(2) *satisfies* $\displaystyle\min_{t\in[c,T-c]} u(t) \ge \frac{c^{n-1}}{T^{n-1}} \max_{t'\in[0,T]} u(t')$.

We can also formulate similar results as Lemma 2.1 - Lemma 2.6 above for the boundary value problem

$$v^{(m)}(t) + h(t) = 0, \quad t \in (0,T), \tag{3}$$

$$v(0) = \sum_{i=1}^{q-2} b_i v(\eta_i), \quad v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(T) = 0, \tag{4}$$

where $0 < \eta_1 < \cdots < \eta_{q-2} < T$, $b_i \geq 0$ for all $i = 1, \ldots, q-2$ and $h \in C([0,T])$. If $e = T^{m-1} - \sum_{i=1}^{q-2} b_i \left(T^{m-1} - \eta_i^{m-1}\right) \neq 0$, we denote by G_2 the Green's function associated to problem (3)-(4) and defined in a similar manner as G_1. We also denote by g_2, θ_2 and J_2 the corresponding functions for (3)-(4) defined in a similar manner as g_1, θ_1 and J_1, respectively.

3. MAIN RESULTS

We present the assumptions that we shall use in the sequel:

$(H1)$ $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $0 < \eta_1 < \cdots < \eta_{q-2} < T$, $a_i \geq 0$, $i = 1, \ldots, p-2$, $b_i \geq 0$, $i = 1, \ldots, q-2$, $d = T^{n-1} - \sum_{i=1}^{p-2} a_i (T^{n-1} - \xi_i^{n-1}) > 0$, $e = T^{m-1} - \sum_{i=1}^{q-2} b_i (T^{m-1} - \eta_i^{m-1}) > 0$.

$(H2)$ The functions $\alpha, \beta : [0,T] \to [0,\infty)$ are continuous and for any $c \in (0, T/2)$ there exist t_0, $\widetilde{t_0} \in [c, T-c]$ such that $\alpha(t_0) > 0$, $\beta(\widetilde{t_0}) > 0$.

$(H3)$ The functions f, $g : [0,\infty) \to [0,\infty)$ are continuous and there exists $r_0 > 0$ such that $f(u) < \dfrac{r_0}{L}$, $g(u) < \dfrac{r_0}{L}$ for all $u \in [0, r_0]$, where

$$L = \max\left\{ \int_0^T J_1(s)\alpha(s)\,ds, \ \int_0^T J_2(s)\beta(s)\,ds \right\}.$$

$(H4)$ The functions f, $g : [0,\infty) \to [0,\infty)$ are continuous and satisfy the conditions $\lim_{u\to\infty} \dfrac{f(u)}{u} = \infty$, $\lim_{u\to\infty} \dfrac{g(u)}{u} = \infty$.

First, we present an existence result for the positive solutions of $(S) - (BC)$.

Theorem 3.1. *Assume that the assumptions $(H1), (H2)$ and $(H3)$ hold. Then the problem $(S)-(BC)$ has at least one positive solution for $a_0 > 0$ and $b_0 > 0$ sufficiently small.*

Proof. We consider the problems

$$\begin{cases} h^{(n)}(t) = 0, \quad t \in (0,T), \\ h(0) = \sum_{i=1}^{p-2} a_i h(\xi_i) + 1, \quad h'(0) = \cdots = h^{(n-2)}(0) = 0, \quad h(T) = 0, \end{cases} \tag{5}$$

$$\begin{cases} w^{(m)}(t) = 0, \quad t \in (0,T), \\ w(0) = \sum_{i=1}^{q-2} b_i w(\eta_i) + 1, \quad w'(0) = \cdots = w^{(m-2)}(0) = 0, \quad w(T) = 0. \end{cases} \tag{6}$$

The above problems (5) and (6) have the solutions

$$h(t) = \frac{T^{n-1} - t^{n-1}}{d}, \quad w(t) = \frac{T^{m-1} - t^{m-1}}{e}, \quad t \in [0, T]. \tag{7}$$

We define the functions $x(t)$ and $y(t)$, $t \in [0, T]$ by

$$x(t) = u(t) - a_0 h(t), \quad y(t) = v(t) - b_0 w(t), \quad t \in [0, T],$$

where (u, v) is a solution of $(S) - (BC)$. Then $(S) - (BC)$ can be equivalently written as

$$\begin{cases} x^{(n)}(t) + \alpha(t) f(y(t) + b_0 w(t)) = 0, & t \in (0, T), \\ y^{(m)}(t) + \beta(t) g(x(t) + a_0 h(t)) = 0, & t \in (0, T), \end{cases} \tag{8}$$

with the boundary conditions

$$\begin{cases} x(0) = \displaystyle\sum_{i=1}^{p-2} a_i x(\xi_i), \quad x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(T) = 0, \\ y(0) = \displaystyle\sum_{i=1}^{q-2} b_i y(\eta_i), \quad y'(0) = \cdots = y^{(m-2)}(0), \quad y(T) = 0. \end{cases} \tag{9}$$

Using the Green's functions given in Section 2, a pair (x, y) is a solution of the problem (8)-(9) if and only if (x, y) is a solution for the nonlinear integral equations

$$\begin{cases} x(t) = \displaystyle\int_0^T G_1(t, s)\alpha(s) f\left(\int_0^T G_2(s, \tau)\beta(\tau) g(x(\tau) + a_0 h(\tau))\, d\tau + b_0 w(s)\right) ds, \\ y(t) = \displaystyle\int_0^T G_2(t, s)\beta(s) g(x(s) + a_0 h(s))\, ds, \quad 0 \le t \le T, \end{cases} \tag{10}$$

where $h(t)$ and $w(t)$ for $t \in [0, T]$ are given by (7).

We consider the Banach space $X = C([0, T])$ with the supremum norm $\|\cdot\|$ and define the set

$$K = \{x \in C([0, T]), \ \ 0 \le x(t) \le r_0, \ \ \forall\, t \in [0, T]\} \subset X.$$

We also define the operator $\mathcal{A} : K \to X$ by

$$\mathcal{A}(x)(t) = \int_0^T G_1(t, s)\alpha(s) f\left(\int_0^T G_2(s, \tau)\beta(\tau) g(x(\tau) + a_0 h(\tau)) d\tau + b_0 w(s)\right) ds,$$
$$0 \le t \le T, \quad x \in K.$$

For sufficiently small $a_0 > 0$ and $b_0 > 0$, by $(H3)$, we deduce

$$f(y(t) + b_0 w(t)) \le \frac{r_0}{L}, \quad g(x(t) + a_0 h(t)) \le \frac{r_0}{L}, \quad \forall\, t \in [0, T], \quad \forall\, x, y \in K.$$

Then, by using Lemma 2.3, we obtain $\mathcal{A}(x)(t) \geq 0$ for all $t \in [0, T]$ and $x \in K$. By Lemma 2.4, for all $x \in K$, we have

$$\int_0^T G_2(s,\tau)\beta(\tau)g(x(\tau) + a_0h(\tau))\,d\tau \leq \int_0^T J_2(\tau)\beta(\tau)g(x(\tau) + a_0h(\tau))\,d\tau$$
$$\leq \frac{r_0}{L}\int_0^T J_2(\tau)\beta(\tau)\,d\tau \leq r_0, \quad \forall\, s \in [0, T],$$

and

$$\mathcal{A}(x)(t) \leq \int_0^T J_1(s)\alpha(s)f\left(\int_0^T G_2(s,\tau)\beta(\tau)g(x(\tau) + a_0h(\tau))\,d\tau + b_0w(s)\right)ds$$
$$\leq \frac{r_0}{L}\int_0^T J_1(s)\alpha(s)\,ds \leq r_0, \quad \forall\, t \in [0, T].$$

Therefore $\mathcal{A}(K) \subset K$.

Using standard arguments, we deduce that $\mathcal{A}$ is completely continuous ($\mathcal{A}$ is compact, that is for any bounded set $B \subset K$, $\mathcal{A}(B) \subset K$ is relatively compact by Arzèla-Ascoli theorem, and $\mathcal{A}$ is continuous). By the Schauder fixed point theorem, we conclude that $\mathcal{A}$ has a fixed point $x \in K$. This element together with y given by

$$y(t) = \int_0^T G_2(t,s)\beta(s)g(x(s) + a_0h(s))\,ds, \quad t \in [0, T]$$

represents a solution for (8)-(9). This shows that our problem $(S)-(BC)$ has a positive solution $u = x + a_0h$, $v = y + b_0w$ for sufficiently small positive a_0 and b_0. □

In what follows, we present sufficient conditions for the nonexistence of the positive solutions of $(S) - (BC)$.

Theorem 3.2. *Let the assumptions* $(H1)$, $(H2)$ *and* $(H4)$ *be satisfied. Then the problem* $(S) - (BC)$ *has no positive solution for* a_0 *and* b_0 *sufficiently large.*

Proof. We suppose that (u, v) is a positive solution of $(S) - (BC)$. Then $x = u - a_0h$, $y = v - b_0w$ is a solution for (8)-(9), where h and w are the solutions of problems (5) and (6) (given by (7)). By Lemma 2.5, we have $x(t) \geq 0$, $y(t) \geq 0$ for all $t \in [0, T]$, and by $(H2)$ we deduce that $\|x\| > 0$, $\|y\| > 0$. Using Lemma 2.6, for $c \in (0, T/2)$, we also have

$$\inf_{t\in[c,T-c]} x(t) \geq \frac{c^{n-1}}{T^{n-1}}\|x\| \text{ and } \inf_{t\in[c,T-c]} y(t) \geq \frac{c^{m-1}}{T^{m-1}}\|y\|.$$

Using now (7), we deduce that $\inf_{t\in[c,T-c]} h(t) = [T^{n-1} - (T-c)^{n-1}]/d$. Therefore

$$\inf_{t\in[c,T-c]} h(t) = \frac{T^{n-1} - (T-c)^{n-1}}{T^{n-1}}\|h\| \geq \frac{c^{n-1}}{T^{n-1}}\|h\|.$$

In a similar manner we obtain $\inf_{t\in[c,T-c]} w(t) \geq c^{m-1}/T^{m-1}\|w\|$.

Therefore, we obtain

$$\begin{aligned}
\inf_{t\in[c,T-c]}(x(t)+a_0h(t)) &\geq \inf_{t\in[c,T-c]}x(t)+a_0\inf_{t\in[c,T-c]}h(t)\\
&\geq \frac{c^{n-1}}{T^{n-1}}(\|x\|+a_0\|h\|)\geq \frac{c^{n-1}}{T^{n-1}}\|x+a_0h\|,\\
\inf_{t\in[c,T-c]}(y(t)+b_0w(t)) &\geq \inf_{t\in[c,T-c]}y(t)+b_0\inf_{t\in[c,T-c]}w(t)\\
&\geq \frac{c^{m-1}}{T^{m-1}}(\|y\|+b_0\|w\|)\geq \frac{c^{m-1}}{T^{m-1}}\|y+b_0w\|.
\end{aligned}$$

We now consider

$$R=\frac{T^{n+m-2}}{c^{n+m-2}}\left(\min\left\{\int_c^{T-c}J_1(s)\alpha(s)\,ds,\ \int_c^{T-c}J_2(s)\beta(s)\,ds\right\}\right)^{-1}>0.$$

By $(H4)$, for R defined above, we deduce that there exists $M>0$ such that $f(u)>2Ru$, $g(u)>2Ru$ for all $u\geq M$.

We consider $a_0>0$ and $b_0>0$ sufficiently large such that

$$\inf_{t\in[c,T-c]}(x(t)+a_0h(t))\geq M \quad\text{and}\quad \inf_{t\in[c,T-c]}(y(t)+b_0w(t))\geq M.$$

By using Lemma 2.4 and the above considerations, we have

$$\begin{aligned}
y(c)&=\int_0^T G_2(c,s)\beta(s)g(x(s)+a_0h(s))\,ds\geq\\
&\geq\int_c^{T-c}G_2(c,s)\beta(s)g(x(s)+a_0h(s))\,ds\geq\\
&\geq\frac{c^{m-1}}{T^{m-1}}\int_c^{T-c}J_2(s)\beta(s)g(x(s)+a_0h(s))\,ds\geq\\
&\geq\frac{2Rc^{m-1}}{T^{m-1}}\int_c^{T-c}J_2(s)\beta(s)(x(s)+a_0h(s))\,ds\geq\\
&\geq\frac{2Rc^{m-1}}{T^{m-1}}\inf_{\tau\in[c,T-c]}(x(\tau)+a_0h(\tau))\int_c^{T-c}J_2(s)\beta(s)\,ds\geq
\end{aligned}$$

$$\geq\frac{2Rc^{n+m-2}}{T^{n+m-2}}\|x+a_0h\|\int_c^{T-c}J_2(s)\beta(s)\,ds\geq 2\|x+a_0h\|\geq 2\|x\|.$$

Therefore, we obtain

$$\|x\|\leq y(c)/2\leq\|y\|/2. \tag{11}$$

In a similar manner, we deduce

$$\begin{aligned}
x(c) &= \int_0^T G_1(c,s)\alpha(s)f(y(s)+b_0w(s))\,ds \geq \\
&\geq \int_c^{T-c} G_1(c,s)\alpha(s)f(y(s)+b_0w(s))\,ds \geq \\
&\geq \frac{c^{n-1}}{T^{n-1}}\int_c^{T-c} J_1(s)\alpha(s)f(y(s)+b_0w(s))\,ds \geq \\
&\geq \frac{2Rc^{n-1}}{T^{n-1}}\int_c^{T-c} J_1(s)\alpha(s)(y(s)+b_0w(s))\,ds \geq \\
&\geq \frac{2Rc^{n-1}}{T^{n-1}}\inf_{\tau\in[c,T-c]}(y(\tau)+b_0w(\tau))\int_c^{T-c} J_1(s)\alpha(s)\,ds \geq \\
&\geq \frac{2Rc^{n+m-2}}{T^{n+m-2}}\|y+b_0w\|\int_c^{T-c} J_1(s)\alpha(s)\,ds \geq 2\|y+b_0w\| \geq 2\|y\|.
\end{aligned}$$

So, we obtain

$$\|y\| \leq x(c)/2 \leq \|x\|/2. \tag{12}$$

By (11) and (12), we obtain $\|x\| \leq \|y\|/2 \leq \|x\|/4$, which is a contradiction, because $\|x\| > 0$. Then, for a_0 and b_0 sufficiently large, our problem $(S)-(BC)$ has no positive solution. □

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References

[1] J.R. Graef, J. Henderson, B. Yang, *Positive solutions of a nonlinear higher order boundary-value problem*, Electron. J. Differ. Equ., **45** (2007), 1-10.

[2] J. Henderson, R. Luca, *Positive solutions for a system of higher-order multi-point boundary value problems*, Comput. Math. Appl., **62** (2011), 3920-3932.

[3] J. Henderson, R. Luca, *Positive solutions for a system of second-order multi-point boundary value problems*, Appl. Math. Comput., **218** (2012), 6083-6094.

[4] J. Henderson, R. Luca, *Existence and multiplicity for positive solutions of a multi-point boundary value problem*, Appl. Math. Comput., **218** (2012), 10572-10585.

[5] J. Henderson, R. Luca, *Positive solutions for a system of second-order multi-point discrete boundary value problems*, J. Difference Equ. Applic., DOI: 10.1080/10236198.2011.582868.

[6] J.Henderson, R. Luca, *Existence and multiplicity for positive solutions of a second-order multi-point discrete boundary value problem*, J. Difference Equ. Appl., DOI: 10.1080/10236198.2011.648187.

[7] J. Henderson, R. Luca, *On a system of higher-order multi-point boundary value problems*, Electron. J. Qual. Theory Diff. Equ., in press.

[8] J. Henderson, R. Luca, *Existence and multiplicity for positive solutions of a system of higher-order multi-point boundary value problems*, to appear.

[9] J. Henderson, S.K. Ntouyas, *Positive solutions for systems of nth order three-point nonlocal boundary value problems*, Electron. J. Qual. Theory Differ. Equ., **18** (2007), 1-12.

[10] J. Henderson, S.K. Ntouyas, *Positive solutions for systems of nonlinear boundary value problems*, Nonlinear Stud., **15** (2008), 51-60.

[11] J. Henderson, S.K. Ntouyas, *Positive solutions for systems of three-point nonlinear boundary value problems*, Aust. J. Math. Anal. Appl., **5** (2008), 1-9.

[12] J. Henderson, S.K. Ntouyas, I.K. Purnaras, *Positive solutions for systems of three-point nonlinear discrete boundary value problems*, Neural Parallel Sci. Comput., **16** (2008), 209-224.

[13] J. Henderson, S.K. Ntouyas, I. Purnaras, *Positive solutions for systems of generalized three-point nonlinear boundary value problems*, Comment. Math. Univ. Carolin., **49** (2008), 79-91.

[14] J. Henderson, S.K. Ntouyas, I.K. Purnaras, *Positive solutions for systems of m-point nonlinear boundary value problems*, Math. Model. Anal., **13** (2008), 357-370.

[15] J. Henderson, S.K. Ntouyas, I.K. Purnaras, *Positive solutions for systems of nonlinear discrete boundary value problems*, J. Difference Equ. Appl., **15** (2009), 895-912.

[16] V.A. Il'in, E.I. Moiseev, *Nonlocal boundary value problems of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects*, Differ. Equ., **23** (1987), 803-810.

[17] V.A. Il'in, E.I. Moiseev, *Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator*, Differ. Equ., **23** (1987), 979-987.

[18] Y. Ji, Y. Guo, *The existence of countably many positive solutions for some nonlinear nth order m-point boundary value problems*, J. Comput. Appl. Math., **232** (2009), 187-200.

[19] Y. Ji, Y. Guo, C. Yu, *Positive solutions to* $(n-1, n)$ *m-point boundary value problemss with dependence on the first order derivative*, Appl. Math. Mech., Engl. Ed., **30**(2009), no.4, 527-536.

[20] W.T. Li, H.R. Sun, *Positive solutions for second-order m-point boundary value problems on times scales*, Acta Math. Sin. (Engl. Ser.), **22** (2006), 1797-1804.

[21] R. Luca, *Positive solutions for* $m+1$*-point discrete boundary value problems*, Libertas Math., **XXIX** (2009), 65-82.

[22] R. Luca, *Existence of positive solutions for a class of higher-order m-point boundary value problems*, Electron. J. Qual. Theory Diff. Equ., **74** (2010), 1-15.

[23] R. Luca, *Positive solutions for a second-order m-point boundary value problem*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **18** (2011), 161-176.

[24] R. Luca, *Existence of positive solutions for a second-order* $m+1$ *point discrete boundary value problem*, J. Difference Equ. Appl., **18** (2012), 865-877.

[25] M. Moshinsky, *Sobre los problemas de condiciones a la frontiera en una dimension de caracteristicas discontinuas*, Bol. Soc. Mat. Mexicana, **7** (1950), 1-25.

[26] H. Su, Z. Wei, X. Zhang, J. Liu, *Positive solutions of n-order and m-order multi-point singular boundary value system*, Appl. Math. Comput., **188** (2007), 1234-1243.

[27] S. Timoshenko, *Theory of elastic stability*, McGraw-Hill, New York, 1961.