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ON AN EXTREMAL PROBLEM IN ANALYTIC SPACES IN TWO SIEGEL DOMAINS IN C^N

Romi F. Shamoyan

Department of Mathematics, Bryansk State Technical University, Bryansk, Russia

rshamoyan@yahoo.com

Abstract New sharp estimates concerning distance function in certain Bergman -type spaces of analytic functions in a certain Siegel domain of first type are obtained. Related sharp new estimates for more general Siegel domains of second type are also provided. For Siegel domains of second type in C^n these are the first results of this type.

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1. INTRODUCTION

In this paper we obtain sharp distance estimates in certain spaces of analytic functions in Siegel domains of first type and of second type. These types of domains are known in literature.They have been studied by many authors during last decades (see for example [10], [6], [7], [24] and references therein). In connection with the study of authomorphic functions of several complex variables, the notion of Siegel domains of the first and second type was introduced by Piatetskii-Shapiro [10], [24]. We recall basic facts that relate them to some well-known domains. The Siegel domain of first type is a particular case of a Siegel domain of second type [10]. In particular there is a Siegel domain isomorphic to unit ball of C^{m+1}, in addition, the simplest case of one dimensional Siegel domain of the first type is the upperhalfspace C_+. Note that our results below were already proved in this case in [21]. Next the Siegel domain of first type is a special type of a actively studied recently general tube domains over symmetric cones (see [19] and various references there concerning tube domains). But there are homogeneous Siegel domains of second type which are not even symmetric domains [10], [24]. Tube domains may be also viewed as special cases of Siegel domains of second type. It is known that every bounded homogeneous domain in C^n can be realized as Siegel domain of the first and the second type and that this realization is unique up to affine transformations. Siegel domains are holomorphically equivalent to a bounded domains. But there is a lot of bounded domains that are not holomorphically isomorphic to Siegel domains [10]. We will provide definitions of Siegel domains of first type and more general of second type below referring also to [24] (see also, for example, [10]).

Our line of investigation in this work can be considered as direct continuation of our previous papers on extremal problems [20], [22] and [23]. Our main two new results are contained in the second and third sections of this note. First we provide a concrete special example of a Siegel domain of first type and we obtain a sharp estimate for distance function in certain Bergman type analytic spaces on it. Next we turn in our final section to Siegel domains of the second type. We remark that here for the first time in literature we consider this extremal problem related with distance estimates in spaces of analytic functions in Siegel domains of second type. The next two sections partially also contain some required preliminaries on analysis on these domains.

In the upperhalfspace C_+ which is one dimensional tubular domain and also in general tubular domains our theorems are not new and they were obtained recently in [21], and then in general form in [19]. Moreover arguments in proofs we provided below are similar to those we have in previous cases and hence our arguments sometimes will be sketchy below. The main tool of the proof is again the so-called Bergman reproducing formula, but in Siegel domains (see, for example, [5], [6] for this integral representation and it is applications). This paper first deals with a concrete example of Siegel domain of first type and based on some results from [5] we present a sharp result in this direction. But then in the final part we turn to more general situation (see [6] for notation which will be constantly needed in this second part) and we obtain some related estimates for distances there also. Note again some results from [6] are crucial here in last section for us.

We now shortly remind the history of this extremal problem.

After the appearance of [26] various papers appeared where arguments which can be seen in [26] were extended and changed in various directions [22], [23], [20].

In particular in the mentioned papers various new results on distances for analytic function spaces in higher dimension (unit ball and polydisk) were obtained. Namely new results for large scales of analytic mixed norm spaces in higher dimension were proved.

Later several new sharp results for harmonic function spaces of several variables in the unit ball and upperhalfplane of Euclidean space were also obtained (see, for example, [20] and references therein). The classical Bergman representation formula in various domains serves as a base in all these papers in proofs of main results. Recently (see[18]) concrete analogues of our theorems were proved also in some spaces of entire functions of one and several variables. Various other extremal problems in analytic function spaces also were considered before in various papers (see for example [1], [15], [16], [14]). In those just mentioned papers other results around this topic and some applications of certain other extremal problems can be found also.

2. NEW SHARP ESTIMATES FOR DISTANCES IN ANALYTIC BERGMAN -TYPE SPACES IN SIEGEL DOMAINS OF FIRST TYPE

This section is devoted to one of the main results of this paper. We remark that our notes, namely this one and [19], are first papers with sharp results on extremal problems in higher dimension in C^n, namely in analytic function spaces in Siegel domains in C^n. We now establish some notation from [5] which will be needed for us. Let $\Omega \subset C^n$ be an open nonempty set. Let $W(\Omega)$ be the set of all weights in Ω. We mean by this a set of all Lebesgue measurable functions acting from Ω to R_+. For each such γ function let $L^2(\Omega, \gamma)$ be the Hilbert space of all f functions from Ω to C so that the quasi-norm $\int_\Omega |f(z)|^2 \gamma(z) dm(z)$ is finite,where $dm(z)$ is a Lebesgue measure on Ω. By $A^2(\Omega)$ we denote the analytic subspace of this space but for so -called special (see [5]) admissible γ weights and with the same quasinorm (see [5]). Note that the class of these weights it is a closed subspace of $L^2(\Omega)$. Next according to the well-known Riesz representation theorem there is a unique function that for all functions from this space a certain integral representation holds with a certain function called Bergman kernel which is from $L^2(\Omega)$ (see [5]and references therein). In certain cases and our case is of them in higher dimension this function called Bergman kernel can be explicitly written. This last fact alone already opens a large way for various investigations in this research area. In the present paper first we look at the family of the following admissible weights $\gamma_\alpha(\tau)$,

$$\gamma_\alpha(\tau) = (\Im\tau_1 - |\tilde{\tau}|^2)^\alpha$$

$\alpha > -1$ on the concrete Siegel domain of the first type (see [5])

$$\Omega = \left\{\tau \in C^n, \Im\tau_1 > |\tilde{\tau}|^2\right\},$$

where we denote by τ and $\tilde{\tau}$ the vectors $\tau = (\tau_1, \ldots, \tau_n)$,$\tilde{\tau} = (\tau_2, \ldots, \tau_n)$. Let w be a vector from C^n. Let also $dm_\beta(w) = (\Im w_1 - |\tilde{w}|^2)^\beta dm(w)$,where $dm(w)$ is a Lebesgue measure on R^{2n} and we also define a Bergman kernel as see[5]

$$B_\beta(\tau, w) = (\tau - \overline{w})^{-\beta-n-1} = (u - 2v)^{-n-1-\beta}$$

$u = i(\overline{\tau_1} - w_1)$,$v = (\tilde{w}\tilde{\tau})$, where the last expression is as usual a scalar product of two vectors in C^{n-1}. These definitions are crucial for our paper. The goal of this section to develop further some ideas from our recent already mentioned papers and to present a new sharp theorem in mentioned Siegel domain of first type .

For formulation of our result we will now need various standard definitions from the theory of these Siegel domains of first type (see [10], [5]).

Let Ω be the Siegel domain . $\mathcal{H}(\Omega)$ denotes the space of all holomorphic functions on Ω. Let further, for all positive β,

$$A^\infty_\beta(\Omega) = \left\{F \in \mathcal{H}(\Omega) : \|F\|_{A^\infty_\beta} = \sup_{x+iy\in\Omega} |F(x+iy)|\gamma_\beta(x+iy) < \infty\right\}, \qquad (1)$$

(we use in this paper the following notation $w = u + iv$ and $z = x + iy$, $w \in \Omega$, $z \in \Omega$). It can be checked that this is a Banach space.

For $1 \leq p < +\infty$, $\alpha > -1$ we denote by $A^p_\alpha(\Omega)$ the weighted Bergman space consisting of analytic functions f in Ω such that

$$\|F\|_{A^p_\alpha} = \left(\int_\Omega |F(z)|^p \gamma_\alpha(z) dm(z)\right)^{1/p} < \infty.$$

This is a Banach space. Below we will restrict ourselves to $p = 2$ case following [5]. Replacing above A by L we will get as usual the corresponding larger space $L^2_\nu(\Omega)$ of all measurable functions in our domain Ω with the same quasi-norm (see [5]). The (weighted) Bergman projection P_β is the orthogonal projection from the Hilbert space $L^2_\nu(\Omega)$ onto its closed subspace $A^2_\nu(\Omega)$ and it is given by the following integral formula (see [5])

$$P_\beta f(z) = C_\beta \int_\Omega B_\beta(z, w) f(w) dm_\beta(w), \tag{2}$$

where C_β is a special constant (see [5]) and $\beta > \frac{\nu-1}{2}$. For these values of β this is a bounded linear operator from L^2_ν to A^2_ν. Hence, by using these facts, we have that for any analytic function from $A^2_\nu(\Omega)$ the following integral formula is valid for all functions from A^2_ν, for all $\beta, \beta > \frac{\nu-1}{2}$ and $\nu > -1$ (see[5])

$$f(z) = C_\beta \int_\Omega B_\beta(z, w) f(w) dm_\beta(w). \tag{3}$$

In this case sometimes below we say simply that the analytic f function allows Bergman representation via Bergman kernel with β index.

We need also the following estimate (A) of Bergman kernel from [5]. Let $t > -1$ and $\beta > 0$. Then there is a positive constant $c = c_{n,t,\beta}$ so that

$$\int_\Omega \gamma_t(\tau) |B_{t+\beta}(\tau, w)| dm(\tau) \leq c\gamma_\beta^{-1}(w),$$

$w \in \Omega$. This estimate of Bergman kernel will be used and not once below during the proof of our first theorem.

Note here also these assertions we just mentioned have direct analogues in simpler cases of analytic function spaces in unit disk, polydisk, unit ball, upperhalfspace C_+ and in spaces of harmonic functions in the unit ball or upperhalfspace of Euclidean space R^n. These classical facts are well- known and can be found, for example, in some items in references (see, for example, [26], [9]).

Above and throughout the paper we write C (sometimes with indexes) to denote positive constants which might be different each time we see them (and even in a chain of inequalities), but is independent of the functions or variables being discussed.

As in case of analytic functions in unit disk, polydisk, unit ball, and upperhalfspace C_+, and tubular domains over symmetric cones, and in case of spaces of harmonic functions in Euclidean space, [26], [21], [20], [22], [23], the role of the Bergman representation formula and estimates for Bergman kernel are crucial in these issues related with our extremal problem and our proof will be heavily based on them.

And as it was mentioned already above a variant of Bergman representation formula is available also in Bergman- type analytic function spaces in Siegel domains and this known fact (see [5], [24], [6]), which is crucial in various problems in analytic function spaces in Siegel domains of both types is also used in our proof below.

Moreover will also need for our proof the following additional facts on integral representation of functions on these Ω domains which follows from assertions we already formulated above. Note first that for all functions from A^∞_α the integral representations of Bergman we mentioned above with Bergman kernel

$$B_\nu(z,w)$$

(with ν index) is valid for large enough ν. This follows directly from the fact that A^∞_α for any α is a subspace of A^2_τ if τ is large enough [5]. Moreover it can be easily shown that we have a continuous embedding $A^2_\alpha \hookrightarrow A^\infty_\beta$ (see, for example, [5] where the proof can be found also) for a concrete β depending on α, $\alpha > -1$ and this naturally leads to a problem of estimating

$$\mathrm{dist}_{A^\infty_\beta}(f, A^2_\alpha)$$

for a given $f \in A^\infty_\beta$,where $\beta = \frac{\alpha+n+1}{2}$, $\alpha > -1$.

This problem on distances we just formulated will be solved in our next theorem below, which is one of the main results of this section. Let us set, for $f \in \mathcal{H}(\Omega)$, $s > 0$ and $\epsilon > 0$ and $z = x + iy \in \Omega$,

$$N_{\epsilon,s}(f) = \{z \in \Omega : |f(z)|\gamma_s(z) \geq \epsilon\}. \tag{4}$$

We denote by N_1 and by N_2 two sets- the first one is $N_{\epsilon,s}(f)$, the other one is the set of all those points, which are in tubular domain Ω, but not in N_1. Note now, to clarify the notation for readers again, by $m(z)$ or by m with only one lower index we denote in this section the Lebesgue measure on R^{2n}.

Theorem 2.1. *Let $t = \frac{\nu+1+n}{2}$. Set, for $f \in A^\infty_{\frac{n}{2}+\frac{\nu+1}{2}}$, $\nu > -1$*

$$l_1(f) = \mathrm{dist}_{A^\infty_{\frac{n}{2}+\frac{\nu+1}{2}}}(f, A^2_\nu), \tag{5}$$

$$l_2(f) = \inf\left\{\epsilon > 0 : \int_\Omega \left(\int_{N_{\epsilon,t}(f)} \frac{\gamma_{\beta-t}(w)dw}{(z-\overline{w})^{\beta+n+1}}\right)^2 \gamma_\nu(z)dm(z) < \infty\right\}. \tag{6}$$

Then there is a positive number β_0, so that for all $\beta > \beta_0$, we have $l_1(f) \asymp l_2(f)$.

Proof. We will start the proof with the following observation, which already was mentioned above. By our arguments before formulation of this theorem for all functions from $A^{\infty}_{\tau_1}$ the integral representations of Bergman with Bergman kernel

$$B_(\tau_2)(z,w)$$

is valid for large enough τ_2.

We denote below the double integral which appeared in formulation of theorem by $G(f)$ and we will show first that $l_1(f) \leq Cl_2(f)$. We assume now that $l_2(f)$ is finite.

We use the Bergman representation formula which we provided above, namely(3), and using conditions on parameters we now have the following equalities.

First we have obviously by the remark from which we started this proof that for large enough β

$$f(z) = C_\beta \int_\Omega B_\beta(z,w) f(w) dm_\beta(w) = f_1(z) + f_2(z),$$

$$f_1(z) = C_\beta \int_{N_2} B_\beta(z,w) f(w) dm_\beta(w,$$

$$f_2(z) = C_\beta \int_{N_1} B_\beta(z,w) f(w) dm_\beta(w).$$

Then we estimate both functions separately using estimate (A) provided above and following some arguments we provided in one dimensional case that is the case of upperhalfspace C_+ [21]. Here our arguments are sketchy since they are parallel to arguments from [21]. Using definitions of N_1 and N_2 above after some calculations following arguments from [21] using the estimate (A) of Bergman kernel we mentioned above we will have immediately

$$f_1 \in A^{\infty}_{\frac{\nu+n+1}{2}}$$

and

$$f_2 \in A^2_\nu.$$

We easily note the last inclusion follows directly from the fact that l_2 is finite.

Moreover it can be easily seen that the norm of f_1 can be estimated from above by $C\epsilon$, for some positive constant C ([21]), since obviously

$$\sup_{N_2} |f(w)| \gamma_t(w) \leq \epsilon.$$

Note this last fact follows directly from the definition of N_2 set and estimate (A) above which leads to the following inequality

$$\int_\Omega \gamma_{\beta-t}(\tau) |B_\beta(\tau,w)| dm(\tau) \leq C\gamma_t^{-1}(w),$$

$w \in \Omega$ for all β so that $\beta > \beta_0$, for some large enough fixed β_0 which depends on n, ν, where

$$t = (\frac{1}{2})(\nu + 1 + n).$$

This gives immediately one part of our theorem. Indeed, we have now obviously

$$l_1 \leq C_2 \|f - f_2\|_{A_t^\infty} = C_3 \|f_1\|_{A_t^\infty} \leq C_4 \epsilon.$$

It remains to prove that $l_2 \leq l_1$. Let us assume $l_1 < l_2$. Then there are two numbers ϵ and ϵ_1,both positive such that there exists f_{ϵ_1} ,so that this function is in A_ν^2 and $\epsilon > \epsilon_1$ and also the condition

$$\|f - f_{\epsilon_1}\|_{A_t^\infty} \leq \epsilon_1$$

holds and $G(f) = \infty$, where G is a double integral in formulation of theorem in l_2 (see (6)).

Next from

$$\|f - f_{\epsilon_1}\|_{A_t^\infty} \leq \epsilon_1$$

we have the following two estimates,the second one is a direct corollary of first one. First we have

$$(\epsilon - \epsilon_1)\tau_{N_{\epsilon,t}}(z)\gamma_t^{-1}(z) \leq C|f_{\epsilon_1}(z)|$$

,where $\tau_{N_{\epsilon,t}}(z)$ is a characteristic function of $N = N_{\epsilon,t}(f)$ set we defined above.

And from last estimate we have directly multiplying both sides by Bergman kernel $B_\beta(z, w)$ and integrating by tube Ω both sides with measure dm_β

$$G(f) \leq C \int_\Omega (L(f_{\epsilon_1}))^2 \gamma_\nu(z) dm(z),$$

where

$$L = L(f_{\epsilon_1}, z)$$

and

$$L(f_{\epsilon_1}, z) = \int_\Omega |f_{\epsilon_1}(w)||B_\beta(z, w)| dm_\beta(w).$$

Denote this expression by I.

Put $\beta + n + 1 = k_1 + k_2$, where $k_1 = \beta + 1 - n - \mu$, $k_2 = \mu + 2n(\frac{1}{2} + \frac{1}{2})$ where the additional parameter will be chosen by us later.

By classical Hölder inequality we obviously have

$$I^2 \leq C I_1 I_2,$$

where

$$I_1(f) = \int_\Omega |f_1(z)|^2 |(z - \overline{w})^s| \gamma_{2\beta}(z) dm(z),$$

$$I_2 = \int_\Omega |(z - \overline{w})^v| dm(z)$$

and where $f_1 = f_{\epsilon_1}$ and

$$s = 2\mu - 2 - 2\beta,$$
$$v = -2n - 2\mu.$$

Choosing finally the μ parameter, so that the estimate (A) namely

$$\int_\Omega \gamma_{\tilde{t}}(\tau)|B_{\tilde{t}+\tilde{\beta}}(\tau, w)|dm(\tau) \leq C\gamma_{\tilde{\beta}}^{-1}(w),$$

$$w \in \Omega$$

can be used twice above with some restrictions on parameters and finally making some additional easy calculations we will get what we need.

Indeed we have now obviously,

$$\int_\Omega (\int_\Omega |f_{\epsilon_1}(z)|B_\beta(z, w)|dm_\beta(z))^2 \gamma_\nu(w)dm(w) \leq C\|f_{\epsilon_1}\|^2_{A^2_\nu}$$

and

$$G(f) \leq C\|f_{\epsilon_1}\|_{A^2_\nu},$$

but we also have

$$f_{\epsilon_1} \in A^2_\nu.$$

This is in contradiction with our previous assumption above that $G(f) = \infty$. So we proved the estimate which we wanted to prove. The proof of our first theorem in Siegel domains of first type is now complete. ∎

3. NEW ESTIMATES FOR DISTANCES IN BERGMAN TYPE SPACES IN SIEGEL DOMAINS OF THE SECOND TYPE

We first recall some basic facts on Siegel domains of second type and then establish notations for our second main theorem. Recall first the explicit formula for the Bergman kernel function is known for very few domains.The explicit forms and zeros of the Bergman kernel function for Hartogs domains and Hartogs type domains (Cartan-Hartogs domains) were found only recently [2]. On the other hand in strictly pseudoconvex domains the principle part of the Bergman kernel can be expressed explicitly by kernels closely related to so-called Henkin -Ramirez kernel see for example [11] and references there. In [10] the Bergman kernel

$$b\left((\tau_1, \tau_2), (\tau_3, \tau_4)\right)$$

for the Siegel domain of the second type was computed explicitly. It is an integral via V^* a convex homogeneous open irreducible cone of rank l in R^n, a conjugate

cone of V cone and which also contains no straight line and in that integral the fixed Hermitian form from definition of D Siegel domain(see below for definition) participates (see details for this [6]). This fact was heavily used in [6] in solutions of several classical problems in Siegel domains of the second type and we will also use one estimate from [6] for this kernel, but we define it otherwise ,representing it otherwise in this paper (see also [6]). We will need now some short, but more concrete review of certain results from [6] to make this exposition more complete. To be more precise the authors in [6] showed that on homogeneous Siegel domain of type 2 under certain conditions on parameters the subspace of a weighted L^p space for all positive p consisting of holomorphic functions is reproduced by a concrete weighted Bergman kernel which we just mentioned. They also obtain some L^p estimates for weighted Bergman projections in this case. The proof relies on direct generalization of the Plancherel-Gindikin formula for the Bergman space A^2 (see[10]). We remind the reader that the Siegel domain of type 2 associated with the open convex homogeneous irreducible cone V of rank l which contains no straight line,$V \in R^n$,and a V-Hermitian homogeneous form F which act from product of two C^m into C^n is a set of points (w, τ) from C^{m+n} so that the difference D of $\Im w$ and the value of F on (τ, τ) is in V cone. This domain is affine homogeneous and we now should recall the following expression for the Bergman kernel of $D = D(V, F)$. Let D be an affine-homogeneous Siegel domain of type 2. Let $dv(z)$ denote the Lebesgue measure on D and let $H(D)$ denote the space of all holomorphic functions on D. The Bergman kernel is given by the following formula (see[6]) for $(\tau_1, \tau_2) \in D$ and $(\tau_3, \tau_4) \in D$

$$b\left((\tau_1, \tau_2), (\tau_3, \tau_4)\right) = (\frac{\tau_1 - \overline{\tau}_3}{2i} - (F(\tau_2, \tau_4))^{2d-q},$$

where two vectors $q = (q_i)$ and $d = (d_i)$ and in addition $n = (n_i)$ here the i index is running from 1 to l are specified via $n_{i,k}$, where these $n_{i,k}$ numbers are dimensions of certain $(R_{i,k})$ and $(C_{i,j})$ subspaces of the certain canonical decomposition of C^{m+n} and R^n via the V cone from definition of our D domain (see for some additional details about this [10] and [6]). We will call this family of triples parameters of a Siegel domain D of second type. They will appear in our main theorem and it is short proof. The standard Bergman projection here on D as usual is denoted by P, it is the orthogonal projection of $L^2(D, dv)$ onto it is analytic subspace $A^2(D, dv)$ consisting of all holomorphic functions. The authors in [6] showed that some well-known facts of much simpler domains holds also here, for example there is an integral operator on L^2 space defined by the certain $b(\tau, z)$ Bergman kernel. And for this types of Siegel domains as it was mentioned above this Bergman kernel was computed explicitly previously in [10]. Further, let ϵ be a real number. Now for all positive finite p we define a space of integrable functions (weighted L^p spaces with $b^{-\epsilon}(z, z)$ weights) for all $\epsilon > \epsilon_0$

$$L^{p,\epsilon}(D) = L^p(D, b^{-\epsilon}(z, z)dv(z))$$

and we denote as usual by $A^{p,\epsilon}$ the analytic subspace of this space with usual modification when $p = \infty$. Note the restriction is meaningful since there is an ϵ_0 so that for all those ϵ which are smaller than this fixed ϵ_0 the $A^{2,\epsilon}$ is an empty class (see [6]). We denote by P_ϵ the corresponding Bergman projection which is the orthogonal projection of $L^{2,\epsilon}$ to it is analytic subspace $A^{2,\epsilon}$. In [6] the authors give a condition on real numbers and vectors r, p, ϵ, so that the weighted Bergman projection reproduces all functions in $A^{p,r}(D)$.This vital property for our theorem was partly deduced by them from Plancherel-Gindikin formula and the fact that

$$P_\epsilon(f)(z) = c_\epsilon \int_D f(u) b^{1+\epsilon}(z,u) b^{-\epsilon}(u) du,$$

so it defines as in simpler cases an integral operator on $L^{2,\epsilon}(D)$ by the kernel $b^{1+\epsilon}(\tau, z)$ (see for this [6]), it is a weighted Bergman projection from $L^{2,\epsilon}$ onto $A^{2,\epsilon}$ (see, for example, [6] and references therein.).

The following several assertions concerning Bergman projection acting in analytic spaces in Siegel domain of the second type and estimates of Bergman kernel which we mentioned above and in addition to this some facts on spaces of integrable functions and their analytic subspaces we defined above on these Siegel domains were proved in [6] and some are crucial for this paper. We will formulate immediately after them our main result on distances in Siegel domains of the second type. Then providing a comment on a proof of that assertion which contains no new ideas when we compare it with the proof of previous theorem we will finish this paper. We use the following notation. The i index below is running from 1 to l everywhere and to make the reading easier we accept this from advance. We also use below everywhere standard rules of calculations between two vectors as they were seen by us for example in [6], also sometimes we write

$$d\tilde{V}(\tau_1, \tau_2)$$

not $dv(\tau)$ meaning

$$\tau = (\tau_1, \tau_2) \in D.$$

In the following assertions

$$(n_i), (q_i), (d_i)$$

will always act as parameters of the Siegel domain D we introduced above and they are playing a crucial role. We write always D below meaning

$$D(n, q, d)$$

,where $n = (n_i)$, $d = (d_i)$, $q = (q_i)$. We write $c_i \leq b_i$ for two vectors from R^l below meaning as usual that this is true for all values of i from 1 to l. If $c \leq bi$ (or $c < b_i$) then all b_i are bigger or equal (or bigger)than c.

Proposition 3.1. *Let $\epsilon \in R^l$, $r \in R^l$, $p \in R_+$, $0 \leq r_j$. Then there are two sets of numbers (k_i), (m_i) depending on parameters of D Siegel domain so that if $1 \leq p < k_i$ and $\epsilon_i > m_i$, then*

$$P_\epsilon f = f$$

for all

$$f \in A^{p,r}(D).$$

Let $\epsilon \in R^l$, $r \in R^l$, $p \in (0, \infty)$, $v_i < r_i$, for some v_i numbers depending on parameters of D domain. Then there are two sets of numbers (k_i^1),(m_i^1), depending from parameters of D Siegel domain, so that if

$$0 < p < k_i^1$$

and if $\epsilon_i > m_i^1$, then

$$P_\epsilon f = f$$

for all

$$f \in A^{p,r}(D).$$

Proposition 3.2. *If*

$$\epsilon_i > \frac{n+2}{2(2d-q)_i}$$

where $\epsilon \in R^l$, then P_ϵ is an integral operator with

$$b^{1+\epsilon}(t_1, z_1)(t_2, z_2)$$

kernel on $L^{2,\epsilon}$ and

$$P_\epsilon f = f$$

for all $f \in A^{p,0}(D)$, when $p \in (0, p_0)$, where

$$p_0 \leq \frac{n_i - 2(2d-q)_i}{n_i}.$$

If there is an index i so that

$$2\epsilon_i \leq \frac{n_i + 2}{(2d-q)_i}$$

then we have $A^{2,\epsilon} = 0$, moreover if the reverse estimate holds for all i and ϵ_i instead of $2\epsilon_i$ then the intersection of $A^{2,\epsilon}$ and $A^{p,r}$ is dense in $A^{p,r}$, if $1 \leq p < \infty$, $0 \leq r_i$, $\epsilon \in R^l$.

The following embedding which is taken also from [6] is important for us. It allows us as in previous simpler case to pose a distance problem in this domain showing that

Bergman spaces $A^{p,r}$ are subspaces of $A^{\infty}_{\frac{1+r}{p}}$ Bergman-type spaces. Let $r \in R^l$ and $p \in (0, \infty)$, then

$$|f(z)|^p \leq Cb^{1+r}(z,z)\|f\|^p_{p,r}, \quad z \in D.$$

Further let ϵ and r are from R^l. If

$$\epsilon_i > \frac{n_i}{-2(2d-q)_i}$$

and

$$r_i > \frac{n_i+2}{2(2d-q)_i} + \epsilon_i,$$

then we have

$$P_r(f) = f$$

as soon as f belongs to A^{∞}_{ϵ}(see[6],[13]). This will also be needed in the proofs of main result of this section (see for this also the parallel proof of our previous theorem from previous section).

Proposition 3.3. *Let $\beta \in R^l$ and all β_i are nonnegative then the following estimate holds*

$$b^{\beta}(z+\tau, z+\tau) \leq b^{\beta}(z,z)$$

and also

$$|b^{\beta}(\tau, z)| \leq b^{\beta}(z,z)$$

for all τ and z from D.

The following estimate to be more precise it is direct analogue can be found in the proof of previous theorem where it was used three times.

Proposition 3.4. *Let α and ϵ be two vectors from R^l and (τ, z) be a point of D. Then if*

$$\frac{n_i+2}{2(2d-q)_i} < \epsilon_i$$

and

$$\epsilon_i - \frac{n_i}{2(2d-q)_i} < \alpha_i,$$

then the integral

$$\int_D |b^{\alpha+1}((\tau,v),(z,u))|b^{-\epsilon}((z,u)(z,u))d\tilde{V}(z,v)$$

is equal with

$$c_{\alpha,\epsilon}b^{\alpha-\epsilon}((\tau,v),(\tau,v)).$$

We are able now based only on last proposition and two comments concerning integral representations before previous proposition to formulate a theorem on distances

in Siegel domains of the second type which is a direct analogue of our previous results (see, for example, [22], [23], [21]) and our previous theorem on distances in this situation. All facts and preliminaries which are needed here for our proof can be found above in assertions from [6] which we just formulated, all lines of arguments for our proof of this theorem can be also found above in the proof of our previous theorem though some not very long technical calculations with indexes should be added. Note that one implication in this theorem below is easier and we just repeat here arguments of our previous theorem.

Theorem 3.1. *Let*

$$N_{\tilde{\epsilon},r}(f) = \left\{z \in D, |f(z)|b^{1+r}(z,z) > \tilde{\epsilon}\right\},$$

where $\tilde{\epsilon}$ is a positive number. Then the following two quantities are equivalent

$$\mathrm{dist}_{A^{\infty}_{1+r}}(f, A^{1,r})$$

and

$$\inf\left\{\tilde{\epsilon} > 0, \int_D (\int_{N_{r,\tilde{\epsilon}}(f)} b^{-k+1+r}(\tau,\tau)|b(\tau,z)|^{k+1} dv(\tau)) b^{-r}(z,z) dv(z) < \infty\right\},$$

for all r and k so that $r \in (r_0, \infty)$ and $k \in (k_0, \infty)$ and for certain fixed vectors r_0 and k_0 depending on parameters of the Siegel D domain (d_i) and (q_i) and (n_i).

We finally remark that the theorem above is probably valid for all $p > 1$ (not only $p = 1$ when calculations are simpler) and the reader can formulate easily that theorem in general case following the formulation of our previous theorem. The proof probably is parallel to the proof of previous theorem and it is based on estimates from propositions above. Note also our assertion is true for all homogeneous Siegel domains not only symmetric Siegel domains of the second type (see [6], [8], [12]). We remark as r_0 we can take $\max(r_1, r_2, 0)$ where r_1 and r_2 are depending on parameters of domain $r_1 = \frac{n_i+2}{2(2d-q)_i}$ and $r_2 = \frac{-n_i}{2(2d-q)_i} - 1$.

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