ROMAI J., v.8, no.2(2012), 189–199

FUNCTIONAL CONTRACTIONS IN LOCAL BRANCIARI METRIC SPACES

Mihai Turinici

"A. Myller" Mathematical Seminar; "A. I. Cuza" University; Iaşi, Romania

mturi@uaic.ro

Abstract A fixed point result is given for a class of functional contractions over local Branciari metric spaces. It extends some contributions in the area due to Fora et al [Mat. Vesnik, 61 (2009), 203-208].

Keywords: Symmetric, polyhedral inequality, local Branciari metric, convergent/Cauchy sequence, Matkowski function, contraction, periodic and fixed point.

2010 MSC: 47H10 (Primary), 54H25 (Secondary).

1. INTRODUCTION

Let X be a nonempty set. By a *symmetric* over X we shall mean any map $d : X \times X \to R_+$ with (cf. Hicks [11])

(a01) $d(x, y) = d(y, x),\ \forall x, y \in X$ (d is *symmetric*);

the couple (X, d) will be referred to as a *symmetric space*. Further, let $T : X \to X$ be a selfmap of X, and put Fix(T) $= \{z \in X; z = Tz\}$; any such point will be called *fixed* under T. According to Rus [24, Ch 2, Sect 2.2], we say that $x \in X$ is a *Picard point* (modulo (d, T)) if **1a)** $(T^n x; n \geq 0)$ is d-convergent, **1b)** each point of $\lim_n(T^n x)$ is in Fix(T). If this happens for each $x \in X$, then T is referred to as a *Picard operator* (modulo d); if (in addition) Fix(T) is a *singleton* ($x, y \in$ Fix$(T) \Longrightarrow x = y$), then T is called a *global Picard operator* (modulo d). [We refer to Section 2 for all unexplained notions]. Sufficient conditions for these properties to be valid require some additional conditions upon d; the usual ones are

(a02) $d(x, y) = 0$ iff $x = y$ (d is *reflexive sufficient*)

(a03) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$ (d is *triangular*);

when both these hold, d is called a *(standard) metric* on X. In this (classical) setting, a basic result to the question we deal with is the 1922 one due to Banach [3]; it says that, whenever (X, d) is complete and (for some λ in $[0, 1[$)

(a04) $d(Tx, Ty) \leq \lambda d(x, y),\ \forall x, y \in X,$

then, T is a global Picard operator (modulo d). This result found various applications in operator equations theory; so, it was the subject of many extensions. A natural way of doing this is by considering "functional" contractive conditions like (a05) $d(Tx, Ty) \leq F(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)), \quad \forall x, y \in X$;

where $F : R_+^5 \to R_+$ is an appropriate function. For more details about the possible choices of F we refer to the 1977 paper by Rhoades [23]; see also Turinici [28]. Another way of extension is that of conditions imposed upon d being modified. For example, in the class of symmetric spaces, a relevant paper concerning the contractive question is the 2005 one due to Zhu et al [29]. Here, we shall be interested in fixed point results established over *generalized* metric spaces, introduced as in Branciari [5]; where, the triangular property (a03) is to be substituted by the *tetrahedral* one:

(a06) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$,
whenever $x, y, u, v \in X$ are distinct to each other.

Some pioneering results in the area were given by Das [7], Miheţ [21], and Samet [25]; see also Azam and Arshad [2]. In parallel to such developments, certain technical problems involving these structures were considered. For example, Sarma et al [27] observed that Branciari's result may not hold, in view of the Hausdorff property for (X, d) being not deductible from (a06). This remark was followed by a series of results founded on this property being *ab initio* imposed; see, in this direction, Chen and Sun [6] or Lakzian and Samet [18]. However, in 2011, Kikina and Kikina [16] noticed that such a regularity condition is ultimately superfluous for the ambient space; so, the initial setting will suffice for these results being retainable. It is our aim in the present exposition to confirm this, within a class of "local" Branciari metric spaces. Further aspects will be delineated elsewhere.

2. PRELIMINARIES

Let $N = \{0, 1, ...\}$ denote the set of all natural numbers. For each $n \geq 1$ in N, let $N(n, >) := \{0, ..., n-1\}$ stand for the *initial interval* (in N) induced by n. Any set P with $P \sim N$ (in the sense: there exists a bijection from P to N) will be referred to as *effectively denumerable*; also denoted as: $\text{card}(P) = \aleph_0$. In addition, given some $n \geq 1$, any set Q with $Q \sim N(n, >)$ will be said to be *n-finite*; and we write this: $\text{card}(Q) = n$; when n is generic here, we say that Q is *finite*. Finally, the (nonempty) set Y is called (at most) *denumerable* iff it is either effectively denumerable or finite.

(A) Let (X, d) be a symmetric space. Given $k \geq 1$, any ordered system $C = (x_1, ..., x_k)$ in X^k will be called a *k-chain* of X; the class of all these will be re-denoted as $\text{chain}(X; k)$. Given such an object, put $[C] = \{x_1, ..., x_k\}$. If $\text{card}([C]) = k$, then C will be referred to as a *regular k-chain* (in X); denote the class of all these as $\text{rchain}(X; k)$. In particular, any point $a \in X$ may be identified with a regular 1-chain of X. For any $C \in \text{chain}(X; k)$, where $k \geq 2$, denote

$$\Lambda(C) = d(x_1, x_2) + ... + d(x_{k-1}, x_k), \text{ whenever } C = (x_1, ..., x_k)$$

(the "length" of C). Given $h \geq 1$ and the h-chain $D = (y_1, ..., y_h)$ in X, let $(C; D)$ stand for the $(k + h)$-chain $E = (z_1, ..., z_{k+h})$ in X introduced as

$$z_i = x_i,\ 1 \le i \le k;\ \ z_{k+j} = y_j,\ 1 \le j \le h;$$

it will be referred to as the "product" between C and D. This operation may be extended to a finite family of such objects.

Having these precise, let us say that the symmetric d is a *local Branciari metric* when it is reflexive sufficient and has the property: for each effectively denumerable $M \subseteq X$, there exists $k = k(M) \ge 1$ such that

(b01) $d(x, y) \le \Lambda(x; C; y)$, for all $x, y \in M$, $x \ne y$, and all $C \in \mathrm{rchain}(M; k)$, with $(x; C; y) \in \mathrm{rchain}(M; 2 + k)$

(referred to as: the $(2 + k)$-*polyhedral inequality*). Note that, the triangular inequality (a03) and the tetrahedral inequality (a06) are particular cases of this one, corresponding to $k = 1$ and $k = 2$, respectively. On the other hand, (b01) is not reducible to (a03) or (a06); because, aside from $k > 2$ being allowed, the index in question depends on each effectively denumerable subset M of X.

Suppose that we introduced such an object. Define a d-convergence structure over X as follows. Given the sequence (x_n) in X and the point $x \in X$, we say that (x_n), d-converges to x (written as: $x_n \xrightarrow{d} x$) provided $d(x_n, x) \to 0$; i.e.,

(b02) $\forall \varepsilon > 0, \exists i = i(\varepsilon)$: $n \ge i \Longrightarrow d(x_n, x) < \varepsilon$.

(This concept meets the standard requirements in Kasahara [14]; we do not give details). The set of all such points x will be denoted $\lim_n(x_n)$; when it is nonempty, (x_n) is called *d-convergent*. Note that, in this last case, $\lim_n(x_n)$ may be not a singleton, even if (a06) holds; cf. Samet [26]. Further, call the sequence (x_n), *d-Cauchy* when $d(x_m, x_n) \to 0$ as $m, n \to \infty$, $m < n$; i.e.,

(b03) $\forall \varepsilon > 0, \exists j = j(\varepsilon)$: $j \le m < n \Longrightarrow d(x_m, x_n) < \varepsilon$.

Clearly, a necessary condition for this is

$$d(x_m, x_{m+i}) \to 0 \text{ as } m \to \infty, \ \text{ for each } i > 0;$$

referred to as: (x_n) is *d-semi-Cauchy*; but the converse is not in general true. Note that, by the adopted setting, a d-convergent sequence need not be d-Cauchy, even if d is tetrahedral; see the quoted paper for details. Despite of this, (X, d) is called *complete*, if each d-Cauchy sequence is d-convergent.

(B) As already precise, the (nonempty) set of limit points for a convergent sequence is not in general a singleton. However, in the usual (metric) fixed point arguments, the convergence property of this sequence comes from the d-Cauchy property of the same. So, we may ask whether this supplementary condition upon (x_n) will suffice for such a property. Call (X, d), *Cauchy-separated* if, for each d-convergent d-Cauchy sequence (x_n) in X, $\lim_n(x_n)$ is a singleton.

Proposition 2.1. *Assume that d is a local Branciari metric (see above). Then, (X, d) is Cauchy-separated.*

Proof. Let (x_n) be a d-convergent d-Cauchy sequence. Assume by contradiction that $\lim_n(x_n)$ has at least two distinct points:

(b04) $\exists u, v \in X$ with $u \neq v$, such that $x_n \overset{d}{\longrightarrow} u$, $x_n \overset{d}{\longrightarrow} v$.

i) Denote $A = \{n \in N; x_n = u\}$, $B = \{n \in N; x_n = v\}$. We claim that both A and B are finite. In fact, if A is effectively denumerable, then $A = \{n(j); j \geq 0\}$, where $(n(j); j \geq 0)$ is strictly ascending (hence $n(j) \to \infty$ as $j \to \infty$) and $x_{n(j)} = u$, $\forall j \geq 0$. Since, on the other hand, $x_{n(j)} \to v$ as $j \to \infty$, we must have $d(u, v) = 0$; so that, $u = v$, contradiction. An identical reasoning is applicable when B is effectively denumerable; hence the claim. As a consequence, there exists $p \in N$, such that: $x_n \neq u$, $x_n \neq v$, for all $n \geq p$. Without loss, one may assume that $p = 0$; i.e.,

$$\{x_n; n \geq 0\} \cap \{u, v\} = \emptyset \ \ [x_n \neq u \text{ and } x_n \neq v, \ \text{ for all } n \geq 0]. \tag{1}$$

ii) Put $h(0) = 0$. We claim that the set $S_0 = \{n \in N; x_n = x_{h(0)}\}$ is finite. For, otherwise, it has the representation $S_0 = \{m(j); j \geq 0\}$, where $(m(j); j \geq 0)$ is strictly ascending (hence $m(j) \to \infty$ as $j \to \infty$) and $x_{m(j)} = x_0$, $\forall j \geq 0$. Combining with (b04) gives $x_0 = u$, $x_0 = v$; hence, $u = v$, contradiction. As a consequence of this, there exists $h(1) > h(0)$ with $x_{h(1)} \neq x_{h(0)}$. Further, by a very similar reasoning, $S_{0,1} = \{n \in N; x_n \in \{x_{h(0)}, x_{h(1)}\}\}$ is finite too; hence, there exists $h(2) > h(1)$ with $x_{h(2)} \notin \{x_{h(0)}, x_{h(1)}\}$; and so on. By induction, we get a subsequence $(y_n := x_{h(n)}; n \geq 0)$ of (x_n) with

$$y_i \neq y_j, \text{ for } i \neq j; \ \ y_n \overset{d}{\longrightarrow} u, y_n \overset{d}{\longrightarrow} v \text{ as } n \to \infty. \tag{2}$$

The subset $M = \{y_n; n \geq 0\} \cup \{u, v\}$ is effectively denumerable; let $k = k(M) \geq 1$ stand for the natural number assured by the local Branciari metric property of d. From the $(2 + k)$-polyhedral inequality (b01) we have, for each $n \geq 0$,

$$d(u, v) \leq d(u, y_{n+1}) + ... + d(y_{n+k}, v).$$

(The possibility of writing this is assured by (1) and (2) above). On the other hand, (y_n) is a d-Cauchy sequence; because, so is (x_n); hence $d(y_m, y_{m+1}) \to 0$ as $m \to \infty$. Passing to limit in the above relation gives $d(u, v) = 0$; whence, $u = v$, contradiction. So, (b04) is not acceptable; and this concludes the argument. ∎

(B) Let $\mathcal{F}(R_+)$ stand for the class of all functions $\varphi : R_+ \to R_+$. Denote

(b05) $\mathcal{F}_r(R_+) = \{\varphi \in \mathcal{F}(R_+); \varphi(0) = 0; \ \varphi(t) < t, \ \forall t > 0\}$;

each $\varphi \in \mathcal{F}_r(R_+)$ will be referred to as *regressive*. Note that, for any such function,

$$\forall u, v \in R_+ : \ v \leq \varphi(\max\{u, v\}) \Longrightarrow v \leq \varphi(u). \tag{3}$$

Call $\varphi \in \mathcal{F}_r(R_+)$, *strongly regressive*, provided

(b06) $\forall \gamma > 0, \exists \beta \in]0, \gamma[, (\forall t)$: $\gamma \le t < \gamma + \beta \Longrightarrow \varphi(t) \le \gamma$;
or, equivalently: $0 \le t < \gamma + \beta \Longrightarrow \varphi(t) \le \gamma$.

Some basic properties of such functions are given below.

Proposition 2.2. *Let $\varphi \in \mathcal{F}_r(R_+)$ be strongly regressive. Then,*

i) *for each sequence $(r_n; n \ge 0)$ in R_+ with $r_{n+1} \le \varphi(r_n)$, $\forall n$, we have $r_n \to 0$ [we then say that φ is iteratively asymptotic]*

ii) *in addition, for each sequence $(s_n; n \ge 0)$ in R_+ with $s_{n+1} \le \varphi(\max\{s_n, r_n\})$, $\forall n$ we have $s_n \to 0$.*

Proof. i) Let $(r_n; n \ge 0)$ be as in the premise of this assertion. As φ is regressive, we have $r_{n+1} \le r_n$, $\forall n$. The sequence $(r_n; n \ge 0)$ is therefore descending; hence $\gamma := \lim_n(r_n)$ exists in R_+. Assume by contradiction that $\gamma > 0$; and let $\beta \in]0, \gamma[$ be the number indicated by the strong regressiveness of φ. As $r_n \ge \gamma > 0$, $\forall n$ (and φ=regressive), one gets $r_{n+1} < r_n$, $\forall n$; hence, $r_n > \gamma$, $\forall n$. Further, as $r_n \to \gamma$, there exists some rank $n(\beta)$ in such a way that (combining with the above) $n \ge n(\beta) \Longrightarrow \gamma < r_n < \gamma + \beta$. The strong regressiveness of φ then gives (for the same ranks, n) $\gamma < r_{n+1} \le \varphi(r_n) \le \gamma$; contradiction. Consequently, $\gamma = 0$; and we are done.

ii) Let $(r_n; n \ge 0)$ and $(s_n; n \ge 0)$ be as in the premise of these assertions. Denote for simplicity $(t_n := \max\{s_n, r_n\}; n \ge 0)$. For each n, we have $r_{n+1} \le r_n \le t_n$ and (as φ is regressive) $s_{n+1} \le \varphi(t_n) \le t_n$; hence $[t_{n+1} \le t_n, \forall n]$. The sequence $(t_n; n \ge 0)$ is therefore descending; wherefrom, $t := \lim_n(t_n)$ exists in R_+ and $t_n \ge t$, $\forall n$. Assume by contradiction that $t > 0$. As $r_n \to 0$, there must be some rank $n(t)$ such that $n \ge n(t) \Longrightarrow r_n < t$. Combining with the above, one gets $t_n \ge t > r_n$, for all $n \ge n(t)$; whence $t_n = s_n$, for all $n \ge n(t)$. But then, the choice of $(s_n; n \ge 0)$ gives $s_{n+1} \le \varphi(s_n)$, for all $n \ge n(t)$. This, along with the first part of the proof, gives $s_n \to 0$; hence $t_n \to 0$; contradiction. Consequently, $t = 0$; and, from this, the conclusion follows. ∎

Now, let us give two basic examples of such functions.

B1) Suppose that $\varphi \in \mathcal{F}_r(R_+)$ is a *Boyd-Wong* function [4]; i.e.

(b07) $\limsup_{t \to s+} \varphi(t) < s$, for all $s > 0$.

Then, φ is strongly regressive. The verification is immediate, by definition; so, we do not give details.

B2) Suppose that $\varphi \in \mathcal{F}_r(R_+)$ is a *Matkowski function* [20]; i.e.

(b08) φ is increasing and $[\varphi^n(t) \to 0$ as $n \to \infty$, for all $t > 0]$.

(Here, for each $n \ge 0$, φ^n stands for the n-th iterate of φ). Then, φ is strongly regressive. The verification of this assertion is to be found to Jachymski [12]; however, for completeness reasons, we shall provide it, with some modifications. Assume by contradiction that φ is not strongly regressive; that is (for some $\gamma > 0$)

$\forall \beta \in]0, \gamma[$, $\exists t \in [\gamma, \gamma + \beta[$: $\varphi(t) > \gamma$ (hence, $\gamma < t < \gamma + \beta$).

As φ=increasing, this yields $\varphi(t) > \gamma$, $\forall t > \gamma$. By induction, we get $\varphi^n(t) > \gamma$, for all n, and all $t > \gamma$. Fixing some $t > \gamma$, we have (passing to limit as $n \to \infty$) $0 \geq \gamma$, contradiction; hence the claim.

3. MAIN RESULT

Let X be a nonempty set; and $d(.,.)$ be a local Branciari metric over it, with

(c01) (X, d) is complete (each d-Cauchy sequence is d-convergent).

Note that, by Proposition 2.1, for each d-Cauchy sequence (x_n) in X, $\lim_n(x_n)$ is a (nonempty) singleton, $\{z\}$; as usually, we write $\lim_n(x_n) = \{z\}$ as $\lim_n(x_n) = z$.

Let $T : X \to X$ be a selfmap of X. We say that $x \in X$ is a *Picard point* (modulo (d, T)) if **3a)** $(T^n x; n \geq 0)$ is d-Cauchy (hence d-convergent), **ii)** $\lim_n(T^n x)$ is in $\mathrm{Fix}(T)$. If this happens for each $x \in X$, then T is referred to as a *Picard operator* (modulo d); if (in addition) $\mathrm{Fix}(T)$ is a singleton, then T is called a *globally Picard operator* (modulo d).

Now, concrete circumstances guaranteeing such properties involve functional contractive (modulo d) conditions upon T. Precisely, denote for $x, y \in X$:

(c02) $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

It is easy to see that

$$M(x, Tx) = \max\{d(x, Tx), d(Tx, T^2x)\}, \quad \forall x, y \in X. \tag{4}$$

Given $\varphi \in \mathcal{F}_r(R_+)$, we say that T is $(d, M; \varphi)$*-contractive* if

(c03) $d(Tx, Ty) \leq \varphi(M(x, y))$, $\forall x, y \in X$.

The main result of this note is

Theorem 3.1. *Suppose that T is $(d, M; \varphi)$-contractive, where $\varphi \in \mathcal{F}_r(R_+)$ is strongly regressive. Then, T is a globally Picard operator (modulo d).*

Proof. First, we check the singleton property. Let $z_1, z_2 \in \mathrm{Fix}(T)$ be arbitrary fixed. By this very choice,

$$M(z_1, z_2) = \max\{d(z_1, z_2), 0, 0\} = d(z_1, z_2).$$

Combining with the contractive condition yields

$$d(z_1, z_2) = d(Tz_1, Tz_2) \leq \varphi(d(z_1, z_2));$$

wherefrom $d(z_1, z_2) = 0$; hence $z_1 = z_2$; so that, $\mathrm{Fix}(T)$ is (at most) a singleton. It remains now to establish the Picard property. Fix some $x_0 \in X$; and put $x_n = T^n x_0$, $n \geq 0$. There are several steps to be passed.

I) If $x_n = x_{n+1}$ for some $n \geq 0$, we are done. So, it remains to discuss the remaining situation; i.e. (by the reflexive sufficiency of d)

(c04) $\rho_n := d(x_n, x_{n+1}) > 0$, for all $n \geq 0$.

By the contractive property and (4), $\rho_{n+1} \leq \varphi(\max\{\rho_n, \rho_{n+1}\})$, for all $n \geq 0$; so that (taking (3) into account)

$$\rho_{n+1} \leq \varphi(\rho_n), \quad \forall n \geq 0. \tag{5}$$

Combining with (c04) one gets that $(\rho_n; n \geq 0)$ is strictly descending; moreover, by Proposition 2.2, $\rho_n \to 0$ as $n \to \infty$.

II) Fix $i \geq 1$, and put $(\sigma_n^i := d(x_n, x_{n+i}); n \geq 0)$. Again by the contractive condition, we get the evaluation

$$\sigma_{n+1}^i = d(Tx_n, Tx_{n+i}) \leq \varphi(M(x_n, x_{n+i})) = \varphi(\max\{\sigma_n^i, \rho_n, \rho_{n+i}\}), \ \forall n \geq 0;$$

wherefrom, by (5)

$$\sigma_{n+1}^i \leq \varphi(\max\{\sigma_n^i, \rho_n\}), \quad \forall n \geq 0. \tag{6}$$

This yields (again via Proposition 2.2) $\sigma_n^i \to 0$, for each $i \geq 1$; that is,

$$d(x_n, x_{n+i}) \to 0 \text{ as } n \to \infty, \text{ for each } i \geq 1; \tag{7}$$

or, in other words: (x_n) is d-semi-Cauchy.

III) Suppose that

(c05) there exists $i, j \in N$ such that $i < j$, $x_i = x_j$.

Denoting $p = j - i$, we thus have $p > 0$ and $x_i = x_{i+p}$; so that (by the very definition of our iterative sequence)

$$x_i = x_{i+np}, \ x_{i+1} = x_{i+np+1}, \ \text{for all } n \geq 0.$$

By the introduced notations this yields (via (c04) and (7))

$$0 < \rho_i = \rho_{i+np} \to 0 \text{ as } n \to \infty;$$

contradiction. Hence, (c05) cannot hold; wherefrom, we must have

$$\text{for all } i, j \in N\text{: } i \neq j \text{ implies } x_i \neq x_j. \tag{8}$$

IV) As a consequence of this fact, the map $n \mapsto x_n$ is injective; so that, $Y := \{x_n; n \geq 0\}$ is effectively denumerable. Let $k = k(Y) \geq 1$ be the natural number attached to it, by the local Branciari property of d. Also, let $\gamma > 0$ be arbitrary fixed; and $\beta \in]0, \gamma[$ be given by the strong regressivity of φ. By the d-semi-Cauchy property (7), there exists $j(\beta) \in N$ such that

$$d(x_n, x_{n+i}) < \beta/2k \ (\leq \beta/2 < \gamma + \beta/2), \ \ \forall n \geq j(\beta), \ \forall i \in \{1, ..., k+1\}. \tag{9}$$

We now claim that

$$(\forall q \geq 1): \ d(x_n, x_{n+q}) < \gamma + \beta/2, \ \ \forall n \geq j(\beta); \tag{10}$$

and, from this, the d-Cauchy property for $(x_n; n \geq 0)$ follows. The case of $q \in \{1, ..., k+1\}$ is clear, via (9). Assume that (10) holds, for all $q \leq p$ (where $p \geq k+1$); we show that it holds as well for $q = p+1$. So, let $n \geq j(\beta)$ be arbitrary fixed. By the inductive hypothesis and (9),

$$d(x_{n+k}, x_{n+p}) < \gamma + \beta/2 < \gamma + \beta$$
$$d(x_{n+k}, x_{n+k+1}) < \beta/2k < \beta < \gamma + \beta$$
$$d(x_{n+p}, x_{n+p+1}) < \beta/2k < \beta < \gamma + \beta;$$

whence, by definition,

$$M(x_{n+k}, x_{n+p}) < \gamma + \beta.$$

This, by the contractive condition and (b06), gives

$$d(x_{n+k+1}, x_{n+p+1}) \leq \varphi(M(x_{n+k}, x_{n+p})) \leq \gamma.$$

Combining with the $(2+k)$-polyhedral inequality (for $C = (x_{n+2}, ..., x_{n+k+1})$),

$$d(x_n, x_{n+p+1}) \leq d(x_n, x_{n+2}) + ... + d(x_{n+k}, x_{n+k+1}) + d(x_{n+k+1}, x_{n+p+1})$$
$$< k\beta/2k + \gamma \leq \beta/2 + \gamma;$$

and the assertion follows. As (X, d) is complete, we have

$$x_n \xrightarrow{d} x \text{ as } n \to \infty, \text{ for some } z \in X; \tag{11}$$

moreover, by Proposition 2.1, z is uniquely determined by this relation. We claim that this is our desired point. Assume by contradiction that $z \neq Tz$; or, equivalently, $\rho := d(z, Tz) > 0$.

V) Denote $A = \{n \in N; x_n = z\}$, $B = \{n \in N; x_n = Tz\}$. If A is effectively denumerable, we have $A = \{m(j); j \geq 0\}$, where $(m(j); j \geq 0)$ is strictly ascending (hence $m(j) \to \infty$). As $x_{m(j)} = z$, $\forall j \geq 0$, we have $x_{m(j)+1} = Tz$, $\forall j \geq 0$. Combining with $x_{m(j)+1} \xrightarrow{d} z$ as $j \to \infty$, we must have $d(z, Tz) = 0$; hence $z = Tz$, contradiction. On the other hand, if B is effectively denumerable, we have $B = \{n(j); j \geq 0\}$, where $(n(j); j \geq 0)$ is strictly ascending (hence $n(j) \to \infty$). As $x_{n(j)} = Tz$, $\forall j \geq 0$, one gets (via $x_{n(j)} \xrightarrow{d} z$ as $j \to \infty$) $d(z, Tz) = 0$; whence $z = Tz$, again a contradiction. It remains to discuss the case of both A and B being finite; i.e.,

(c06) there exists $h \geq 0$ such that: $\{x_n; n \geq h\} \cap \{z, Tz\} = \emptyset$.

The subset $Y := \{x_n; n \geq h\} \cup \{z, Tz\}$ is therefore effectively denumerable. Let $k = k(Y) \geq 1$ be the natural number attached to it, by the local Branciari property of

d. We have, for each $n \geq k$ (by the $(2+k)$-polyhedral inequality applied to $C := (x_{n+2}, ..., x_{n+k+1})$)

$$\rho \leq d(z, x_{n+2}) + ... + d(x_{n+k}, x_{n+k+1}) + d(x_{n+k+1}, Tz) \tag{12}$$

By (7) and (11), there exists $j(\rho) \geq h$ in such a way that

$$n \geq j(\rho) \Longrightarrow d(x_n, z), d(x_n, x_{n+1}) < \rho/2.$$

As a consequence, we must have

$$M(x_{n+k}, z) = \rho, \quad \forall n \geq j(\rho).$$

so that, by the contractive condition,

$$d(x_{n+k+1}, Tz) \leq \varphi(\rho), \forall n \geq j(\rho).$$

Replacing in (12), we get an evaluation like

$$\rho \leq d(z, x_{n+2}) + ... + d(x_{n+k}, x_{n+k+1}) + \varphi(\rho), \quad \forall n \geq j(\rho).$$

Passing to limit as n tends to infinity gives $\rho \leq \varphi(\rho)$; wherefrom (as φ is regressive) $\rho = 0$; contradiction. Hence, $z = Tz$; and the proof is complete. ∎

In particular, when the regressive function φ is a Boyd-Wong one, our main result covers the one due to Das and Dey [8]; note that, by the developments in Jachymski [13], it includes as well the related statements in Di Bari and Vetro [9]. On the other hand, when $d(.,.)$ is a standard metric, Theorem 3.1 reduces to the statement in Leader [19]. Further aspects may be found in Kikina et al [17]; see also Khojasteh et al [15].

4. FURTHER ASPECTS

A direct inspection of the proof above shows that conclusion of Theorem 3.1 is retainable even if one works with orbital completeness of the ambient space. Some conventions are in order. Let X be a nonempty set; and $d(.,.)$ be a reflexive sufficient symmetric over it; supposed to be a local Branciari metric. Further, take a selfmap T of X. Call the sequence $(y_n; n \geq 0)$ in X, *T-orbital* when $y_n = T^n x$, $n \geq 0$, for some $x \in X$. In this case, let us say that (X, d) is *T-orbital complete* when each T-orbital d-Cauchy sequence is d-convergent.

The following extension of Theorem 3.1 is available. Let the general conditions above be fulfilled; as well as (in place of (c01))

(d01) (X, d) is T-orbital complete.

Theorem 4.1. *Suppose that T is $(d, M; \varphi)$-contractive, where $\varphi \in \mathcal{F}_r(R_+)$ is strongly regressive. Then, T is a global Picard operator (modulo d).*

The proof mimics the one of Theorem 3.1; so, we omit it.

Call the regressive function $\varphi \in \mathcal{F}_r(R_+)$, *admissible* provided

(d02) φ is increasing, usc and $\sum_n \varphi^n(t) < \infty$, $\forall t > 0$.

Clearly, φ is a Matkowski function; hence, in particular, a strongly regressive one. This, in the particular case of d fulfilling the tetrahedral inequality, tells us that the main result in Fora et al [10] is a particular case of Theorem 4.1 above. In addition, we note that the usc condition posed by the authors may be removed. Note that, the introduced framework may be also used to get an extension of the contributions due to Akram and Siddiqui [1]; see also Moradi and Alimohammadi [22]. These will be discussed elsewhere.

References

[1] M. Akram and A. A. Siddiqui, *A fixed-point theorem for A-contractions on a class of generalized metric spaces,* Korean J. Math. Sci., 10 (2003), 1-5.

[2] A. Azam and M. Arshad, *Kannan fixed point theorem on generalized metric spaces,* J. Nonlinear Sci. Appl., 1 (2008), 45-48.

[3] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales,* Fund. Math., 3 (1922), 133-181.

[4] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions,* Proc. Amer. Math. Soc., 20 (1969), 458-464.

[5] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces,* Publ. Math. Debrecen, 57 (2000), 31-37.

[6] C. M. Chen and W. Y. Sun, *Periodic points and fixed points for the weaker $(\phi - \varphi)$-contractive mappings in complete generalized metric spaces,* J. Appl. Math., Volume 2012, Article Id: 856974.

[7] P. Das, *A fixed point theorem on a class of generalized metric spaces,* Korean J. Math. Sci., 9 (2002), 29-33.

[8] P. Das and L. K. Dey, *Fixed point of contractive mappings in generalized metric spaces,* Math. Slovaca, 59 (2009), 499-504.

[9] C. Di Bari and C. Vetro, *Common fixed points in generalized metric spaces,* Appl. Math. Comput., 218 (2012), 7322-7325.

[10] A. Fora, A. Bellour and A. Al-Bsoul, *Some results in fixed point theory concerning generalized metric spaces,* Mat. Vesnik, 61 (2009), 203-208.

[11] T. L. Hicks, *Fixed-point theory in symmetric spaces with applications to probabilistic spaces,* Nonlin. Anal., 36 (1999), 331-344.

[12] J. Jachymski, *Common fixed point theorems for some families of mappings,* Indian J. Pure Appl. Math., 25 (1994), 925-937.

[13] J. Jachymski, *Equivalent conditions for generalized contractions on (ordered) metric spaces,* Nonlin. Anal., 74 (2011), 768-774

[14] S. Kasahara, *On some generalizations of the Banach contraction theorem,* Publ. Res. Inst. Math. Sci. Kyoto Univ., 12 (1976), 427-437.

[15] F. Khojasteh, A. Razani, and S. Moradi, *A fixed point of generalized TF-contraction mappings in cone metric spaces,* Fixed Point Th. Appl., 2011, 2011:14.

[16] L. Kikina and K. Kikina, *Fixed points on two generalized metric spaces,* Int. J. Math. Anal., 5 (2011), 1459-1467.

[17] L. Kikina, K. Kikina and K. Gjino, *A new fixed point theorem on generalized quasimetric spaces,* ISRN Math. Analysis, Volume 2012, Article ID 457846.

[18] H. Lakzian and B. Samet, *Fixed point for $(\psi - \varphi)$-weakly contractive mappings in generalized metric spaces,* Appl. Math. Lett., 25 (2011), 902-906.

[19] S. Leader, *Fixed points for general contractions in metric spaces,* Math. Japonica, 24 (1979), 17-24.

[20] J. Matkowski, *Fixed point theorems for mappings with a contractive iterate at a point,* Proc. Amer. Math. Soc., 62 (1977), 344-348.

[21] D. Miheţ, *On Kannan fixed point principle in generalized metric spaces,* J. Nonlinear Sci. Appl., 2 (2009), 92-96.

[22] S. Moradi and D. Alimohammadi, *New extensions of Kannan fixed-point theorem on complete metric and generalized metric spaces,* Int. J. Math. Analysis, 5 (2011), 2313 - 2320.

[23] B. E. Rhoades, *A comparison of various definitions of contractive mappings,* Trans. Amer. Math. Soc., 336 (1977), 257-290.

[24] I. A. Rus, *Generalized Contractions and Applications,* Cluj University Press, Cluj-Napoca, 2001.

[25] B. Samet, *A fixed point theorem in a generalized metric space for mappings satisfying a contractive condition of integral type,* Int. J. Math. Anal., 3 (2009), 1265-1271.

[26] B. Samet, *Discussion on "A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces" by A. Branciari,* Publ. Math. Debrecen, 76 (2010), 493-494.

[27] I. R. Sarma, J. M. Rao and S. S. Rao, *Contractions over generalized metric spaces,* J. Nonlinear Sci. Appl., 2 (2009), 180-182.

[28] M. Turinici, *Fixed points in complete metric spaces,* in "Proc. Inst. Math. Iaşi" (Romanian Academy, Iaşi Branch), pp. 179-182, Editura Academiei R.S.R., Bucureşti, 1976.

[29] J. Zhu, J. J. Cho and S. M. Kang, *Equivalent contractive conditions in symmetric spaces,* Comput. Math. Appl., 50 (2005), 1621-1628.