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WEAKLY CONTRACTIVE MAPS IN ALTERING METRIC SPACES

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Abstract The weakly contractive metric type fixed point result in Berinde [Nonlin. Anal. Forum, 9 (2004), 45-53] is "almost" covered by the related altering metric one due to Khan et al [Bull. Austral. Math. Soc., 30 (1984), 1-9]. Further extensions of both these results are then provided.

Keywords: complete metric space, contraction, fixed point, altering metric, subunitary and right Boyd-Wong function.

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1. INTRODUCTION

Let (X, d) be a complete metric space; and $T \in \mathcal{F}(\mathcal{X})$ be a selfmap of X. [Here, for each couple A, B of nonempty sets, $\mathcal{F}(\mathcal{A}, \mathcal{B})$ stands for the class of all functions from A to B; when $A = B$, we simply denote $\mathcal{F}(\mathcal{A}, \mathcal{A})$ as $\mathcal{F}(\mathcal{A})$]. Put $\text{Fix}(T) = \{z \in X; z = Tz\}$; each element of this set is called *fixed* under T. In the metrical fixed point theory, such points are to be determined by a limit process as follows. Let us say that $x \in X$ is a *Picard point* (modulo (d, T)) when **i)** $(T^n x; n \geq 0)$ is d-convergent, **ii)** $\lim_n(T^n x)$ belongs to $\text{Fix}(T)$. If this happens for each $x \in X$, then T is called a *Picard operator* (modulo d); and, if in addition, **iii)** $\text{Fix}(T)$ is a *singleton* ($z_1, z_2 \in \text{Fix}(T)$ implies $z_1 = z_2$), then T is referred to as a *strong Picard operator* (modulo d); cf. Rus [13, Ch 2, Sect 2.2]. In this perspective, a basic result to the question we deal with is the 1922 one due to Banach [2]: it states that, whenever T is *α-contractive* (modulo d), i.e.,

(a01) $d(Tx, Ty) \leq \alpha d(x, y), \ \forall x, y \in X,$

for some $\alpha \in [0, 1[$, then T is a strong Picard operator (modulo d). This result found a multitude of applications in operator equations theory; so, it was the subject of many extensions. For example, a natural way of doing this is by considering "functional" contractive conditions of the form

(a02) $d(Tx, Ty) \leq F(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)), \ \ \forall x, y \in X;$

where $F : R_+^5 \to R_+$ is an appropriate function. For more details about the possible choices of F we refer to the 1977 paper by Rhoades [12]; see also Turinici [15]. Here, we shall be concerned with a 2004 contribution in the area due to Berinde [4]. Given $\alpha, \lambda \geq 0$, let us say that T is a *weak* (α, λ)*-contraction* (modulo d) provided

(a03) $d(Tx, Ty) \leq \alpha d(x, y) + \lambda d(Tx, y)$, for all $x, y \in X$.

Theorem 1.1. *Suppose that T is a weak (α, λ)-contraction (modulo d), where $\alpha \in [0, 1[$. Then, T is a Picard operator (modulo d).*

In a subsequent paper devoted to the same question, Berinde [3] claims that this class of contractions introduced by him is for the first time considered in the literature. Unfortunately, his assertion is not true: conclusions of Theorem 1.1 are "almost" covered by a related 1984 statement due to Khan et al [9], in the context of altering distances. This, among others, motivated us to propose an appropriate extension of the quoted statement; details are given in Section 3. The preliminary material for our device is listed in Section 2. Finally, in Section 4, a "functional" extension of Berinde's result is established. Further aspects will be delineated elsewhere.

2. PRELIMINARIES

Let (X, d) be a metric space. Let us say that the sequence (x_n) in X, *d-converges* to $x \in X$ (and write: $x_n \overset{d}{\longrightarrow} x$) iff $d(x_n, x) \to 0$; that is

(b01) $\forall \varepsilon > 0, \exists p = p(\varepsilon)$: $n \geq p \Longrightarrow d(x_n, x) \leq \varepsilon$.

Denote $\lim_n(x_n) = \{x \in X; x_n \overset{d}{\longrightarrow} x\}$; when this set is nonempty, (x_n) is called *d-convergent*. Note that, in this case, $\lim_n(x_n)$ is a singleton, $\{z\}$; as usually, we write $\lim_n(x_n) = z$. Further, let us say that (x_n) is *d-Cauchy* provided $d(x_m, x_n) \to 0$ as $m, n \to \infty$, $m < n$; that is

(b02) $\forall \varepsilon > 0, \exists q = q(\varepsilon)$: $q \leq m < n \Longrightarrow d(x_m, x_n) \leq \varepsilon$.

Clearly, any d-convergent sequence is d-Cauchy too; when the reciprocal holds too, (X, d) is called *complete*. Concerning this aspect, note that any d-Cauchy sequence $(x_n; n \geq 0)$ is *d-semi-Cauchy*; i.e.,

(b03) $\rho_n := d(x_n, x_{n+1}) \to 0$ (hence, $d(x_n, x_{n+i}) \to 0, \forall i \geq 1$), as $n \to \infty$.

The following result about such sequences is useful in the sequel. For each sequence $(z_n; n \geq 0)$ in R and each $z \in R$, put $z_n \downarrow z$ iff $[z_n > z, \forall n]$ and $z_n \to z$.

Proposition 2.1. *Suppose that $(x_n; n \geq 0)$ is d-semi-Cauchy, but not d-Cauchy. There exists then $\eta > 0$, $j(\eta) \in N$ and a couple of rank sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, in such a way that*

$$j \leq m(j) < n(j), \quad \alpha(j) := d(x_{m(j)}, x_{n(j)}) > \eta, \ \forall j \geq 0 \tag{1}$$

$$n(j) - m(j) \geq 2, \ \beta(j) := d(x_{m(j)}, x_{n(j)-1}) \leq \eta, \ \forall j \geq j(\eta) \tag{2}$$

$$\alpha(j) \downarrow \eta \text{ (hence, } \alpha(j) \to \eta) \text{ as } j \to \infty \tag{3}$$

$$\alpha_{p,q}(j) := d(x_{m(j)+p}, x_{n(j)+q}) \to \eta, \text{ as } j \to \infty, \ \forall p, q \in \{0, 1\}. \tag{4}$$

A proof of this may be found in Khan et al [9]. For completeness reasons, we supply an argument which differs, in part, from the original one.

Proof. **(Proposition 2.1)** As (b02) does not hold, there exists $\eta > 0$ with

$$A(j) := \{(m, n) \in N \times N; j \le m < n, d(x_m, x_n) > \eta\} \neq \emptyset, \ \forall j \ge 0.$$

Having this precise, denote, for each $j \ge 0$,

$$m(j) = \min \mathrm{Dom}(A(j)), \ n(j) = \min A(m(j)).$$

As a consequence, the couple of rank-sequences $(m(j); j \ge 0)$, $(n(j); j \ge 0)$ fulfills (1). On the other hand, letting the index $j(\eta) \ge 0$ be such that

$$d(x_k, x_{k+1}) < \eta, \ \ \forall k \ge j(\eta), \tag{5}$$

it is clear that (2) holds too. Finally, by the triangular property,

$$\eta < \alpha(j) \le \beta(j) + \rho_{n(j)-1} \le \eta + \rho_{n(j)-1}, \ \ \forall j \ge j(\eta);$$

and this yields (3); hence, the case $(p = 0, q = 0)$ of (4). Combining with

$$\alpha(j) - \rho_{n(j)} \le d(x_{m(j)}, x_{n(j)+1}) \le \alpha(j) + \rho_{n(j)}, \ \forall j \ge j(\eta)$$

establishes the case $(p = 0, q = 1)$ of the same. The remaining situations are deductible in a similar way. ∎

3. MAIN RESULT

Let X be a nonempty set; and $d(.,.)$ be a metric over it [in the usual sense]. Further, let $\varphi \in \mathcal{F}(\mathcal{R}_+)$ be an *altering function*; i.e.

(c01) φ is continuous, increasing, and reflexive-sufficient [$\varphi(t) = 0$ iff $t = 0$].

The associated map (from $X \times X$ to R_+)

(c02) $e(x, y) = \varphi(d(x, y)), \ x, y \in X$

has the immediate properties

$$e(x, y) = e(y, x), \ \forall x, y \in X \ (e \text{ is symmetric}) \tag{6}$$

$$e(x, y) = 0 \Longleftrightarrow x = y \ (e \text{ is reflexive-sufficient}). \tag{7}$$

So, it is a (reflexive sufficient) *symmetric*, under Hicks' terminology [8]. In general, $e(.,.)$ is not endowed with the triangular property; but, in compensation to this, one has (as φ is increasing and continuous)

$$e(x, y) > e(u, v) \Longrightarrow d(x, y) > d(u, v) \tag{8}$$

$$x_n \xrightarrow{d} x,\ y_n \xrightarrow{d} y \text{ implies } e(x_n, y_n) \to e(x, y). \tag{9}$$

Suppose in the following that

(c03) (X, d) is complete (each d-Cauchy sequence is d-convergent).

Let $T \in \mathcal{F}(\mathcal{X})$ be a selfmap of X. The formulation of the problem involving $\mathrm{Fix}(T) = \{x \in X; x = Tx\}$ is the already sketched one. In the following, we are trying to solve it in the precise context. Denote, for $x, y \in X$,

(c04) $M_1(x, y) = e(x, y)$, $M_2(x, y) = (1/2)[e(x, Tx) + e(y, Ty)]$,
$M_3(x, y) = \min\{e(x, Ty), e(Tx, y)\}$,
$M(x, y) = \max\{M_1(x, y), M_2(x, y), M_3(x, y)\}$.

Further, given $\psi \in \mathcal{F}(\mathcal{R}_+)$, we say that T is *$(d, e; M, \psi)$-contractive*, provided

(c05) $e(Tx, Ty) \le \psi(d(x, y))M(x, y)$, $\forall x, y \in X$, $x \ne y$.

The properties of ψ to be used here write

(c06) ψ is strictly subunitary on $R^0_+ :=]0, \infty[$: $\psi(s) < 1$, $\forall s \in R^0_+$

(c07) ψ is right Boyd-Wong on R^0_+: $\limsup_{t \to s+} \psi(t) < 1$, $\forall s \in R^0_+$.

This is related to the developments in Boyd and Wong [6]; we do not give details.

The main result of this exposition is

Theorem 3.1. *Suppose that T is $(d, e; M, \psi)$-contractive, where $\psi \in \mathcal{F}(\mathcal{R}_+)$ is strictly subunitary and right Boyd-Wong on R^0_+. Then, T is a strong Picard operator (modulo d).*

Proof. First, let us check the singleton property for $\mathrm{Fix}(T)$. Let $z_1, z_2 \in \mathrm{Fix}(T)$ be such that $z_1 \ne z_2$; hence $\delta := d(z_1, z_2) > 0$, $\varepsilon := e(z_1, z_2) > 0$. By definition,

$$M_1(z_1, z_2) = \varepsilon,\ M_2(z_2, z_2) = 0,\ M_3(x, y) = \varepsilon; \text{ hence } M(x, y) = \varepsilon.$$

By the contractive condition (written at (z_1, z_2))

$$\varepsilon = e(z_1, z_2) = e(Tz_1, Tz_2) \le \psi(\delta)M(z_1, z_2) = \psi(\delta)\varepsilon;$$

hence, $1 \le \psi(\delta) < 1$; contradiction. This established the singleton property. It remains now to verify the Picard property. Fix some $x_0 \in X$; and put $x_n = T^n x_0$, $n \ge 0$. If $x_n = x_{n+1}$ for some $n \ge 0$, we are done; so, without loss, one may assume

(c08) $\rho_n := d(x_n, x_{n+1}) > 0$ (hence, $\sigma_n := e(x_n, x_{n+1}) > 0$), for all n.

There are several steps to be passed.

I) For the arbitrary fixed $n \geq 0$, we have

$$
\begin{aligned}
&M_1(x_n, x_{n+1}) = \sigma_n,\\
&M_2(x_n, x_{n+1}) = (1/2)[\sigma_n + \sigma_{n+1}] \leq \max\{\sigma_n, \sigma_{n+1}\},\\
&M_3(x_n, x_{n+1}) = 0;\ \text{hence } M(x_n, x_{n+1}) \leq \max\{\sigma_n, \sigma_{n+1}\}.
\end{aligned}
$$

By the contractive condition (written at (x_n, x_{n+1})),

$$\sigma_{n+1} \leq \psi(\rho_n) \max\{\sigma_n, \sigma_{n+1}\},\ \forall n.$$

This, along with (c08), yields (as ψ is strictly subunitary on R^0_+)

$$\sigma_{n+1}/\sigma_n \leq \psi(\rho_n) < 1,\ \forall n. \tag{10}$$

As a direct consequence,

$$\sigma_n > \sigma_{n+1} \text{ (hence, } \rho_n > \rho_{n+1}\text{), for all } n.$$

The sequence $(\rho_n; n \geq 0)$ is therefore strictly descending in R_+; hence, $\rho := \lim_n(\rho_n)$ exist in R_+ and $\rho_n > \rho$, $\forall n$. Likewise, the sequence $(\sigma_n = \varphi(\rho_n); n \geq 0)$ is strictly descending in R_+; hence, $\sigma := \lim_n(\sigma_n)$ exists; with, in addition, $\sigma = \varphi(\rho)$. We claim that $\rho = 0$. Assume by contradiction that $\rho > 0$; hence $\sigma > 0$. Passing to lim sup as $n \to \infty$ in (10) yields

$$1 \leq \limsup_n \psi(\rho_n) \leq \limsup_{t \to \rho+} \psi(t) < 1;$$

contradiction. Hence, $\rho = 0$; i.e.,

$$\rho_n := d(x_n, x_{n+1}) \to 0,\ \text{as } n \to \infty. \tag{11}$$

II) We now show that $(x_n; n \geq 0)$ is d-Cauchy. Suppose that this is not true. By Proposition 2.1, there exist $\eta > 0$, $j(\eta) \in N$ and a couple of rank sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, in such a way that (1)-(4) hold. Denote for simplicity $\zeta = \varphi(\eta)$; hence, $\zeta > 0$. By the notations used there, we may write as $j \to \infty$

$$\lambda_j := e(x_{m(j)+1}, x_{n(j)+1}) = \varphi(\alpha_{1,1}(j)) \to \zeta.$$

In addition, we have (again under $j \to \infty$)

$$
\begin{aligned}
&M_1(x_{m(j)}, x_{n(j)}) = \varphi(\alpha(j)) \to \zeta\\
&M_2(x_{m(j)}, x_{n(j)}) = (1/2)[\varphi(\rho_{m(j)}) + \varphi(\rho_{n(j)})] \to 0\\
&M_3(x_{m(j)}, x_{n(j)}) = \min\{\varphi(\alpha_{0,1}(j)), \varphi(\alpha_{1,0}(j))\} \to \zeta;
\end{aligned}
$$

and this, by definition, yields

$$\mu_j := M(x_{m(j)}, x_{n(j)}) \to \zeta \text{ as } j \to \infty.$$

From the contractive condition (written at $(x_{m(j)}, x_{n(j)})$)

$$\lambda_j/\mu_j \leq \psi(\alpha(j)),\ \forall j \geq j(\eta);$$

so that, passing to lim sup as $j \to \infty$

$$1 \leq \limsup_j \psi(\alpha(j)) \leq \limsup_{t\to\eta+} \psi(t) < 1;$$

contradiction. Hence, $(x_n; n \geq 0)$ is d-Cauchy, as claimed.

III) As (X, d) is complete, there exists a (uniquely determined) $z \in X$ with $x_n \overset{d}{\longrightarrow} z$; hence $\gamma_n := d(x_n, z) \to 0$ as $n \to \infty$.

Two assumptions are open before us:

i) For each $h \in N$, there exists $k > h$ with $x_k = z$. In this case, there exists a sequence of ranks $(m(i); i \geq 0)$ with $m(i) \to \infty$ as $i \to \infty$ such that $x_{m(i)} = z$, $\forall i$; hence, $x_{m(i)+1} = Tz$, $\forall i$. Letting i tends to infinity and using the fact that $(y_i := x_{m(i)+1}; i \geq 0)$ is a subsequence of $(x_i; i \geq 0)$, we get $z = Tz$.

ii) There exists $h \in N$ such that $n \geq h \Longrightarrow x_n \neq z$. Suppose that $z \neq Tz$; i.e., $\theta := d(z, Tz) > 0$; hence, $\omega := e(z, Tz) > 0$. Note that, in such a case, $\delta_n := d(x_n, Tz) \to \theta$. From our previous notations, we have (as $n \to \infty$)

$$\lambda_n := e(x_{n+1}, Tz) = \varphi(\delta_{n+1}) \to \varphi(\theta) = \omega.$$

In addition (again under $n \to \infty$),

$$M_1(x_n, z) = \varphi(\gamma_n) \to 0,\ M_2(x_n, z) = (1/2)[\sigma_n + \omega] \to \omega/2$$
$$M_3(x_n, z) = \min\{\varphi(\delta_n), \varphi(\gamma_{n+1})\} \to 0;$$

wherefrom,

$$\mu_n := M(x_n, z) \to \omega/2, \text{ as } n \to \infty.$$

By the contractive condition (written at (x_n, z))

$$\lambda_n \leq \psi(\gamma_n)\mu_n < \mu_n,\ \forall n \geq h$$

we then have (passing to limit as $n \to \infty$), $\omega \leq \omega/2$; hence $\omega = 0$. This yields $\theta = 0$; contradiction. Hence, z is fixed under T and the proof is complete. ∎

In particular, the right Boyd-Wong on R^0_+ property of ψ is assured when this function fulfills (c06) and is decreasing on R^0_+. As a consequence, the following particular version of our main result may be stated.

Theorem 3.2. *Suppose that T is $(d, e; M, \psi)$-contractive, where $\psi \in \mathcal{F}(\mathcal{R}_+)$ is strictly subunitary and decreasing on R^0_+. Then, T is a strong Picard operator (modulo d).*

Let $a, b, c \in \mathcal{F}(\mathcal{R}_+)$ be a triple of functions. We say that the selfmap T of X is $(d, e; a, b, c)$-*contractive* if

(c09) $e(Tx, Ty) \leq a(d(x, y))e(x, y) + b(d(x, y))[e(x, Tx) + e(y, Ty)]+$
$c(d(x, y)) \min\{e(x, Ty), e(Tx, y)\}, \ \forall x, y \in X, \ x \neq y.$

Denote for simplicity $\psi = a + 2b + c$; it is clear that, under such a condition, T is $(d, e; M; \psi)$-contractive. Consequently, the following statement is a particular case of Theorem 1.1 above:

Theorem 3.3. *Suppose that T is $(d, e; a, b, c)$-contractive, where the triple of functions $a, b, c \in \mathcal{F}(\mathcal{R}_+)$ is such that their associated function $\psi = a + 2b + c$ is strictly subunitary and right Boyd-Wong on R^0_+. Then, conclusions of Theorem 1.1 hold.*

In particular, when a, b, c are all decreasing on R^0_+, the right Boyd-Wong property on R^0_+ holds; note that, in this case, Theorem 3.3 is also reducible to Theorem 3.2. This is just the 1984 fixed point result in Khan et al [9].

Finally, it is worth mentioning that the nice contributions of these authors was the starting point for a series of results involving altering contractions, like the one in Dutta and Choudhury [7] or Nashine et al [10]. Some other aspects may be found in Akkouchi [1]; see also Pathak and Shahzad [11].

4. FURTHER ASPECTS

Let again (X, d) be a complete metric space and $T \in \mathcal{F}(\mathcal{X})$ be a selfmap of X. A basic particular case of Theorem 3.3 corresponds to the choices φ=identity and $[a, b, c$=constants$]$. The corresponding form of Theorem 3.3 is comparable with Theorem 1.1. However, the inclusion between these is not complete. This raises the question of determining proper extensions of Theorem 1.1, close enough to Theorem 3.3. A direct answer to this is provided by

Theorem 4.1. *Let the numbers $a, b \in R_+$ and the function $K \in \mathcal{F}(\mathcal{R}_+)$ be such that*

(d01) $d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + K(d(Tx, y)), \forall x, y \in X$

(d02) $a + 2b < 1$ *and* $K(t) \to 0$ *as* $t \to 0$.

Then, T is a Picard map (modulo d).

Proof. Take an arbitrary fixed $u \in X$. By the very contractive condition (written at $(T^n u, T^{n+1} u)$), we have the evaluation

$$d(T^{n+1}u, T^{n+2}u) \leq \lambda d(T^n u, T^{n+1}u), \ \ \forall n \geq 0. \tag{12}$$

where $\lambda := (a + b)/(1 - b) < 1$. This yields

$$d(T^n u, T^{n+1}u) \leq \lambda^n d(u, Tu), \ \ \forall n \geq 0. \tag{13}$$

Consequently, $(T^n u; n \geq 0)$ is d-Cauchy; whence (by completeness)

$$T^n u \xrightarrow{d} z := T^\infty u, \text{ for some } z \in X.$$

From the contractive condition (written at $(T^n u, z)$),

$$d(T^{n+1}u, Tz) \leq ad(T^n u, z) + b[d(T^n u, T^{n+1}u) + d(z, Tz)] + K(d(T^{n+1}u, z)),\ \forall n.$$

Passing to limit as $n \to \infty$ gives (via (d02)) $d(z, Tz) \leq bd(z, Tz)$; so that, if $z \neq Tz$, one gets $1 \leq b \leq 1/2$, contradiction. Hence $z = Tz$; and the proof is complete. ∎

In particular, when $b = 0$ and $K(.)$ is linear ($K(t) = \lambda t$, $t \in R_+$, for some $\lambda \geq 0$), this result is just Theorem 1.1. Note that, from (13), one has for these "limit" fixed points, the error approximation formula (which – under the accepted conditions for our data – is available as well in case of Theorem 3.3)

$$d(T^n u, T^\infty u) \leq [\lambda^n/(1-\lambda)]d(u, Tu),\quad \forall n \in N. \tag{14}$$

However, the non-singleton property of Fix(T) makes this "local" evaluation to be without practical effect in Theorem 4.1, by the highly unstable character of the map $u \mapsto T^\infty u$: even if the distance $d(u, v)$ between two initial approximations would decrease, the distance $d(T^\infty u, T^\infty v)$ between the associated fixed points may not decrease.

Finally, another interesting particular case to consider is that of φ being an arbitrary altering function and $[a, b, c$=constants]; we do not give details. Further aspects may be found in Bhaumik et al [5] see also Sastry and Babu [14];

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