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STIFFNESS OF THE LINEAR DIFFUSION AND WAVE-TYPE PARTIAL DIFFERENTIAL EQUATIONS

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Abstract After applying the Finite Element Method (FEM) to the one-dimensional diffusion-type and wave-type Partial Differential Equations (PDEs) with boundary conditions and initial conditions, a first order and a second order ODE systems are obtained respectively. The latter can be reduced to a first order ODE system. These first order ODE systems usually present high stiffness, so numerical methods with good stability properties are required in their resolution. In this paper, we have studied the stiffness of the resulting first order ODE systems as function of the number of elements considered in the discretization, the length of the domain in which the PDE is applied and the thermal diffusivity (in the case of the diffusion-type PDE) and the wave speed propagation (in the case of the wave-type PDE).

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1. INTRODUCTION

We will consider the diffusion equation (or heat equation) and the wave equation given in terms of Partial Differential Equations (PDEs). In the case of the diffusion equation, we will consider a thin rod of length L. We will assume that the ends of the rod are kept at the same fixed temperature. Let $u(x,t)$ represent the temperature at the point x along the rod at time t, and assume that an initial temperature distribution $u(x,0) = f(x)$ is given. The following PDE is used to model the one-dimensional linear temperature evolution:

- Diffusion equation: $$\begin{cases} u_t = \alpha^2 u_{xx}, & 0 < x < L,\ t > 0 \\ BC : u(0,t) = 0 = u(L,t), & t > 0 \\ IC : u(x,0) = f(x), & 0 \le x \le L \end{cases} \tag{1}$$

where $\alpha^2 = \frac{k}{c\rho}$ is the thermal diffusivity, k the thermal conductivity, c the thermal capacity and ρ the density, u_t is the rate of change in temperature with respect to time, u_{xx} is the concavity of the temperature profile which compares the temperature of one point to the temperature at neighbouring points. BC and IC are the boundary conditions and the initial conditions respectively.

In the case of the wave equation, a string of length L will be considered, where its two ends held fixed at height zero. Assume that its initial position and speed are given, $f(x)$ and $g(x)$ respectively. Let $u(x,t)$ denote the vertical displacement of the string from the x axis in time t. It is assumed that the string is undergoing small amplitude transverse vibrations so that $u(x,t)$ obeys the wave equation. The one-dimensional linear wave equation with boundary conditions (BC) and initial conditions (IC) is given by:

- Wave equation: $$\begin{cases} u_{tt} = \alpha^2 u_{xx}, & 0 < x < L,\ t > 0 \\ BC : u(0,t) = 0 = u(L,t), & t > 0 \\ IC : u(x,0) = f(x), u_t(x,0) = g(x), & 0 \le x \le L \end{cases} \tag{2}$$

where $\alpha = \sqrt{\frac{T}{\rho}}$ is the speed propagation of the wave, T is the applied tension in the string and ρ the linear mass density.

The continuous solution of both equations, (1) and (2), can be found using the method of separation of variables [6]. The solution of the diffusion equation (1) is given by:

$$u(x,t) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{L}\right) e^{-(k\pi\alpha)^2 t} \tag{3}$$

where $A_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx$.

And the solution of the wave equation (2) is given by:

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi x}{L}\right)\left[A_k \sin\left(\frac{k\pi\alpha t}{L}\right) + B_k \cos\left(\frac{k\pi\alpha t}{L}\right)\right] \tag{4}$$

where A_k and B_k are given by: $\begin{cases} A_k = \frac{2}{k\pi\alpha}\int_0^L g(x)\sin\left(\frac{k\pi x}{L}\right)dx \\ B_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx \end{cases}$

It is also possible to find the approximate solutions of equations (1) and (2) numerically using the Finite Element Method (FEM), in which the process of finding the solution $u(x,t)$ consists of discretizing the domain L in elements and nodes. The solution approach is based on the elimination of the spatial derivatives of the PDE and this leads to a system of Ordinary Differential Equations (ODEs). The resulting system of ODEs can be solved using standard numerical methods [2], such as Runge-Kutta methods [3], Backward Differentiation Formulae [8], or using the ode solvers implemented in MATLAB [1]. The *ode45*, based on an embedded Runge-Kutta method DOPRI(5,4) [5], and the *ode15s*, based on BDFs [8], are two of the ode solvers offered by MATLAB.

The ODE system that results after the FEM discretization presents high stiffness usually. Stiffness is a delicate as well as important concept when solving ODEs. Various authors [12, 14, 15] agree saying that there is no a rigorous definition of stiffness.

It depends on the ODE, on its initial conditions, on the numerical method used for its resolution and on the time interval in which the ODE is solved. In this article we will use the definition of stiffness given in [10, 13], which says that stiffness occurs when different magnitude eigenvalues exist in the solution, where this difference in the magnitude could happen in the real part or in the imaginary part of the eigenvalues. The aim of this article is to study the stiffness of the ODE system which results after the FEM discretization of the one-dimensional diffusion and wave PDEs.

The article is organized as follows: in Section (2) the formulation of the Finite Element Method is given which enables us to obtain an ODE system from a PDE; in Sections (3) and (4) the study of the stiffness of the resulting ODE system is done and in Section (5) some conclusions are given.

2. THE FINITE ELEMENT METHOD

The Finite Element Method consists of finding solutions in a finite dimensional space. Having chosen a basis of functions and having defined finite dimensional subspaces, the PDE solution is written as linear combination of the functions of the basis. To do this it is necessary to check that the scalar product of the differential operator with all the functions of the subspace is zero. This requires a variational formulation of the problem, which is obtained by integrating by parts.

2.1. APPLICATION OF THE FEM METHOD TO THE DIFFUSION EQUATION

We will show the application of the Finite Element Method to the diffusion PDE (1). A step size $h > 0$ of the spatial mesh will be considered, being $h = L/(n+1)$ where $n \in \mathbb{N}$. In this way the partition defined by the nodes:

$$x_j = jh, \;\; j = 0, 1, ..., (n+1) \tag{5}$$

breaks up the spatial interval $[0, L]$ in $(n+1)$ subintervals of length h: $I_j = \left[x_j, x_{j+1}\right]$, $j = 0, ..., n$. Observe that the first and the last nodes correspond to the end of the interval $[0, L]$: $x_0 = 0$, $x_{n+1} = L$.

Each internal node x_j, $j = 1, ..., n$, is associated a piecewise continuous and linear basis function $N_j(x)$, that verifies $N_j(x_i) = \delta_{ji}$ for $j = 1, ..., n$ and $i = 0, ..., (n+1)$, being δ the Kronecker delta. Next, we introduce the vectorial subspace of dimension n spanned by the functions $\left\{N_j\right\}_{j=1,...,n}$:

$$V_h = span\left\{N_j : j = 1, ..., n\right\} \tag{6}$$

Approximate solutions of the diffusion PDE (1) in the subspace $C([0, \infty), V_h)$ are found so that:

$$u(x, t) \approx u_h(x, t) = \sum_{j=1}^{n} d_j(t) N_j(x) \tag{7}$$

Observe that function (7) depends on the spatial variable x as well as on the temporary variable t. The function (7) will be univocally defined if its coefficients $d_j(t)$ are defined precisely. It has to be taken into account that because of the election of the basis functions $\left\{N_j\right\}_{j=1,\dots,n}$, $d_j(t)$ is the value of the function $u_h(x,t)$ in the point $x = x_j$.

The weak formulation to calculate the approximate solution u_h is given by [11]:

find $u_h \in C^1\left([0,\infty);V_h\right)$ that verifies:

$$\int_0^L u_{h,t}(x,t)w_h(x)dx = -\int_0^L \alpha^2 u_{h,x}(x,t)w_h'(x)dx,\ t>0,\ \forall w_h \in V_h \tag{8}$$

and the initial condition:

$$u_h(x,0) = f_h(x),\ 0 \le x \le L. \tag{9}$$

In (9) an initial value $u_{h,0}$ has been taken. This value can be taken in different ways in order to approximate the initial value u_0 of the diffusion equation. One of the simplest ways to chose f_h is as the orthogonal projection of u_0 in V_h.

By substituting the expression (7) of $u_h(x,t)$ and its derivative $u_h'(x,t)$ in (8), and applying this equality for all the functions N_i of the basis, the following system of n first order Ordinary Differential Equations is obtained:

$$\sum_{j=1}^{n}\left(\underbrace{\int_0^L N_i(x)N_j(x)dx}_{m_{ij}}\right)d_j'(t) = -\sum_{j=1}^{n}\left(\underbrace{\int_0^L \alpha^2 N_i'(x)N_j'(x)dx}_{k_{ij}}\right)d_j(t), \qquad i=1,2,\dots,n \tag{10}$$

which in matricial form can be written as:

$$\begin{cases} M\mathbf{d}'(t) = -K\mathbf{d}(t) \\ \mathbf{d}(0) = (f(x_1),\dots,f(x_n))^T \end{cases} \tag{11}$$

where $M = (m_{ij})$ and $K = (k_{ij})$ are the mass and stiffness matrices of the FEM discretization, and $d_j(t)$, $j = 1,\dots,n$, are the unknowns.

2.2. APPLICATION OF THE FEM METHOD TO THE WAVE EQUATION

The same partition $\left\{x_j\right\}_{j=1,\dots,n}$ as in the previous section, being $x_j = jh$ the nodes of the interval $[0,L]$ will be considered, where $h = L/(n+1)$, and $I_j = [x_j, x_{j+1}]$, $j = 0,\dots,n$, the elements in which the whole interval $[0,L]$ has been partitioned. In this case the variational formulation to calculate the approximate solution u_h is given by:

find $u_h \in C^2([0,\infty);V_h)$ *that verifies:*

$$\int_0^L u_{h,tt}(x,t)w_h(x)dx = -\int_0^L \alpha^2 u_{h,x}(x,t)w_h'(x)dx,\ t>0,\ \forall w_h \in V_h \tag{12}$$

and the initial conditions:

$$u_h(x,0) = f_h(x),\ \ u_h'(x,0) = g_h(x),\ 0 \le x \le L \tag{13}$$

where f_h and g_h *are approximations of the initial conditions* (f,g) *in* V_h.

Proceeding as before, a second order ODE system is obtained:

$$\sum_{j=1}^{n}\left(\underbrace{\int_0^L N_i(x)N_j(x)dx}_{m_{ij}}\right)d_j''(t) = -\sum_{j=1}^{n}\left(\underbrace{\int_0^L \alpha^2 N_i'(x)N_j'(x)dx}_{k_{ij}}\right)d_j(t),$$

$$i = 1,2,...,n \tag{14}$$

which in matricial form can be written using the mass $M=(m_{ij})$ and stiffness matrices $K=(k_{ij})$ of the FEM:

$$\begin{cases} M\mathbf{d}''(t) = -K\mathbf{d}(t) \\ \mathbf{d}(0) = (f(x_1),...,f(x_n))^T,\ \mathbf{d}'(0) = (g(x_1),...,g(x_n))^T \end{cases} \tag{15}$$

and $d_j(t),\ j=1,...,n$ are the unknowns. Equation (15) can be reduced to a first order ODE system with the form $y' = f(t,y)$:

$$\begin{pmatrix} \mathbf{y}_1'(t) \\ \mathbf{y}_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & 0 \end{pmatrix}\begin{pmatrix} \mathbf{y}_1(t) \\ \mathbf{y}_2(t) \end{pmatrix}, \tag{16}$$

where: $\begin{cases} \mathbf{y}_1(t) = \mathbf{d}(t),\ \mathbf{y}_2(t) = \mathbf{d}'(t) \\ \mathbf{y}_1(0) = \mathbf{d}(0),\ \mathbf{y}_2(0) = \mathbf{d}'(0) \end{cases}$.

3. STIFFNESS OF THE DIFFUSION PDE

We will consider the one-dimensional heat equation given by (1). In the previous section we have seen that the FEM discretization of this equation with $(n+2)$ nodes and $(n+1)$ elements leads to a system of ODEs which in the form $y' = f(t,y)$ is given by (11):

$$\mathbf{d}'(t) = -M^{-1}K\mathbf{d}(t),\ \ \mathbf{d}(0) = \mathbf{d}_0 = (f(x_1),...,f(x_n))^T$$

In this section the stiffness of the system (11) is studied, which means that the eigenvalues of the jacobian matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{d}} = -M^{-1}K$ are studied depending on three variables: the number of elements of the discretization, the length of the rod L and the thermal diffusivity α^2.

3.1. STIFFNESS AS FUNCTION OF THE ELEMENTS OF THE DISCRETIZATION

Two different materials have been considered: an epoxy called Araldite and an epoxy/alumina (100/188), being their experimental thermal diffusivities $\alpha^2 = 0.129 \cdot 10^{-2}\ cm^2/s$ and $\alpha^2 = 0.449 \cdot 10^{-2}\ cm^2/s$ respectively; they coincide with the theoretical values given in [9] and [16] respectively. Having fixed α^2 and the length of the rod (L = 20 cm), we change the number of elements of the FEM discretization and we analyse the values of the greatest and smallest eigenvalues in module, see Table (1). The greatest eigenvalues in module of each case are represented in Figure (1).

Number of elements	$\lvert\lambda_{max}\rvert$ epoxy	$\lvert\lambda_{min}\rvert$ epoxy	$\lvert\lambda_{max}\rvert$ epoxy/alum.	$\lvert\lambda_{min}\rvert$ epoxy/alum.
5	$7.3478 \cdot 10^{-4}$	$3.2890 \cdot 10^{-5}$	$2.5575 \cdot 10^{-3}$	$1.1448 \cdot 10^{-4}$
10	$3.5991 \cdot 10^{-3}$	$3.2092 \cdot 10^{-5}$	$1.2527 \cdot 10^{-2}$	$1.1170 \cdot 10^{-4}$
20	$1.5198 \cdot 10^{-2}$	$3.1895 \cdot 10^{-5}$	$5.2897 \cdot 10^{-2}$	$1.1101 \cdot 10^{-4}$
40	$6.1635 \cdot 10^{-2}$	$3.1846 \cdot 10^{-5}$	$2.1453 \cdot 10^{-1}$	$1.1084 \cdot 10^{-4}$
80	$2.4739 \cdot 10^{-1}$	$3.1834 \cdot 10^{-5}$	$8.6108 \cdot 10^{-1}$	$1.1080 \cdot 10^{-4}$
160	$9.9043 \cdot 10^{-1}$	$3.1830 \cdot 10^{-5}$	$3.4473 \cdot 10^{0}$	$1.1079 \cdot 10^{-4}$
320	$3.9626 \cdot 10^{0}$	$3.1830 \cdot 10^{-5}$	$1.3792 \cdot 10^{1}$	$1.1079 \cdot 10^{-4}$
640	$1.5851 \cdot 10^{1}$	$3.1830 \cdot 10^{-5}$	$5.5172 \cdot 10^{1}$	$1.1079 \cdot 10^{-4}$
1280	$6.3406 \cdot 10^{1}$	$3.1829 \cdot 10^{-5}$	$2.2069 \cdot 10^{2}$	$1.1079 \cdot 10^{-4}$

Table 1: Eigenvalues of the ODE system (11) vs. number of elements.

Conclusions:

- When the number of elements of the discretization is increased, the greatest eigenvalue of the ODE system (11) grows and the smallest one does not change.

- If an initial discretization of m elements is multiplied by n, obtaining a new discretization of $n \cdot m$ elements, the greatest eigenvalue of the ODE system (11) which corresponds to the second discretization is equal to the greatest eigenvalue of the first discretization's ODE system multiplied by the coefficient n^2. For instance, in the material epoxy with 10 elements, the greatest eigenvalue is given by $\lvert\lambda_{max}\rvert = 3.5991 \cdot 10^{-3}$. If the number of elements is multiplied by 2, having in this way a discretization of 20 elements, the greatest eigenvalue is multiplied by 2^2: $\lvert\lambda_{max}\rvert = 4 \cdot 3.5991 \cdot 10^{-3} = 1.4396 \cdot 10^{-2} \approx 1.5198 \cdot 10^{-2}$.

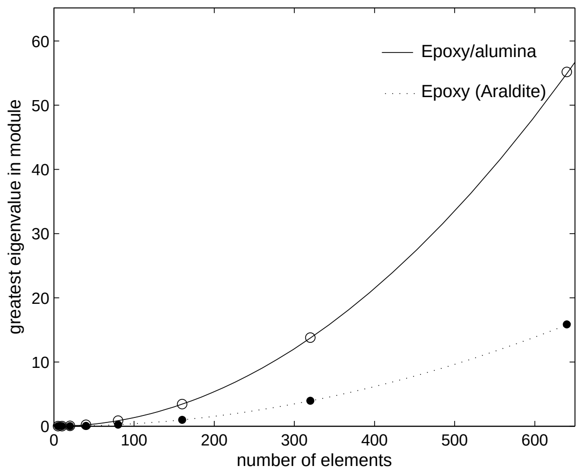

Fig. 1.: The greatest eigenvalue in module of the ODE system (11) vs. number of elements.

3.2. STIFFNESS AS FUNCTION OF THE LENGTH OF THE ROD

Having fixed α^2 and the number of elements of the discretization, the length of the rod has been changed and the values of the greatest and smallest eigenvalues in module have been analysed. A discretization of 20 elements and the materials epoxy (Araldite) and epoxy/alumina have been considered in this analysis. The greatest and smallest eigenvalues of each case have been computed in Table (2).

Length of the rod (cm)	$\lvert\lambda_{max}\rvert$ epoxy	$\lvert\lambda_{min}\rvert$ epoxy	$\lvert\lambda_{max}\rvert$ epoxy/alum.	$\lvert\lambda_{min}\rvert$ epoxy/alum.
2	$1.5198 \cdot 10^{0}$	$3.1895 \cdot 10^{-3}$	$5.2897 \cdot 10^{0}$	$1.1101 \cdot 10^{-2}$
20	$1.5198 \cdot 10^{-2}$	$3.1895 \cdot 10^{-5}$	$5.2897 \cdot 10^{-2}$	$1.1101 \cdot 10^{-4}$
200	$1.5198 \cdot 10^{-4}$	$3.1895 \cdot 10^{-7}$	$5.2897 \cdot 10^{-4}$	$1.1101 \cdot 10^{-6}$
2000	$1.5198 \cdot 10^{-6}$	$3.1895 \cdot 10^{-9}$	$5.2897 \cdot 10^{-6}$	$1.1101 \cdot 10^{-8}$
20000	$1.5198 \cdot 10^{-8}$	$3.1895 \cdot 10^{-11}$	$5.2897 \cdot 10^{-8}$	$1.1101 \cdot 10^{-10}$

Table 2: Eigenvalues of the ODE system (11) vs. length of the rod.

Conclusions:

- The shorter the length of the rod, the greater all the eigenvalues of the ODE system (11).
- Given a rod of length l, if this is multiplied by x obtaining a new rod of length $l \cdot x$, the eigenvalues of the resulting new ODE system are multiplied by $\frac{1}{x^2}$.

3.3. STIFFNESS AS FUNCTION OF THE THERMAL DIFFUSIVITY

The number of elements of the discretization and the length of the rod are fixed, and two materials with thermal diffusivities α_1^2 and α_2^2 are considered, which verify $\alpha_1^2 = k \cdot \alpha_2^2$. Making use of (10) it is easy to verify that the resulting matrices $-M_1^{-1}K_1$ and $-M_2^{-1}K_2$ of the system (11) verify the same relation as the thermal diffusivities

$$-M_1^{-1}K_1 = k \cdot \left(-M_2^{-1}K_2\right). \tag{17}$$

Given two proportional matrices A and B, we will see which is the connection of their characteristic polynomials. Given a matrix A, the notation that we will use to denote its characteristic polynomial will be $\chi_A(\lambda) = det(\lambda I - A)$.

Theorem 3.1. *Given A a square matrix of dimension n and the matrix $B = k \cdot A$, which is obtained by multiplying A by an scalar $k \neq 0$, the characteristic polynomials of both matrices verify:*

$$\chi_B(\lambda) = k^n \chi_A\left(\frac{\lambda}{k}\right) \tag{18}$$

As a consequence, the eigenvalues of B are the eigenvalues of A multiplied by the scalar k.

Proof. Taking into account the definition of the characteristic polynomial:

$$\chi_B(\lambda) = det(\lambda I - B) = \begin{vmatrix} \lambda - b_{11} & -b_{12} & \cdots & -b_{1n} \\ -b_{21} & \lambda - b_{22} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \cdots & \lambda - b_{nn} \end{vmatrix} \tag{19}$$

where: $b_{ij} = k \cdot a_{ij}$.

Dividing each of the rows of the determinant (19) by the constant k, the value of the determinant is divided by k^n:

$$\chi_B(\lambda) = k^n \begin{vmatrix} \frac{\lambda - b_{11}}{k} & \frac{-b_{12}}{k} & \cdots & \frac{-b_{1n}}{k} \\ \frac{-b_{21}}{k} & \frac{\lambda - b_{22}}{k} & \cdots & \frac{-b_{2n}}{k} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-b_{n1}}{k} & \frac{-b_{n2}}{k} & \cdots & \frac{\lambda - b_{nn}}{k} \end{vmatrix} = k^n \begin{vmatrix} \frac{\lambda}{k} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \frac{\lambda}{k} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \frac{\lambda}{k} - a_{nn} \end{vmatrix} \tag{20}$$

From (20) we have that:

$$\chi_B(\lambda) = k^n det\left(\frac{\lambda}{k}I - A\right) = k^n \chi_A\left(\frac{\lambda}{k}\right) \Rightarrow \chi_B(\lambda) = k^n \chi_A\left(\frac{\lambda}{k}\right) \tag{21}$$

So the relation (18) has been proved. Once proved (18), the next chain of equivalences is obtained:

$$\lambda \text{ eigenvalue of } B \Leftrightarrow \ \chi_B(\lambda) = 0 \Leftrightarrow \chi_A\left(\frac{\lambda}{k}\right) = 0 \Leftrightarrow \ \frac{\lambda}{k} \text{ eigenvalue of } A \tag{22}$$

Thus, if λ_j is eigenvalue of A with multiplicity p, $\lambda_j \cdot k$ is eigenvalue of B with multiplicity p. ∎

We have experimentally proved that the equality (18) is verified. The analysis has been done using a rod of copper of length $L = 20cm$ and another rod of the same length but made of epoxy/alumina (100/188). The thermal diffusivities of both rods are $\alpha^2_{ep.al.} = 0.449 \cdot 10^{-2}\ cm^2/s$ and $\alpha^2_{copper} = 1.0745\ cm^2/s$. Both values are experimental but they agree with the values given in [16] and [4]. The relation between these diffusivities is given by:

$$k = \alpha^2_{copper}/\alpha^2_{ep.al.} = 2.39 \cdot 10^2 \tag{23}$$

The eigenvalues of the resulting ODE systems have been obtained, see Table (3). When the eigenvalues obtained for the material epoxy/alumina are multiplied by the constant k (23), the eigenvalues of the copper are obtained.

Number of elements	$\lvert\lambda_{max}\rvert$ epoxy/alum.	$\lvert\lambda_{min}\rvert$ epoxy/alum.	$\lvert\lambda_{max}\rvert$ copper	$\lvert\lambda_{min}\rvert$ copper
5	$2.5575 \cdot 10^{-3}$	$1.1448 \cdot 10^{-4}$	$6.1203 \cdot 10^{-1}$	$2.7395 \cdot 10^{-2}$
10	$1.2527 \cdot 10^{-2}$	$1.1170 \cdot 10^{-4}$	$2.9979 \cdot 10^{0}$	$2.6731 \cdot 10^{-2}$
20	$5.2897 \cdot 10^{-2}$	$1.1101 \cdot 10^{-4}$	$1.2659 \cdot 10$	$2.6567 \cdot 10^{-2}$
40	$2.1453 \cdot 10^{-1}$	$1.1084 \cdot 10^{-4}$	$5.1338 \cdot 10$	$2.6526 \cdot 10^{-2}$
80	$8.6108 \cdot 10^{-1}$	$1.1080 \cdot 10^{-4}$	$2.0607 \cdot 10^{2}$	$2.6516 \cdot 10^{-2}$
160	$3.4473 \cdot 10^{0}$	$1.1079 \cdot 10^{-4}$	$8.2498 \cdot 10^{2}$	$2.6513 \cdot 10^{-2}$
320	$1.3792 \cdot 10^{1}$	$1.1079 \cdot 10^{-4}$	$3.3006 \cdot 10^{3}$	$2.6512 \cdot 10^{-2}$
640	$5.5172 \cdot 10^{1}$	$1.1079 \cdot 10^{-4}$	$1.3203 \cdot 10^{4}$	$2.6512 \cdot 10^{-2}$
1280	$2.2069 \cdot 10^{2}$	$1.1079 \cdot 10^{-4}$	$5.2814 \cdot 10^{4}$	$2.6512 \cdot 10^{-2}$

Table 3: Eigenvalues of the ODE system (11) vs. number of elements.

Conclusions:

- The greater the thermal diffusivity, the greater the eigenvalues of the resulting ODE system (11).
- Given a value of a thermal diffusivity α^2, if this value is multiplied by a constant k, all the eigenvalues of the new ODE system are multiplied by the same constant k. As the relation between the thermal diffusivities of the materials epoxy/alumina and copper is given by (23), the connection that exists among the eigenvalues of the jacobian matrices is exactly the same:

$$\begin{cases} \left|\lambda_{max,copper}\right| = 2.3931 \cdot 10^2 \cdot \left|\lambda_{max,ep.al.}\right| \\ \left|\lambda_{min,copper}\right| = 2.3931 \cdot 10^2 \cdot \left|\lambda_{min,ep.al.}\right| \end{cases} \tag{24}$$

4. STIFFNESS OF THE WAVE PDE

In Section (2) we have already seen that the FEM discretization of the wave PDE (2) with $(n+2)$ nodes and $(n+1)$ elements leads to the ODE system (16). In this section the eigenvalues of the jacobian matrix of the ODE system (16) have been studied.

First of all, the relation between the eigenvalues of the ODE systems derived by the diffusion and wave-type PDEs is proved. The previous relation between the eigenvalues of the jacobian matrices of these problems and the conclusions achieved in the case of the diffusion PDE have been used in order to conclude the stiffness of the problem (16) as function of the number of elements of the discretization, the length of the string and the speed of propagation of the wave.

4.1. RELATION BETWEEN THE EIGENVALUES OF THE ODE SYSTEMS DERIVED FROM THE DIFFUSION AND WAVE-TYPE PDES

The jacobian matrices of the diffusion and wave-type PDEs are given by:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{d}} = A, \qquad \frac{\partial \mathbf{f}}{\partial \mathbf{d}} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \tag{25}$$

where $A = -M^{-1}K$. In the following theorem the relation between the eigenvalues of the matrices that verify (25) is proved.

Theorem 4.1. *Let be A a square matrix of dimension n and $B = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$. Then, the characteristic polynomials of both matrices verify:*

$$\chi_B(\lambda) = \chi_A(\lambda^2) \tag{26}$$

As a consequence of this result, the eigenvalues of the matrix B are the square root of the eigenvalues of the matrix A.

Proof. The characteristic polynomial of B is given by:

$$\chi_B(\lambda) = det\begin{pmatrix} \lambda I & -I \\ -A & \lambda I \end{pmatrix} \tag{27}$$

As A is a square matrix of dimension n, the matrix B has dimension $2n \times 2n$. If in the determinant (27) the $(n + i)$ column multiplied by λ is added to column i, and this operation is repeated for the indexes $i = 1, 2, 3, ..., n$, obtaining the following determinant:

$$\chi_B(\lambda) = det\begin{pmatrix} 0 & -I \\ C & \lambda I \end{pmatrix} \tag{28}$$

where:

$$C = \begin{pmatrix} -a_{11} + \lambda^2 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} + \lambda^2 & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{nn} + \lambda^2 \end{pmatrix} = \lambda^2 I - A \tag{29}$$

Taking into account (29), expression (28) results:

$$\chi_B(\lambda) = det\begin{pmatrix} 0 & -I \\ \lambda^2 I - A & \lambda I \end{pmatrix} \tag{30}$$

We will develope the determinant (30) starting from the first row. All the elements of this row except from the element of position $(n + 1)$ are zero. Observe that the element of this position is -1. Then, we will continue developing the resulting determinant from row 2, until we reach the row n and we develope from this row. After these developments we obtain:

$$\begin{aligned} \chi_B(\lambda) &= (-1)^{1+(n+1)}(-1)(-1)^{2+(n+2)}(-1)...(-1)^{n+2n}(-1)det(\lambda^2 I - A) \\ &= (-1)^n(-1)(-1)^n(-1)...(-1)^n(-1)det(\lambda^2 I - A) \\ &= (-1)^{n^2+n}det(\lambda^2 I - A) = det(\lambda^2 I - A) = \chi_A(\lambda^2) \end{aligned} \tag{31}$$

During the proof (31), it has to be taken into account that $n^2 + n = n(n + 1)$ is even. Once the equality (26) is proved, the following chain of equivalences is obtained:

$$\lambda \text{ eigenvalue of B} \Leftrightarrow \chi_B(\lambda) = 0 \Leftrightarrow \chi_A(\lambda^2) = 0 \Leftrightarrow \lambda^2 \text{ eigenvalue of A} \tag{32}$$

Thus, if λ_j is eigenvalue of A with multiplicity p, then, $\pm\sqrt{\lambda_j}$ is eigenvalue of B with multiplicity p. ∎

In the Table (4) the greatest and smallest eigenvalues of the ODE systems derived from the diffusion and the wave-type PDEs when $\alpha^2 = 1$ and $L = 8$, are tabulated. We hereby confirm that the eigenvalues of the ODE system (16) resulted from the wave-type PDE are the square root of the eigenvalues of the ODE system obtained from the diffusion PDE (11).

Number of elements	λ_{max} diffusion	λ_{min} diffusion	λ_{max} wave	λ_{min} wave
50	$-4.6737 \cdot 10^2$	$-1.5426 \cdot 10^{-1}$	$\pm 2.1619 \cdot 10^1 i$	$\pm 3.9270 \cdot 10^{-1} i$
200	$-7.4986 \cdot 10^3$	$-1.5422 \cdot 10^{-1}$	$\pm 8.6595 \cdot 10^1 i$	$\pm 3.9270 \cdot 10^{-1} i$
800	$-1.2000 \cdot 10^5$	$-1.5421 \cdot 10^{-1}$	$\pm 3.4641 \cdot 10^2 i$	$\pm 3.9270 \cdot 10^{-1} i$

Table 4: Eigenvalues of the ODE systems derived from the diffusion and wave-type PDEs, being $\alpha^2 = 1$ and $L = 8$.

4.2. STIFFNESS AS FUNCTION OF THE NUMBER OF ELEMENTS OF THE DISCRETIZATION

Having fixed α^2 and the length of the string, we change the number of elements of the discretization. We are interested in analysing the greatest and the smallest eigenvalues in module. Real data of a string of a guitar made of carbon fiber wire is considered [7]: length $L = 0.648m$, diameter $d = 0.254 \cdot 10^{-3}m$, area of the section $S = 0,25 \cdot \pi \cdot d^2$, frequency $f = 329.60Hz$ and tension $T = 4 \cdot \rho \cdot L^2 \cdot f^2$, being $\rho = 1750 \cdot S$ the mass per unit length and $1750kg/m^3$ the mass per unit volume of the carbon fiber. It is considered as if the string movement was linear.

Number of elements	$\lvert\lambda_{max}\rvert$	$\lvert\lambda_{min}\rvert$
50	$1.1366 \cdot 10^5$	$2.0649 \cdot 10^3$
100	$2.2757 \cdot 10^5$	$2.0647 \cdot 10^3$
200	$4.5526 \cdot 10^5$	$2.0646 \cdot 10^3$
400	$9.1062 \cdot 10^5$	$2.0646 \cdot 10^3$
800	$1.8212 \cdot 10^6$	$2.0646 \cdot 10^3$

Table 5: Eigenvalues of the problem (16) for the cited string of a guitar.

The eigenvalues of the resulting first ODE system have been computed in Table (5), and these are the conclusions:

- The greatest eigenvalue of the ODE system (16) increases as the number of elements of the discretization increases.
- Given a discretization of m elements, if this quantity is multiplied by n obtaining in this way a discretization of $n \cdot m$ elements, the greatest eigenvalue of the ODE system (16) is multiplied by n.

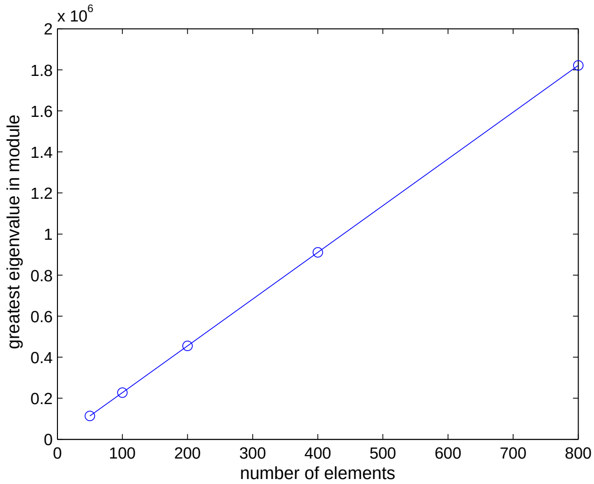

Fig. 2.: The greatest eigenvalue of the problem (16) for a string of a guitar.

4.3. STIFFNESS AS FUNCTION OF THE LENGTH OF THE STRING

Having fixed α^2 and the number of elements of the discretization, strings of different lengths are considered and the greatest and smallest eigenvalues in module are analysed. Taking into account the relation between the jacobian matrices of the ODE systems (11) and (16) obtained after having discretized the corresponding PDEs, and considering the conclusions achieved in Section (3.2), it can be concluded that:

- The smaller the length of the string, the greater the eigenvalues of the ODE system (16).
- Given a string of length l, if this length is multiplied by x obtaining in this way a new string of length $l \cdot x$, the eigenvalues of the ODE system (16) are multiplied by $\frac{1}{x}$.

4.4. STIFFNESS AS FUNCTION OF THE WAVE SPEED OF PROPAGATION

Having fixed the length of the string and the number of elements of the discretization, the parameter α^2 is changed. Taking into account the relation (26) and the conclusions achieved in Section (3.3), we have:

- The greater α^2, the greater the eigenvalues of the ODE system (16).
- Given a value of α^2, if this value is multiplied by k, all the eigenvalues of the ODE system (16) are multiplied by $\sqrt{k}$.

5. CONCLUSIONS

In this paper the one-dimensional diffusion-type and wave-type linear Partial Differential Equations (PDEs) with boundary conditions and initial conditions have been considered. After applying the FEM discretization to these equations, in the case of the diffusion-type PDE a first order ODE system is obtained and a second order ODE system in the case of the wave-type PDE, which can be reduced to a first order ODE system doubling the number of unknowns and equations.

The stiffness of the resulting ODE systems has been studied. It has been numerically proved that the ODE system that results after having discretized the diffusion-type PDE presents more stiffness when the number of elements of the discretization is increased or when the length of the rod is shortened. For the same system, it has been proved analytically that the stiffness of the ODE system increases with the increase of the thermal diffusivity.

It has also been proved the relation between the eigenvalues of the ODE systems obtained from the diffusion-type and the wave-type PDEs, the eigenvalues of the latter ODE system being the square root of the eigenvalues of the first one. Taking into account this result and the conclusions achieved for the ODE system that results from the diffusion PDE, it is concluded that the ODE system that results after discretizing the wave-type PDE presents more stiffness as the number of elements or the speed of propagation are increased, or when the length of the string is shortened.

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