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SOME REMARKS ON EIGENVALUE PROBLEMS FOR THE LAPLACE OPERATOR

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Abstract This paper deals with the eigenvalue problems in some bounded convex domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, under Dirichlet and Neumann conditions. In particular another proof of Payne - Weinberger's inequality $\mu_1 \geq \left(\frac{\pi}{D}\right)^2$ will be provided, where μ_1 is the first nontrivial eigenvalue of the free membrane problem, and D is the diameter of Ω.

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1. INTRODUCTION

This paper deals with the following basic eigenvalue problems

$$\Delta u + \lambda u = 0, \quad x \in \Omega, \qquad u = 0, \quad x \in \partial\Omega, \tag{1}$$

and

$$\Delta v + \mu v = 0, \quad x \in \Omega, \qquad \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega, \tag{2}$$

where Ω is a bounded convex domain in $\mathbb{R}^N$, $N \geq 2$. In (2) $\frac{\partial v}{\partial n}$ is the outward normal derivative of v on the boundary $\partial\Omega$.

It is well known that these problems have nontrivial solutions only for specific values λ and μ called eigenvalues satisfying

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \to \infty, \tag{3}$$

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \to \infty, \tag{4}$$

The corresponding eigenfunctions are denoted by u_k (associated to λ_k) and v_k (associated to μ_k). These eigenvalues and eigenfunctions depend on the geometry and the size of Ω.

These quantities are known only for particular domains including parallelepipeds and balls. The goal of this paper is to derive the following result:

Let Ω be a bounded convex domain in $\mathbb{R}^N$, $N \geq 2$. Then we have

$$|u|_{\max}^{-2} \int_\Omega u^2 dx \le \frac{1}{2}|\Omega|, \tag{5}$$

$$|v|_{\max}^{-2} \int_\Omega v^2 dx \le \frac{1}{2}|\Omega|, \tag{6}$$

where $|\Omega| := \int_\Omega dx$ is the volume of Ω. Assuming that Ω is bounded strictly convex in $\mathbb{R}^2$, we have the inequalities

$$u^2(P) \le \frac{1}{2}|u|_{\max}^2, \quad \text{with } |\nabla u(P)| = \max_\Omega |\nabla u|, \tag{7}$$

$$v^2(P) \le \frac{1}{2}|v|_{\max}^2, \quad \text{with } |\nabla v(P)| = \max_\Omega |\nabla v|, \tag{8}$$

Inequalities (5), (7) are valid for all eigenfunctions $u_k(x)$ and (6), (8) are valid for all nontrivial eigenfunctions $v_k(x)$. Finally, a new proof of Payne-Weinberger's inequality [5]

$$\mu_1 \ge \left(\frac{\pi}{D}\right)^2 \tag{9}$$

will be provided. Inequality (9) holds for a bounded convex domain $\Omega \subset \mathbb{R}^N$, $n \ge 2$, D is the diameter of Ω. Section 2 of this paper reviews Payne-Stakgold's maximum principle ([4]) which is the main tool of this investigation. The proofs of (5)-(9) are presented in Section 3.

2. PAYNE-STAKGOLD'S MAXIMUM PRINCIPLE

Payne and Stakgold have established the following maximum principle ([4]):

Theorem 2.1. *Let $u(x)$ be an arbitrary eigenfunction of the problem (1) associated to the eigenvalue λ. Assuming Ω bounded convex with a smooth boundary $\partial\Omega$, the auxiliary function*

$$\Psi(x) := |\nabla u|^2 + \lambda u^2 \tag{10}$$

takes its maximum value at a critical point of $u(x)$, i.e. we have

$$|\nabla u|^2 \le \lambda\left\{|u|_{\max}^2 - u^2\right\}, \quad x \in \Omega. \tag{11}$$

Payne and Stakgold have formulated Theorem 2.1 only for the first eigenvalue and eigenfunction, but the result and the proof extends to all (λ_k, u_k). A similar maximum principle holds for all eigenfunctions v_k of the problem (2).

Theorem 2.2. *Let $v(x)$ be an arbitrary eigenfunction of the problem (2) associated to the nontrivial eigenvalue μ. Assuming Ω is bounded and convex with a smooth boundary $\partial\Omega$, the auxiliary function*

$$\Phi(x) := |\nabla v|^2 + \mu v^2, \qquad x \in \Omega, \tag{12}$$

takes its maximum value at a critical point of $v(x)$, i.e. we have

$$|\nabla v|^2 \leq \mu\left\{|v|^2_{\max} - v^2\right\}, \quad x \in \Omega. \tag{13}$$

Proof. : The proof of Theorem 2.2 is similar to that of Theorem 2.1. We have to show that the auxiliary function Φ satisfies the following two conditions

$$\frac{\partial \Phi}{\partial n} \leq 0, \quad x \in \partial\Omega, \tag{14}$$

and

$$\Delta\Phi + |\nabla v|^{-2}\vec{W}\cdot\nabla\Phi \geq 0, \quad x \in \Omega - \omega, \tag{15}$$

where $\bar{W}$ is some regular vector field defined on Ω, and $\omega := \{x \in \Omega |\ \nabla v(x) = 0\}$ is the set of critical points of $v(x)$. The conclusion of Theorem 2.2 is a straightforward consequence of (14), (15), together with Hopf's first ([1], [6]) and second ([2], [6]) principles. The differential inequality (15) is established in [4]. It remains to justify (14). Let $\vec{n} = (n_1, \ldots, n_N)$ be the outward unit normal vector on $\partial\Omega$. Making use of the notation $v_{,k} = \frac{\partial v}{\partial x_k}$, $k = 1, \ldots, N$ and adopting the summation convention on repeated subscripts we compute

$$\frac{\partial \Phi}{\partial n} = \frac{1}{2}\frac{\partial}{\partial n}\left(|\nabla v|^2\right) = v_{,k}v_{,ki}n_i = v_{,k}\left(v_{,i}n_i\right)_{,k} - v_{,k}v_{,i}n_{i,k} \tag{16}$$

with $\frac{\partial v}{\partial n} = v_{,i}n_i$ we have for $x \in \partial\Omega$

$$v_{,k}\left(v_{,i}n_i\right)_{,k} = v_{,k}\left\{n_l\left(n_l\frac{\partial}{\partial x_k} - n_k\frac{\partial}{\partial x_l}\right)\left(\frac{\partial v}{\partial n}\right) + n_l n_k\left(\frac{\partial v}{\partial n}\right)_{,l}\right\} = 0 \tag{17}$$

Note that the first term on the right side of (17) vanishes since it is a linear combination of tangential derivatives of $\frac{\partial v}{\partial n}$. Moreover, we have

$$v_{,i}v_{,k}n_{i,k} = v_{,i}v_{,k}\left\{n_{i,k} - n_{i,j}n_j n_k\right\} = v_{,i}v_{,k}B_{ik}, \tag{18}$$

where

$$B_{ik} := n_{i,k} - n_{i,j}n_j n_k \tag{19}$$

is the curvature matrix associated to $\partial\Omega$. This matrix depends only on the geometry of Ω. If $\partial\Omega$ is represented locally by the equation $\varphi(x) = 0$, then B_{ik} has the form

$$B_{ik} = \frac{\varphi_{,ik}}{|\nabla\varphi|} - \frac{1}{|\nabla\varphi|^3}\{\varphi_{,lk}\varphi_{,l}\varphi_{,i} + \varphi_{,li}\varphi_{,l}\varphi_{,k}\} + \frac{\varphi_{,lj}\varphi_{,l}\varphi_{,j}}{|\nabla\varphi|^5}\varphi_{,i}\varphi_{,k}. \tag{20}$$

This shows that B_{ik} is symmetric. The eigenvalues of B_{ik} are therefore real. Moreover we have $B_{ik}n_i n_k = 0$ so that $\vec{n}$ is an eigenvector associated to the eigenvalue zero. The $N-1$ other eigenvalues $k_1, \ldots, k_{N-1}$ are the principal curvatures of $\partial\Omega$. By the

convexity of Ω we have $k_i \geq 0$ on $\partial\Omega$, $i = 1, \ldots, N-1$. The matrix B_{ik} is therefore positive semi-definite. From this fact together with (16), (17), (18) we obtain

$$\frac{\partial \Phi}{\partial n} = -B_{ik} v_{,i} v_{,k} \leq 0, \quad x \in \partial\Omega. \tag{21}$$

This achieves the proof of Theorem 2.2. ∎

3. APPLICATIONS

1. Integrating (13) over Ω we obtain

$$\int_\Omega |\nabla v|^2 dx + \mu \int_\Omega v^2 dx \leq \mu |\Omega| \, |v|^2_{\max}, \tag{22}$$

where $|\Omega| := \int_\Omega dx$ is the volume of Ω. Dividing (22) by $\int_\Omega v^2 dx$ and noting that $\frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega v^2 dx} = \mu$, we get

$$|v|^{-2}_{\max} \int_\Omega v^2 dx \leq \frac{1}{2} |\Omega|, \tag{23}$$

in agreement with (6). The proof of (5) is similar.

2. For the proof of (7), (8) we need the following two results of Payne (see [3]).

Theorem 3.1. *Let $u(x)$ be an arbitrary function in $C^2(\Omega)$ with $u = \Delta u = 0$ on a portion γ of $\partial\Omega$ on which the average curvature is positive. Then $|\nabla u|$ takes its maximum value at $P \in \gamma$ if and only if $u \equiv 0$ in Ω.*

Theorem 3.2. *Let $v(x)$ be an arbitrary function in $C^2(\Omega)$ with $\frac{\partial v}{\partial n} = 0$ on a portion γ of $\partial\Omega$ on which the $(N-1)$ principal curvatures k_i are positive.*

Then $|\nabla v|$ takes its maximum value at $P \in \gamma$ if and only if $v \equiv const$ in Ω.

It follows from Theorem 3.1 that each point P at which an eigenfunction $u(x)$ of (1) satisfies $|\nabla u(P)| = |\nabla u|_{\max}$ is an interior point of Ω assumed bounded and convex, so that we must have at P

$$\frac{1}{2}\Big(|\nabla u|^2\Big)_{,k} = u_{,ik} u_{,i} = 0, \quad k = 1, \ldots, N \tag{24}$$

and

$$\frac{1}{2}\Delta\Big(|\nabla u|^2\Big) = u_{,ik} u_{,ik} + 2(\Delta u)_{,k} u_{,k} = u_{,ik} u_{,ik} - \lambda |\nabla u|^2 \leq 0, \tag{25}$$

where λ is the corresponding eigenvalue. With $|\nabla u| \neq 0$ at P, (24) implies

$$\det |u_{,ik}| = 0 \quad \text{at } P \tag{26}$$

In the particular case of a plane domain Ω, (26) leads to

$$u_{,ik} u_{,ik} = (\Delta u)^2 = \lambda^2 u^2 \quad \text{at } P. \tag{27}$$

From (25) and (27) we obtain

$$|\nabla u|^2 \geq \lambda u^2 \ \text{ at } P. \tag{28}$$

Inequalities (11) and (28) lead to

$$2\lambda u^2 \leq |\nabla u|^2 + \lambda u^2 \leq \lambda |u|^2_{\max} \quad \text{at } P, \tag{29}$$

in agreement with (7). The proof of (8) is similar except that we make use of Theorem 3.2 instead of Theorem 3.1.

3. To conclude this section we want to produce another proof of Payne-Weinberger's inequality (9), based on the maximum principle (13). We note that the first nonconstant eigenfunction $v(x)$ of (2) changes sign in Ω. Its nodal set $\Gamma := \{x \in \Omega : v(x) = 0\}$ separates Ω in two subdomains $\Omega_\pm$ with $v > 0$ in Ω_+ and $v < 0$ in Ω_-. Γ has no self intersection and extends up to the boundary $\partial\Omega$. In $\mathbb{R}^2$, Γ is a curve inside Ω joining two distinct boundary points. From Hopf's second principle ([2]. [6]), $|\nabla v|$ does not vanish on Γ, so that the auxiliary function Φ defined by (12) cannot take its maximum value on Γ. Theorem 2.2 is therefore separately applicable on both subdomains $\Omega_\pm$. This lead to the somewhat sharper version of (13)

$$|\nabla v|^2 \leq \begin{cases} \mu\{v^2_{\max} - v^2\}, & x \in \Omega_+, \\ \mu\{v^2_{\min} - v^2\}, & x \in \Omega_-. \end{cases} \tag{30}$$

Let P_+ be a point in $\bar{\Omega}_+$ at which $v(x)$ takes its maximum value , and Q_+ be a point of Γ nearest to P_+. Integrating (30) along the segment $|P_+Q_+|$ located in Ω_+ we obtain

$$\sqrt{\mu}\int_{Q_+}^{P_+} dx = \sqrt{\mu}d_+ \geq \int_0^{v_{\max}} \frac{dv}{\sqrt{v^2_{\max} - v^2}} = \frac{\pi}{2}, \tag{31}$$

where d_+ is the length of $|P_+Q_+| \in \Omega_+$. In the same way we obtain

$$\sqrt{\mu}d_- \geq \frac{\pi}{2}, \tag{32}$$

where d_- is the length of $|P_-Q_-| \in \Omega_-$ joining P_- where $v(x)$ takes its minimum value to the nearest point Q_- of Γ. Adding the inequalities (31) and (32) we obtain

$$\pi \leq \sqrt{\mu}(d_- + d_+) \leq \sqrt{\mu}D \tag{33}$$

where D is the diameter of Ω. This achieves the proof of (9).

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