

COMMON FIXED POINTS FOR BANACH-CARISTI CONTRACTIVE PAIRS

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Abstract Further extensions are given for the common fixed point statement in Dien [J. Math. Anal. Appl., 187 (1994), 76-90], involving Banach-Caristi contractive pairs.

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1. INTRODUCTION

Let (X, d) be a complete metric space; and $T \in \mathcal{F}(X)$ be a selfmap of X . [Here, for each couple of sets $\{A, B\}$, $\mathcal{F}(A, B)$ stands for the class of all functions from A to B ; if $A = B$, one writes $\mathcal{F}(A, A)$ as $\mathcal{F}(A)$]. The following fixed point statement in Caristi and Kirk [6] (referred to as: Caristi-Kirk theorem) is our starting point.

Theorem 1.1. *Assume that there exists a function $\alpha : X \rightarrow R_+ := [0, \infty[$ with*

$$(a01) \quad d(x, Tx) \leq \alpha(x) - \alpha(Tx), \text{ for each } x \in X$$

$$(a02) \quad \alpha(\cdot) \text{ is lsc on } X \text{ (} \liminf_n \alpha(x_n) \geq \alpha(x), \text{ whenever } x_n \rightarrow x \text{)}.$$

Then, T has at least one fixed point in X .

Note that, in terms of the [associated to $\alpha(\cdot)$] order

$$(a03) \quad (x, y \in X): x \leq y \text{ iff } d(x, y) \leq \alpha(x) - \alpha(y)$$

the contractive condition (a01) becomes

$$(a04) \quad x \leq Tx, \text{ for each } x \in X \text{ (i.e.: } T \text{ is } \textit{progressive} \text{ on } X \text{)}.$$

So, Theorem 1.1 is deductible from the arguments in Ekeland's variational principle [8]. Further aspects may be found in Brezis and Browder [4]; see also Turinici [12].

Now, the Caristi-Kirk theorem found (especially via Ekeland's approach) some basic applications to control and optimization, generalized differential calculus, critical point theory and normal solvability; see the above references for details. As a consequence, many extensions of this result were proposed. Here, we shall concentrate on the 1981 statement in this area due to Bhakta and Basu [3]. Let $\{S, T\}$ be a couple of selfmaps in $\mathcal{F}(X)$. We say that $z \in X$ is a *common fixed point* of $\{S, T\}$ if $Sz = Tz = z$.

Sufficient conditions guaranteeing such a property are obtainable as below. Call the selfmap U in $\mathcal{F}(\mathcal{X})$, *orbitally continuous* (on X), if

$$(a05) \quad z = \lim_n U^{i(n)}x \text{ implies } Uz = \lim_n U^{i(n)+1}x;$$

here, $(i(n); n \geq 0)$ is a rank sequence with $i(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Theorem 1.2. *Suppose that*

$$(a06) \quad \text{both } S \text{ and } T \text{ are orbitally continuous;}$$

and let the functions $\alpha, \beta : X \rightarrow \mathbb{R}_+$ be such that

$$(a07) \quad d(Sx, Ty) \leq \alpha(x) - \alpha(Sx) + \beta(y) - \beta(Ty), \text{ for all } x, y \in X.$$

Then,

- i)** S and T have a unique common fixed point $z \in X$,
- ii)** $S^n x \rightarrow z$ and $T^n x \rightarrow z$ as $n \rightarrow \infty$, for each $x \in X$.

A partial extension of this result was given in the 1994 paper by Dien [7]:

Theorem 1.3. *Suppose that (a06) holds. In addition, let the number $q \in [0, 1[$ and the function $\alpha : X \rightarrow \mathbb{R}_+$ be such that*

$$(a08) \quad d(Sx, Ty) \leq qd(x, y) + \alpha(x) - \alpha(Sx) + \alpha(y) - \alpha(Ty), \forall x, y \in X.$$

Then, conclusions of Theorem 1.2 are retainable.

[As a matter of fact, the original result is with $\alpha = \alpha_1 + \dots + \alpha_k$, where $\{\alpha_i; 1 \leq i \leq k\}$ is a finite system in $\mathcal{F}(\mathcal{X}, \mathbb{R}_+)$. But it gives, practically, the same amount of information as the result in question].

Note that, when $\alpha(\cdot)$ is a constant function and $S = T$, then Theorem 1.3 implies the Banach contraction principle [2]. In addition, (a01) follows from (a08) when $S = I$ (=the *identity map* of $\mathcal{F}(\mathcal{X})$) and $x = y$; for this reason, the couple $\{S, T\}$ above will be referred to as *Banach-Caristi contractive*. It is to be stressed that Theorem 1.1 does not follow from Theorem 1.3; because, the (essential for Theorem 1.1) condition (a02) is not obtainable from the conditions of Theorem 1.3. However, the underlying relationship between these results holds whenever (a06) is accepted, in place of (a02). (This clarifies an assertion made in Ume and Yi [13]; we do not give details). On the other hand, Dien's result cannot be deduced from Caristi-Kirk's; because (a06) cannot be deduced from the conditions of Theorem 1.1. Finally, Theorem 1.2 cannot be viewed as a particular case of Theorem 1.3 (when $q = 0$); because the functions $\alpha(\cdot)$ and $\beta(\cdot)$ may be distinct.

Having this precise, it is our aim in the following to establish (cf. Section 2) a common extension of these statements; as well as (in Section 3) a sum approach of it. Some other aspects will be delineated elsewhere.

2. MAIN RESULT

Let $\varphi \in \mathcal{F}(\mathcal{R}_+)$ be a function; call it *regressive* provided $\varphi(0) = 0$ and $\varphi(t) < t$, $\forall t \in \mathcal{R}_+^0 :=]0, \infty[$; the class of all these will be denoted as $\mathcal{F}(\mathcal{R}_+)$. For example, any function $\varphi = qJ$ where $q \in [0, 1[$, is regressive; here, J is the *identity function* of $\mathcal{F}(\mathcal{R}_+)$ ($J(t) = t, t \in \mathcal{R}_+$).

Now, fix some $\varphi \in \mathcal{F}(\mathcal{R}_+)$. Denote $\psi := J - \varphi$; and call it, the *complement* of φ . Clearly, $\psi \in \mathcal{F}(\mathcal{R}_+)$; precisely,

$$\psi(0) = 0; 0 < \psi(t) \leq t, \forall t \in \mathcal{R}_+^0. \tag{1}$$

For an easy reference, we list our basic hypotheses. The former of these is

$$(b01) \ \varphi \text{ is super-additive: } \varphi(t + s) \geq \varphi(t) + \varphi(s), \text{ for all } t, s \geq 0;$$

clearly, φ must be increasing in such a case. And the latter condition writes:

$$(b02) \ \psi := J - \varphi \text{ is coercive: } \psi(t) \rightarrow \infty \text{ as } t \rightarrow \infty;$$

referred to as: φ is *complementary coercive*. Note that (by this very definition)

$$g(r) := \sup\{t \geq 0; \psi(t) \leq r\} < \infty, \text{ for each } r \in \mathcal{R}_+; \tag{2}$$

whence, $g(\cdot)$ is an element of $\mathcal{F}(\mathcal{R}_+)$. Moreover (from (1) above)

$$g(0) = 0; g(r) \geq r, \forall r \in \mathcal{R}_+. \tag{3}$$

The following auxiliary fact will be useful for us.

Lemma 2.1. *Let $\varphi \in \mathcal{F}(\mathcal{R}_+)$ be super-additive and complementary coercive. Further, let the sequence $(\theta_n; n \geq 0)$ in \mathcal{R}_+ be such that*

$$(b03) \ \theta_{m+1} \leq \varphi(\theta_m) + \delta_m - \delta_{m+1}, \text{ for all } m \geq 0;$$

where $(\delta_n; n \geq 0)$ is a sequence in \mathcal{R}_+ . Then, the series $\sum_n \theta_n$ converges.

Proof. Let $(\sigma_i := \theta_0 + \dots + \theta_i; i \geq 0)$ be the partial sum sequence attached to $(\theta_n; n \geq 0)$. For each $j \geq 0$, we have (summing in (b03) from $m = 0$ to $m = j$)

$$\theta_1 + \dots + \theta_{j+1} \leq \varphi(\theta_0) + \dots + \varphi(\theta_j) + \delta_0 - \delta_{j+1}.$$

This, along with the super-additivity of φ , gives $\sigma_j \leq \varphi(\sigma_j) + \theta_0 + \delta_0$. But then, (2) yields $[\sigma_n \leq g(\theta_0 + \delta_0) < \infty, \forall n]$; wherefrom, all is clear. ■

We now state the promised result. Let (X, d) be a complete metric space; and $\{S, T\}$ be a pair in $\mathcal{F}(X)$.

Theorem 2.1. *Suppose that (a06) holds. In addition, let the function $\varphi \in \mathcal{F}(\mathcal{R}_+)$ as in (b01)+(b02), and the map $\gamma : X \times X \rightarrow \mathcal{R}_+$, be taken so as*

$$(b04) \quad d(Sx, Ty) \leq \varphi(d(x, y)) + \gamma(x, y) - \gamma(Sx, Ty), \text{ for all } x, y \in X.$$

Then, conclusions of Theorem 1.2 are retainable.

Proof. Given $x_0, y_0 \in X$, put $(x_n = S^n x_0; n \geq 0)$, $(y_n = T^n y_0; n \geq 0)$. From (b04), one has (by a finite induction argument), the iterative type relations

$$d(x_{m+1}, y_{m+1}) \leq \varphi(d(x_m, y_m)) + \gamma(x_m, y_m) - \gamma(x_{m+1}, y_{m+1}), \text{ for all } m \geq 0.$$

Combining with Lemma 2.1 (and the adopted notations), one derives that the series $\sum_n d(S^n x_0, T^n y_0)$ converges.

Further, let us develop the same reasoning by starting from the points $u_0 = Sx_0$ and y_0 ; one derives that the series $\sum_n d(S^n u_0, T^n y_0)$ converges; or, equivalently: the series $\sum_n d(S^{n+1} x_0, T^n y_0)$ converges. This, along with

$$d(S^n x_0, S^{n+1} x_0) \leq d(S^n x_0, T^n x_0) + d(S^{n+1} x_0, T^n y_0), \quad \forall n \geq 0$$

tells us that the series $\sum_n d(S^n x_0, S^{n+1} x_0)$ converges; wherefrom, $(S^n x_0; n \geq 0)$ is a d -Cauchy sequence. In a similar way (starting from the points x_0 and $v_0 = Ty_0$) one proves that the series $\sum_n d(T^n y_0, T^{n+1} y_0)$ converges; whence, $(T^n y_0; n \geq 0)$ is a d -Cauchy sequence. As (X, d) is complete, we have that $S^n x_0 \rightarrow z$ and $T^n y_0 \rightarrow w$, for some $z, w \in X$. Combining with the orbital continuity of both S and T , gives $S(S^n x_0) \rightarrow Sz$, $T(T^n y_0) \rightarrow Tw$. But, $S(S^n x_0) = S^{n+1} x_0 \rightarrow z$, $T(T^n y_0) = T^{n+1} y_0 \rightarrow w$; and this yields $z = Sz$, $w = Tw$. Finally, from (b04) again, we have $d(z, w) \leq \varphi(d(z, w))$; so that, $z = w$. Hence, z is a common fixed point of $\{S, T\}$. Its uniqueness is obtainable by the argument we just developed for (z, w) ; and, from this, we are done. ■

In particular, when $\varphi \in \mathcal{F}(\mathcal{R}_+)$ is taken as

$$(b05) \quad \varphi(t) = qt, \quad t \geq 0, \text{ for some } q \in [0, 1[,$$

conditions (b01)+(b02) hold; and then, under the choice

$$(b06) \quad \gamma(x, y) = \alpha(x) + \alpha(y), \quad x, y \in X \text{ (where } \alpha \in \mathcal{F}(X, \mathcal{R}_+))$$

the corresponding version of Theorem 2.1 is just Theorem 1.3. Note that, under the same framework, a more general choice for γ is

$$(b07) \quad \gamma(x, y) = \alpha(x) + \beta(y), \quad x, y \in X \text{ (where } \alpha, \beta \in \mathcal{F}(X, \mathcal{R}_+)).$$

This version of Theorem 2.1 is (under $\varphi = 0$) just Theorem 1.2 above. Further aspects may be found in Alimohammady et al [1]; see also Kadelburg et al [10].

3. FURTHER EXTENSIONS

A simple inspection of the argument we just developed shows that it depends (via Lemma 2.1) on the super-additivity of the function $\varphi \in \mathcal{F}(\mathcal{R}_+)$; so, we may ask

whether this cannot be removed. An appropriate answer is available, if we arrange for the sums given by the argument of Theorem 2.1 being obtainable in a direct way from the contractive conditions.

Let (X, d) be a complete metric space; and $\{S, T\}$ be a pair of selfmaps in $\mathcal{F}(X)$.

Theorem 3.1. *Suppose that (a06) holds. In addition, let the function $\varphi \in \mathcal{F}(\mathcal{R}_+)$ as in (b02) and the map $\gamma : X \times X \rightarrow \mathcal{R}_+$ be taken so as*

$$(c01) \quad \sum_{j=1}^n d(S^j x, T^j y) \leq \varphi\left(\sum_{j=0}^{n-1} d(S^j x, T^j y)\right) + \gamma(x, y) - \gamma(S^n x, T^n y),$$

for all $x, y \in X$ and all $n \geq 1$. Then, conclusions of Theorem 1.3 are retainable.

Proof. Given $x_0, y_0 \in X$, put $(x_n = S^n x_0; n \geq 0)$, $(y_n = T^n y_0; n \geq 0)$. Further, denote $(\theta_i = d(x_i, y_i); i \geq 0)$. By (c01) one has, for each $n \geq 1$,

$$\theta_1 + \dots + \theta_n \leq \varphi(\theta_0 + \dots + \theta_{n-1}) + \gamma(x_0, y_0) - \gamma(x_n, y_n);$$

wherefrom (after some transformations)

$$\theta_0 + \dots + \theta_{n-1} \leq \varphi(\theta_0 + \dots + \theta_{n-1}) + \theta_0 + \gamma(x_0, y_0), \quad \forall n \geq 1.$$

This, from (b02) (and the notations in Section 2), gives

$$\theta_0 + \dots + \theta_{n-1} \leq g[\theta_0 + \gamma(x_0, y_0)] < \infty, \quad \text{for all } n \geq 1;$$

so that (by the adopted notations) the series $\sum_n d(S^n x_0, T^n y_0)$ converges. Further, let us develop the same reasoning by starting from the points $u_0 = S x_0$ and y_0 ; one derives that the series $\sum_n d(S^n u_0, T^n y_0)$ converges; or, equivalently: the series $\sum_n d(S^{n+1} x_0, T^n y_0)$ converges. This, along with

$$d(S^n x_0, S^{n+1} x_0) \leq d(S^n x_0, T^n x_0) + d(S^{n+1} x_0, T^n y_0), \quad \forall n \geq 0$$

tells us that the series $\sum_n d(S^n x_0, S^{n+1} x_0)$ converges; wherefrom $(S^n x_0; n \geq 0)$ is a d -Cauchy sequence. In a similar way (starting from the points x_0 and $v_0 = T y_0$), one proves that the series $\sum_n d(T^n y_0, T^{n+1} y_0)$ converges; whence, $(T^n y_0; n \geq 0)$ is a d -Cauchy sequence. As (X, d) is complete, $S^n x_0 \rightarrow z$ and $T^n y_0 \rightarrow w$, for some $z, w \in X$. The remaining part of the argument runs as in Theorem 2.1, because (c01) \implies (b04); and, from this, all is clear. ■

Now, concrete examples of complementary coercive functions $\varphi \in \mathcal{F}(\mathcal{R}_+)$ are obtainable by starting from the choice

$$(c02) \quad \varphi(t) = t\chi(t), \quad t \geq 0,$$

where $\chi \in \mathcal{F}(\mathcal{R}_+)$ fulfills the regularity conditions

(c03) χ is increasing on \mathcal{R}_+^0 and $\chi(t) < 1, \forall t \in \mathcal{R}_+^0$.

The standard case is $[\chi(t) = q, t \geq 0]$, for some $q \in [0, 1[$; when Theorem 3.1 includes, in a direct way, Theorem 1.3; due, as above said, to Dien [7]. A technical extension of this one may be constructed according to

(c04) $\chi(t) = r_{n+1}$, when $t \in [t_n, t_{n+1}[$, for each $n \geq 0$;

where the sequence $(r_n; n \geq 1)$ in $]0, 1[$ and the strictly ascending sequence $(t_n; n \geq 0)$ in \mathcal{R}_+ with $t_0 = 0$ and $t_n \rightarrow \infty$ are to be determined. To this end, we have

$$t - \varphi(t) = t(1 - r_{n+1}), \quad t \in [t_n, t_{n+1}[, \quad n \geq 0.$$

Assume that $(t_n; n \geq 1)$ is a strictly ascending sequence in $]1, \infty[$ with

(c05) $t_n / \sqrt{t_{n+1}} \rightarrow \infty$ (hence $t_n \rightarrow \infty$).

Then, choose the sequence $(r_n; n \geq 1)$ in $]0, 1[$ according to

(c06) $1 - r_n = 1 / \sqrt{t_n}$, for each $n \geq 1$.

Note that, as a consequence of this, $(r_n; n \geq 1)$ is strictly ascending in $]0, 1[$ (hence, (c03) holds) and $r_n \rightarrow 1$ as $n \rightarrow \infty$. Replacing in a preceding formula, yields

$$t - \varphi(t) = t / \sqrt{t_{n+1}}, \quad \text{when } t \in [t_n, t_{n+1}[, \quad n \geq 0.$$

This gives an evaluation like

$$t - \varphi(t) \geq t_n / \sqrt{t_{n+1}}, \quad \text{for } t \in [t_n, t_{n+1}[, \quad n \geq 0;$$

wherefrom (by (c05)), $\psi := J - \varphi$ is coercive. On the other hand, some useful superadditivity tests for the functions $\varphi \in \mathcal{F}(\mathcal{R}_+)$ like before are obtainable from the methods developed by Bruckner [5]. Some other aspects may be found in Liu, Xu and Cho [11]; see also Fisher [9].

References

- [1] M. Alimohammady, J. Balooee, S. Radojević, V. Rakočević, M. Roohi, *Conditions of regularity in cone metric spaces*, Appl. Math. Comput., 217 (2011), 6259-6263.
- [2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*. Fund. Math., 3 (1922), 133-181.
- [3] P. C. Bhakta, T. Basu, *Some fixed point theorems on metric spaces*, J. Indian Math. Soc., 45 (1981), 399-404.
- [4] H. Brezis, F. E. Browder, *A general principle on ordered sets in nonlinear functional analysis*, Advances Math., 21 (1976), 355-364.
- [5] A. M. Bruckner, *Tests for the superadditivity of functions*, Proc. Amer. Math. Soc., 13 (1962), 126-130.

- [6] J. Caristi, W. A. Kirk, *Geometric fixed point theory and inwardness conditions*, in "The Geometry of Metric and Linear Spaces" (Michigan State Univ., 1974), pp. 74-83, Lecture Notes Math., vol. 490, Springer, Berlin, 1975.
- [7] N. H. Dien, *Some remarks on common fixed point theorems*, J. Math. Anal. Appl., 187 (1994), 76-90.
- [8] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. (New Series), 1 (1979), 443-474.
- [9] B. Fisher, *Four mappings with a common fixed point*, J. Univ. Kuwait Sci., 8 (1981), 131-139.
- [10] Z. Kadelburg, S. Radenović, S. Simić, *Abstract metric spaces and Caristi-Nguyen-type theorems*, Filomat, 25 (2011), 111-124.
- [11] Z. Liu, Y. Xu, Y. J. Cho, *On characterizations of fixed and common fixed points*, J. Math. Anal. Appl., 222 (1998), 494-504.
- [12] M. Turinici, *Functional versions of the Caristi-Kirk theorem*, Revista Union Mat. Argentina, 50 (2009), 89-97.
- [13] J. S. Ume, S. Yi, *Common fixed point theorems for a weak distance in complete metric spaces*, Internat. J. Math. Mathemat. Sci., 30 (2002), 605-611.