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ON THE EXISTENCE OF BERGE EQUILIBRIUM WITH PSEUDOCONTINUOUS PAYOFFS

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Abstract In this paper, we generalize the existence results of Berge equilibrium in [16], [13], [1], [2], [9] and [15] in the case where the payoff function of each player is pseudocontinuous (see [14]). Moreover, we study essential games (see [10] and [20]).

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1. INTRODUCTION

The concept of Berge equilibrium goes back to the book of Berge [5] and was later formalized in [21] for differential games. For a non cooperative game with finite number of persons, this equilibrium means that if each person plays his strategy at a Berge equilibrium, then he obtains the maximum payoff if all the remaining players play their strategy in the Berge equilibrium. It is worth noticing that the Berge equilibrium is totally different from the Nash equilibrium since the Nash equilibrium is stable with respect the deviation of any unique player. For the concept of Nash equilibrium, we refer the reader to the paper of [17]. The existence of Berge equilibrium has been studied in [16], [13], [1, 2] and [9]. More recently, [15] have established the existence of Berge equilibrium without using Nash equilibrium. Previously mentioned works, the authors have assumed that payoffs of persons are continuous. However, many games as the oligopolies defined in [6] and [12] have discontinuous payoffs. Several authors have studied the existence of Nash equilibrium where payoffs are not necessarily continuous. Let us quote for example, [14]. In that paper, the authors have proved the existence of Nash equilibrium with pseudocontinuous payoffs. In this paper, we prove the existence and the essential stability [20] and [10] of the Berge equilibrium with pseudocontinuous payoffs.

This paper is organized as follows: In section 2, we define the Berge equilibrium. In section 3, we give definitions of pseudocontinuous functions and better reply secure games. Moreover, we prove the existence of Berge equilibrium. In section 4, we study the essential stability of Berge equilibrium (called essential games) for two models. In the first model, we consider games parametrized by payoff profiles. In the second model, games are parametrized by payoff profiles and strategies sets.

2. DEFINITION AND EXISTENCE OF BERGE EQUILIBRIUM

2.1. BERGE EQUILIBRIUM OF NONCOOPERATIVE GAME

Let us consider the following noncooperative game in normal form:

$$G = (I,\ (X_i,\ u_i)_{i\in I})$$

where $I = \{1, ..., n\}$ is a finite set of players, X_i is a set of strategies of player i, $X = \prod_{i=1}^{n} X_i$ is a set of issues (joint strategy) of the game G and $u_i : X \to \mathbb{R}$ is a payoff function of the player i. For each player i, we let $I\setminus\{i\} = \{1, ..., i-1, i+1, ..., n\} = \{j \in I, j \neq i\}$ and we denote $X_{-i} = \prod_{j\neq i} X_j = \prod_{j\in I\setminus\{i\}} X_j$ and if $x \in X = \prod_{i=1}^{n} X_i$, then $x_{-i} = (x_j)_{j\neq i} \in X_{-i}$. Choosing a strategy $x_i \in X_i$, the aim of each player in the game G is to maximize his payoff function. Recall that $\bar{x} \in X$ is a Nash equilibrium of the game G if for every $i \in I$, for all $x_i \in X_i, u_i(\bar{x}) \geq u_i(x_i, \bar{x}_{-i})$. The following definition is due to ([5]).

Definition 2.1. *A **Berge equilibrium** of the game G is an n-tuple of strategies $\bar{x} \in X$ such that $\forall i \in I, \forall\ y_{-i} \in X_{-i}, u_i(\bar{x}) \geq u_i(\bar{x}_i, y_{-i})$.*

3. PSEUDOCONTINUOUS FUNCTIONS AND BETTER REPLY SECURE GAMES

In this section, we introduce the definitions of pseudocontinuous functions and better reply secure games.

Definition 3.1. *([14]). Let f be a real valued function defined on a topological vector space E. The function f is said to be upper pseudocontinuous at $x_0 \in E$ if for all $x \in E$ such that $f(x_0) < f(x)$, it follows that:*

$$\lim_{y\to x_0} \sup f(y) < f(x).$$

The function f is said to be upper pseudocontinuous on E if it is upper pseudocontinuous at all $x_0 \in E$.

Definition 3.2. *Let f be a real valued function defined on a topological vector space E. The function f is said to be lower pseudocontinuous at $x_0 \in E$ if $-f$ is upper pseudocontinuous at x_0 and the function f is said to be lower pseudocontinuous on E if it is lower pseudocontinuous at all $x_0 \in E$.*

Definition 3.3. *Let f be a real valued function defined on a topological vector space E. The function f is said to be pseudocontinuous if it is both upper and lower pseudocontinuous.*

Definition 3.4. *([18]). The game $G = (I,\ (X_i,\ u_i)_{i\in I})$ is called better reply secure if for every non Nash's equilibrium z and every vector v such that (z, v) belongs to the closure of the graph of $(u_{1,...,}u_n)$, then there exists some player i with strategy x_i and $u_i(x_i, t_{-i}) > v_i + \epsilon$ for all t_{-i} in some neighborhood of $z_{_{-i}}$ and suitable ϵ.*

Remark 3.1. *It is worth noticing that payoff continuity assumption is stronger than pseudocontinuity (see Example 4, [14]) and every game with pseudocontinuous payoffs is better reply secure (see Proposition 4.1, [14]).*

In the following, we introduce the analogous definition of the better reply secure games given in the Definition 3.4 to the context of the Berge equilibrium.

Definition 3.5. *The game $G = (I,\ (X_i,\ u_i)_{i\in I})$ is called better reply secure if for every non Berge equilibrium z and every vector v such that (z, v) belongs to the closure of the graph of $(u_{1,...,}u_n)$, then there exists some player i and a strategy $x_{-i} \in X_{-i}$ such that $u_i(t_i, x_{-i}) > v_i + \epsilon$ for all t_i in some neighborhood of $z_{_i}$ and suitable ϵ.*

Next, in the proposition below, we prove that every game G where each player payoff function is pseudocontinuous verifies the better reply secure games of the Definition 3.5. The steps are semilar to that of the proof in the Proposition 4.1 of [14] except for some variables permutations in payoffs functions of players. This proposition will be used later-on to prove the Theorem 4.1 of subsection 4.1.

Proposition 3.1. *Let $G = (I,\ (X_i,\ u_i)_{i\in I})$ be a game. Suppose that for each $i \in I = \{1, ..., n\}$, the payoff function u_i of the player i is pseudocontinuous, then the game G verifies the better reply secure of the Definition 3.5.*

Proof. Let z be a non Berge equilibrium for the game G and v a vector such that (z, v) belongs to the closure of the graph of $(u_{1,...,}u_n)$. So, there exists some player i and a strategy $x_{-i} \in X_{-i}$ such that $u_i(z_i, z_{-i}) < u_i(z_i, x_{-i})$.

We consider two cases:

In the first case, we suppose that there exists $(t_i, t_{-i}) \in X$ such that:

$$u_i(z_i, z_{-i}) < u_i(t_i, t_{-i}) < u_i(z_i, x_{-i}).$$

Since the function u_i is upper pseudocontinuous at the point (z_i, z_{-i}) one has:

$$\limsup_{(w_i,w_{-i})\longrightarrow(z_i,z_{-i})} u_i(w_i, w_{-i}) < u_i(t_i, t_{-i}). \tag{1}$$

Using that the point (z, v) belongs to the closure of the graph of $(u_{1,...,}u_n)$ and the inequation (1), we obtain: $v_i < u_i(t_i, t_{-i})$. Now, the function u_i is lower pseudocontinuous at the point (z_i, x_{-i}), so:

$$u_i(t_i, t_{-i}) < \liminf_{(w_i,w_{-i})\longrightarrow(z_i,x_{-i})} u_i(w_i, w_{-i}).$$

It follows that:
$$v_i < u_i\left(t_i, t_{-i}\right) < \liminf_{(w_i,w_{-i})\longrightarrow(z_i,x_{-i})} u_i\left(w_i, w_{-i}\right).$$

Since,
$$\liminf_{(w_i,w_{-i})\longrightarrow(z_i,x_{-i})} u_i\left(w_i, w_{-i}\right) < u_i\left(t_i, x_{-i}\right)$$

for all t_i in some neighborhood of z_i, we deduce:

$$u_i\left(t_i, x_{-i}\right) > v_i + \epsilon.$$

It follows that the game G verifies the Definition 3.5.

In the second case, we suppose that:

$$u_i\left(X\right) \cap \left]u_i\left(z_i, z_{-i}\right), u_i\left(z_i, x_{-i}\right)\right[= \emptyset.$$

Since the function u_i is lower semicontinuous at the point (z_i, x_{-i}), then:

$$u_i\left(z_i, z_{-i}\right) < \liminf_{(w_i,w_{-i})\longrightarrow(z_i,x_{-i})} u_i\left(w_i, w_{-i}\right).$$

Since there are not values of the payoff function u_i between $u_i\left(z_i, z_{-i}\right)$ and $u_i\left(z_i, x_{-i}\right)$, it follows that:

$$u_i\left(z_i, z_{-i}\right) < u_i\left(z_i, x_{-i}\right) < \liminf_{(w_i,w_{-i})\longrightarrow(z_i,x_{-i})} u_i\left(w_i, w_{-i}\right).$$

Using the same arguments as in the first case, we deduce that $u_i\left(t_i, x_{-i}\right) > v_i + \epsilon$ for all t_i in some neighborhood of z_i. Then, the game G verifies also the Definition 3.5. ∎

Next, we give sufficient conditions for the existence of the Berge equilibrium. From now, we assume that each strategy set X_i is a subset of a locally convex topological vector space E_i. For each i, *we call the best reply correspondence for the player i*, the correspondence $\Gamma_i : X \to X$ defined by:

$$\Gamma_i\left(x\right) = \left\{y \in X : u_i\left(x_i, y_{-i}\right) \geq u_i\left(x_i, t_{-i}\right) \ \forall t_{-i} \in X_{-i}\right\}.$$

For each $x \in X$, we set
$$\Gamma\left(x\right) = \bigcap_{i\in I} \Gamma_i\left(x\right).$$

With these notations, a Berge equilibrium is a fixed point of the correspondence Γ, that it is an n-tuple $\bar{x} \in \Gamma\left(\bar{x}\right)$.

Theorem 3.1. *Assume the following assumptions on the game G:*

1 $\forall i \in I$, X_i is a nonempty, compact and convex subset of E_i;

2 $\forall i \in I$, $\forall x_i \in X_i$, the function $y_{-i} \to u_i(x_i, y_{-i})$ is quasi-concave on X_{-i};

3 $\forall i \in I$, *the function* u_i *is pseudocontinuous on* $X_i \times X_{-i}$;

4 $\forall x \in X, \Gamma(x) \neq \emptyset$.

Then there exists a Berge equilibrium.

Proof. Let us assume that each strategy space E_i is locally convex. We remark that under the assumptions (1)-(3) of the Theorem 3.1, each correspondence Γ_i is convex valued but may have empty values. Under these assumptions, the correspondence Γ is closed (for definition and properties of the different continuity concepts for correspondences, we refer the reader to [3], [4] or to the Appendix of [11]) while for statements of classical fixed point theorems, we refer to [3], [11]. Thus Γ is upper semicontinuous with compact values and the set $F = \{x \in X : \Gamma(x) \neq \emptyset\}$ is closed in X. Indeed, the correspondence $x_i \to X_{-i}$ is obviously continuous and an easy adaptation of the proof of the Berge maximum theorem (see Theorem 3.1, [14]) shows that the correspondence:

$$x_i \to \left\{ y : u_i(x_i, y_{-i}) = \max_{t_{-i} \in X_{-i}} u_i(x_i, t_{-i}) \right\},$$

has a closed graph. As intersection of closed correspondences, each correspondence Γ_i is closed, thus upper semicontinuous with compact values. Consequently, if $x^\nu \to x$ with, for each ν of a directed set, $x^\nu \in F$, that is, $y^\nu \in \Gamma(x^\nu)$, in view of the compactness of X, one can assume without loss of generality that $y^\nu \to y \in \Gamma(y)$. It now follows from the Kakutani-Fan theorem that Γ has a fixed point. Let x denote a fixed point of the correspondence $\Gamma : X \to X$ defined as previously by:

$$\forall x \in X, \Gamma(x) = \bigcap_{i \in I} \Gamma_i(x),$$

where $\forall i \in I$, the correspondence $\Gamma_i : X \to X$ is the best reply correspondence of the palyer i. Then, $\forall i \in I$, $x \in \Gamma_i(x)$. From the definition of the best reply correspondence Γ_i, we deduce:

$$\forall i \in I, \forall t_{-i} \in X_{-i} : \ u_i(x_i, x_{-i}) \geq u_i(x_i, t_{-i}).$$

It follows from the above inequation that the fixed point x of the correspondence Γ is a Berge equilibrium (see Definition 1.1) for the game G. ∎

3.1. BERGE EQUILIBRIUM OF AN ABSTRACT ECONOMY

We now consider the following generalized game that will be called *abstract economy*. For this definition, we refer the reader to [7]:

$$H = (I, (X_i, F_i, u_i)_{i \in I}),$$

where $I = \{1, ..., n\}$ is a (finite set) of agents, X_i is the strategy set of the agent i, and if $X = \prod_{i \in I} X_i$, then $u_i : X \to \bar{\mathbb{R}}$ is the payoff function of the player i, while $F_i : X \to X_i$ denotes a feasibility correspondence for the player i given the strategies of the other agents. The following definition extends the Definition 2.1 to abstract economies.

Definition 3.6. *A* ***Berge equilibrium*** *of H is an n-tuple of strategies $\bar{x}$ such that*

$$\forall i \in I, \bar{x}_i \in F_i(\bar{x}),$$

and

$$\forall i \in I, \forall y_{-i} \in \prod_{j \in I\setminus\{i\}} F_j(\bar{x}), u_i(\bar{x}) \geq u_i(\bar{x}_i, y_{-i}).$$

The first condition guarantees that $\bar{x}$ is a vector of feasible strategies. If, as previously, we call for each $i \in I$, *the best reply correspondence for the player i*, the correspondence $\Gamma_i : X \to X$ defined by:

$$\Gamma_i(x) = \left\{ \begin{array}{c} y \in X : y_{-i} \in \prod_{j \in I\setminus\{i\}} F_j(x), u_i(x_i, y_{-i}) \geq u_i(x_i, t_{-i}) \\ \forall t_{-i} \in \prod_{j \in I\setminus\{i\}} F_j(x) \end{array} \right\},$$

both the above conditions jointly express that

$$\bar{x} \in \bigcap_{i \in I} \Gamma_i(\bar{x}).$$

Theorem 3.2. *Assume the following assumptions on the game H :*

1 $\forall i \in I$, E_i is a locally convex topological vector space and X_i is a nonempty, compact and convex subset of E_i;

2 $\forall i \in I$, $\forall x_i \in X_i$, the function $y_{-i} \to u_i(x_i, y_{-i})$ is quasi-concave on X_{-i};

3 $\forall i \in I$, the function u_i is pseudocontinuous on $X_i \times X_{-i}$;

4 $\forall i \in I$, the correspondence $F_i : X \to X_i$ is continuous with nonempty, convex and compact values;

5 $\forall x \in X, \bigcap_{i \in I} \Gamma_i(x) \neq \emptyset$.

Then H has a Berge equilibrium.

Proof. The proof is a slight modification of the proof given for the Theorem 3.1 in the case where each strategy space is locally convex topological vector space. For every $i \in I$, let us denote by $F_{-i} : X \to \prod_{j \in I\setminus\{i\}} X_j$ the correspondence defined by:

$$F_{-i}(x) = \prod_{j \in I\setminus\{i\}} F_j(x).$$

Since F_j is continuous with compact values, we have the Berge maximum theorem:

$$x_i \to \left\{ y : u_i(x_i, y_{-i}) = \max_{t_{-i} \in F_{-i}(x)} u_i(x_i, t_{-i}) \right\},$$

has a closed graph. Once again, as intersection of closed correspondences, each correspondence Γ_i is closed, thus upper semicontinuous with compact values. The same is true for the correspondence $\Gamma : X \to X$ defined by $\Gamma(x) = \bigcap_{i \in I} \Gamma_i(x)$. As by assumptions (2), (4) and (5), Γ is nonempty and convex valued, the existence of the Berge equilibrium of H follows from the Kakutani-Fan theorem. ∎

4. ESSENTIAL GAMES

4.1. GAMES PARAMETRIZED BY PAYOFF PROFILES

Let us assume that, defined on the same strategy spaces the games:

$$\left(\left(I,\ (X_i)_{i=1}^n,\ (u_i)_{i=1}^n\right)\right)$$

are parametrized by the payoff function profiles $g = (u_1, ..., u_n)$. More precisely, let U be the set of payoff function profiles $g = (u_1, ..., u_n)$ that satisfy the assumptions of Theorem 3.1 and verify $\sum_{i \in I} \sup_{x \in X} | u_i(x) | < \infty$, endowed with the distance ρ defined as follows. For each $g^1 = \left(u_1^1, ..., u_n^1\right)$ and $g^2 = \left(u_1^2, ..., u_n^2\right) \in U$

$$\rho(g^1, g^2) = \sum_{i \in I} \sup_{x \in X} | u_i^1(x) - u_i^2(x) | .$$

It is easy to see that U endowed with the distance ρ is a complete metric space. Now, define the Berge equilibria correspondence $J : U \to X$ where for each $g \in U$, $J(g) \subset X$ is the set of Berge equilibria of the game g.

In the following theorem, we give some properties of the correspondence J which require the study of essential games.

Theorem 4.1. *The Berge equilibria correspondence J is upper semicontinuous with nonempty and compact values.*

Proof. It follows from the Theorem 3.1 that the correspondence J has nonempty values. We now prove that the graph of G is closed. Since the correspondence J has values in a space X without a countable basis of neighborhoods, then we have to use a net instead of a sequence. By contradiction, assume that there is a net $g^\alpha \to g$ and a net $x^\alpha \to x \notin J(g), x^\alpha \in J(g^\alpha)$. Let $u_i = \lim_n \sup u_i(x^\alpha)$. Then, $(x, u = (u_1, ..., u_n)) \in$ *cl graph* $(u_{1,...,} u_n)$. It follows from the Proposition 3.1 that:

$$\exists i \in I,\ \exists z^{\alpha} \in X, u_i\left(t_i, z_{-i}^{\alpha}\right) > u_i + \epsilon,\ \forall t_i \in V(x_i), \epsilon > 0.$$

Since $\lim \rho(g, g^{\alpha}) = 0$, we have $u_i\left(t_i, z_{-i}^{\alpha}\right) > u_i + \rho(g, g^{\alpha}) + \epsilon$.
We obtain:

$$u_i\left(t_i, z_{-i}^{\alpha}\right) > u_i + u_i^{\alpha}(x_i^{\alpha}, x_{-i}^{\alpha}) - u_i(x_i^{\alpha}, x_{-i}^{\alpha}) + \epsilon$$

Since $x^{\alpha} \in J(g^{\alpha})$, then:

$$u_i^{\alpha}(x_i^{\alpha}, x_{-i}^{\alpha}) \geq u_i^{\alpha}(x_i^{\alpha}, z_{-i}^{\alpha}).$$

So,

$$u_i\left(t_i, z_{-i}^{\alpha}\right) > u_i + u_i^{\alpha}(x_i^{\alpha}, z_{-i}^{\alpha}) - u_i(x_i^{\alpha}, x_{-i}^{\alpha}) + \epsilon$$

We have

$$\rho(g, g^{\alpha}) \geq u_i\left(t_i, z_{-i}^{\alpha}\right) - u_i^{\alpha}(t_i, z_{-i}^{\alpha})$$

and

$$u_i\left(t_i, z_{-i}^{\alpha}\right) - u_i^{\alpha}(t_i, z_{-i}^{\alpha}) > u_i + u_i^{\alpha}(x_i^{\alpha}, z_{-i}^{\alpha}) - u_i(x_i^{\alpha}, x_{-i}^{\alpha}) - u_i^{\alpha}(t_i, z_{-i}^{\alpha}) + \epsilon.$$

Since $x_i^{\alpha} \to x_i$ and $t_i \in V(x_i)$, then

$$\rho(g, g^{\alpha}) > u_i - u_i(x_i^{\alpha}, x_{-i}^{\alpha}) + \epsilon > u_i - u_i(x_i^{\alpha}, x_{-i}^{\alpha}) + \epsilon > \epsilon$$

Contradiction. ∎

4.2. GAMES PARAMETRIZED BY PAYOFF PROFILES AND THE STRATEGIES SETS

For any $i \in I = \{1, ..., n\}$, let X_i be a closed subset of a Hausdorff locally convex topological vector space E_i and let $CK(X_i)$ be the set of all non empty, convex and compact subsets S_i of X_i equipped with the Vietoris topology (see ([20])). We endowed $\prod_{i=1}^{n} CK(X_i)$ with the topology product of Vietoris topologies on each X_i. Let $C = \{g = (u_1, ..., u_n)\colon \sup_{x \in X} \mid u_i(x) \mid < \infty\}$, where $X = \prod_{i=1}^{n} X_i$ is the set profiles such that for each $i \in I = \{1, ..., n\}$, the payoff function u_i of the player i verifies the hypothesis in the Theorem 3.1 and it is endowed with the metric:

$$\rho(g^1, g^2) = \sum_{i \in I} \sup_{x \in X} \mid u_i^1(x) - u_i^2(x) \mid$$

Let $Y = C\times\prod_{i=1}^{n} CK(X_i)$ be the space of games $g = (u_1, ...u_n, S_1, ..., S_n) \in Y$, parametrized by payoff function profiles and strategy sets.

Recall that $\bar{x} = (\bar{x}_1, ..., \bar{x}_n) \in \prod_{i=1}^{n} S_i$ is a Berge equilibrium for the game $g = (u_1, ...u_n, S_1, ..., S_n) \in Y$ if:

$$\forall i \in I,\ u_i(\bar{x}, \bar{x}_{-i}) = \max_{u_{-i}\in S_{-i}} u_i(\bar{x}, u_{-i}).$$

Let:

$$K : Y \to \prod_{i=1}^{n} S_i$$

$$g = (u_1, ...u_n, S_1, ..., S_n) \to K(g)$$

the correspondence of the Berge equilibria. As in subsection 4.1, we shall prove in the following theorem that the correspondence of the Berge equilibria K is upper semicontinuous with nonempty and compact values.

Theorem 4.2. *Assume the following assumptions on H :*

1 $\forall i \in I$, E_i is a Hausdorff locally convex topological vector space and X_i is a nonempty, closed and convex subset of E_i;

2 $\forall i \in I$, $\forall x_i \in X_i$, the function $y_{-i} \to u_i(x_i, y_{-i})$ is quasi-concave on X_{-i};

3 $\forall i \in I$, the function u_i is pseudocontinuous on $X_i \times X_{-i}$;

4 $\forall i \in I$, the correspondence $F_i : X \to X_i$ is continuous with nonempty, convex and compact values;

5 $\forall x \in X, \bigcap_{i\in I} \Gamma_i(x) \neq \emptyset$.

Then the Berge equilibria correspondence K is upper semicontinuous with nonempty and compact values.

Proof. Let us consider the following noncooperative games parametrized by payoff function profiles and strategy sets $\left(I,\ (S_i)_{i=1}^{n},\ (u_i)_{i=1}^{n}\right)$. From the Theorem 3.1 or the Theorem 3.2, K has non empty values. As we have noticed in the proof of the Theorem 4.1, we have to take a net instead of sequence.
Let $x^\alpha \in K(g)$ be a net where $g = (u_1, ..., u_n, S_1, ..., S_n) \in Y$ and $\lim_n x^\alpha = x$. Then, for each $i \in I$, $u_i(x_i^\alpha, x_{-i}^\alpha) = \max_{t_{-i}\in S_{-i}} u_i(x_i^\alpha, t_{-i})$. We prove that K has closed values in a compact set $\prod_{i=1}^{n} S_i$. Assume that $x \notin K(g)$, then there exists i_0 such that

$$u_{i_0}(x_{i_0}, x_{-i_0}) < u_{i_0}(x_{i_0}, u^0_{-i_0})$$

where $u^0_{-i_0} \in S_{-i_0}$. Since the function u_{i_0} is pseudocontinuous, we have:

$$\forall(z_{i_0}, z_{-i_0}) \in V(x_{i_0}) \times V(x_{-i_0}), \forall(t_{i_0}, t^0_{-i_0}) \in V'(x_{i_0}) \times V\left(u^0_{-i_0}\right)$$

$$u_{i_0}(z_{i_0}, z_{-i_0}) < u_{i_0}(t_{i_0}, t^0_{-i_0})$$

Let $W(x_{i_0}) = V(x_{i_0}) \cap V'(x_{i_0})$, then $u_{i_0}(z_{i_0}, z_{-i_0}) < u_{i_0}(z_{i_0}, t^0_{-i_0})$ for all $(z_{i_0}, z_{-i_0}) \in W(x_{i_0}) \times V(x_{-i_0})$.

It follows that, $(W(x_{i_0}) \times V(x_{-i_0})) \cap K(g) = \varnothing$. Then K has closed values in a compact set $\prod_{i=1}^{n} S_i$. Now, we prove that K is upper semicontinuous. If it is not true at a point $y \in Y$, then there exists an open set O of X and a net $g^\alpha \in Y$ such that $O \supset K(g), \lim_n g^\alpha = g$ and $x^\alpha \in K(g^\alpha), x^\alpha \notin O$. Thus $\lim_n \rho(g^\alpha, g) = 0$ and for each $i \in I, \lim_n S_i^\alpha = S_i$ for the Vietoris topology on $CK(X_i)$. In view of Lemma 2.3 in ([20])), let x be a cluster point of x^α. It is obvious that $x \notin O \supset K(g)$ and hence there exists $i_0 \in I$, such that

$$u_{i_0}(x_{i_0}, x_{-i_0}) < u_{i_0}(x_{i_0}, u^0_{-i_0})$$

where $u^0_{-i_0} \in S_{-i_0}$. Let X be a topological space and $t \in X$, we denote by $O(t)$ any open set of X which contains a point t. By the pseudocontinuity of the function u_{i_0}(see Proposition 2.2, [19], there exists x^0, x^1 such that:

$$u_{i_0}(x_{i_0}, x_{-i_0}) < u_{i_0}(x^0_{i_0}, x^0_{-i_0}) < u_{i_0}(x^1_{i_0}, x^1_{-i_0}) < u_{i_0}(x_{i_0}, u^0_{-i_0}).$$

Using the upper pseudocontinuous of the function u_{i_0} at the point (x_{i_0}, x_{-i_0}), we obtain:

$$\forall(x'_{i_0}, x'_{-i_0}) \in O(x_{i_0}) \times O(x_{-i_0})$$

$$u_{i_0}(x'_{i_0}, x'_{-i_0}) < u_{i_0}(x^0_{i_0}, x^0_{-i_0}) < u_{i_0}(x^1_{i_0}, x^1_{-i_0}) < u_{i_0}(x_{i_0}, u^0_{-i_0}).$$

Let $V_{i_0}\left(u^0_{i_0}\right) \subset X_{i_0}$ be an open set such that $u^0_{i_0} \in S_{i_0}$ and $\prod_{j \in I \setminus \{i_0\}} V_j\left(u^0_j\right) \subset O(u^0_{-i_0})$. Since $V_{i_0}\left(u^0_{i_0}\right) \cap S_{i_0} \neq \varnothing$, then for $\alpha \geq \alpha_0, \rho(g^\alpha, g)$ converges to 0, $V_{i_0}\left(u^0_{i_0}\right) \cap S^\alpha_{i_0} \neq \varnothing$ and $x^\alpha \in O(x_{i_0}) \times O(x_{-i_0})$. Take $u^{0\alpha}_{-i_0} \in O(u^0_{-i_0}) \cap \prod_{j \in I \setminus \{i_0\}} S^\alpha_j$. It is obvious that

$$u_{i_0}(x'_{i_0}, x'_{-i_0}) + \rho(g^\alpha, g) < u_{i_0}(x^1_{i_0}, x^1_{-i_0}) < u_{i_0}(x_{i_0}, u^0_{-i_0}).$$

Then,

$$u_{i_0}(x^\alpha_{i_0}, x^\alpha_{-i_0}) + u^\alpha_{i_0}(x^\alpha_{i_0}, x^\alpha_{-i_0}) - u_{i_0}(x^\alpha_{i_0}, x^\alpha_{-i_0}) < u_{i_0}(x^1_{i_0}, x^1_{-i_0}) < u_{i_0}(x_{i_0}, u^0_{-i_0}),$$

we got:

$$u^\alpha_{i_0}(x^\alpha_{i_0}, x^\alpha_{-i_0}) < u_{i_0}(x^1_{i_0}, x^1_{-i_0}) < u_{i_0}(x_{i_0}, u^0_{-i_0}) < u_{i_0}(x^\alpha_{i_0}, u^{0\alpha}_{-i_0}).$$

From the Proposition 2.3 (see [19]), we have:

$$u_{i_0}^{\alpha}(x_{i_0}^{\alpha}, x_{-i_0}^{\alpha}) < u_{i_0}(x_{i_0}^{1}, x_{-i_0}^{1}) < u_{i_0}^{\alpha}(x_{i_0}^{\alpha}, u_{-i_0}^{0\alpha}).$$

Since $u_{-i_0}^{0\alpha} \in S_{-i_0}^{\alpha}$, we obtain a contradiction with x^{α} is a net of the Berge equilibria. ∎

Remark 4.1. *The pseudocontinuity in Theorem 4.2 cannot be relaxed by the better reply secure games given in the Definition 3.5 (see Example 3.1, [19]). However, this latter has been used in the Theorem 4.1 of Subsection 4.1.*

Now, we introduce the definitions of essential equilibrium and essential games.

Definition 4.1. *([20]). Let M be a nonempty and closed subset of U defined in subsection 4.1 or Y defined in Subsection 4.2 and $g \in M$. An element $x \in J(g)$ or $x \in K(g)$ is called* ***essential equilibrium*** *of the game g relative to M if for any $O \in \mathcal{V}(x)$ there exists $W \in \mathcal{W}(g)$ such that for each $g_1 \in M \cap W$, there exists $x_1 \in J(g_1)$ or $x_1 \in K(g_1)$ with $x_1 \in O$.*

Definition 4.2. *([20]). Let M be a nonempty and closed subset of U defined in subsection 4.1 or Y defined in Subsection 4.2 and $g \in M$. The game $g \in M$ is said to be essential relative to M if all its equilibria are essential relative to M.*

It follows from the Definition 4.2 that the game $g \in M$ is essential if and only if the correspondence $J\colon M \to K(X)$ or $K\colon M \to K(X)$ is lower semicontinuous at g (see Theorem 4.1, [20]), where $K(X)$ denotes the space of all nonempty compact subsets of X.

In the following theorem, we establish that most of games in subsections 4.1 and 4.2 are essential in the sense of Baire category.

Theorem 4.3. *Assume that for each $i \in I$, the strategy space E_i is a normed space. Then most of the games in Subsections 4.1 and 4.2 are essential in the sens of Baire category.*

Proof. The proof is similar as in [10]. ∎

5. CONCLUSION

In this paper, we have proved in subsection 4.1 that games in normal form having essential Berge equilibria are the generic case in the space of discontinuous games. And in Subsection 4.2, we have proved that abstract economies having essential Berge equilibria are also the generic case in the space of discontinuous games. We have used weakening of continuity called pseudocontinuity in [14] In Subsection 4.1, we have showed in Theorem 4.1 that the pseudocontinuity could be weakened by the better reply secure games given in the Definition 3.5 to the context of the Berge equilibrium. However, for games parametrized by payoff function profiles and strategy sets, the same assumption could not be relaxed in the Theorem 4.2 of subsection 4.2

using the Definition 3.5, because the correspondence of the Berge equilibria is defined on an abstract space endowed with a product of metric and a Vietoris topology and has values in a non-fixed compact space. The maximum theorem in the setting of the pseudocontinuity functions given in the paper of [14], the fixed point theorem of Kakutani and the Baire theorem have played a central role in the main results of this paper.

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