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PPF DEPENDENT FIXED POINTS IN A-CLOSED RAZUMIKHIN CLASSES

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Abstract The PPF dependent fixed point result in algebraically closed Razumikhin classes due to Agarwal et al [Fixed Point Theory Appl., 2013, 2013:280] is identical with its constant class counterpart; and this, in turn, is reducible to a fixed point principle involving SVV type contractions (over the subsequent metric space of initial Banach structure), without any regularity conditions about the Razumikhin classes. The conclusion remains valid for all PPF dependent fixed point results founded on such global conditions.

Keywords: metric space, Picard operator, fixed point, SVV type contraction, Banach space, Razumikhin functional class, algebraical and topological closeness, PPF dependent fixed point, nonself contraction, iterative process.
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1. INTRODUCTION

Let X be a nonempty set. Call the subset Y of X, *almost singleton* (in short: *asingleton*) provided $y_1, y_2 \in Y$ implies $y_1 = y_2$; and *singleton*, if, in addition, Y is nonempty; note that, in this case, $Y = \{y\}$, for some $y \in X$. Take a metric $d : X \times X \to R_+ := [0, \infty[$ over it; as well as a selfmap $T \in \mathcal{F}(X)$. [Here, for each couple A, B of nonempty sets, $\mathcal{F}(A, B)$ denotes the class of all functions from A to B; when $A = B$, we write $\mathcal{F}(A)$ in place of $\mathcal{F}(A, A)$]. Denote $\mathrm{Fix}(T) = \{x \in X; x = Tx\}$; each point of this set is referred to as *fixed* under T. The determination of such elements is to be performed under the directions below, comparable with the ones in Rus [21, Ch 2, Sect 2.2] and Turinici [27]:

Pic-1) We say that T is a *Picard operator* (modulo d) if, for each $x \in X$, the iterative sequence $(T^n x; n \geq 0)$ is d-convergent; and a *globally Picard operator* (modulo d) if, in addition, $\mathrm{Fix}(T)$ is an asingleton

Pic-2) We say that T is a *strong Picard operator* (modulo d) if, for each $x \in X$, the iterative sequence $(T^n x; n \geq 0)$ is d-convergent with $\lim_n(T^n x)$ belonging to $\mathrm{Fix}(T)$; and a *globally strong Picard operator* (modulo d) if, in addition, $\mathrm{Fix}(T)$ is an asingleton (hence, a singleton).

The basic result in this area is the 1922 one due to Banach [2]. Given $k \geq 0$, let us say that T is $(d; k)$-*contractive*, if;

(a01) $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$.

Theorem 1.1. *Assume that T is $(d;k)$-contractive, for some $k \in [0,1[$. In addition, let (X,d) be complete. Then, T is globally strong Picard (modulo d).*

This statement (referred to as: Banach's fixed point principle) found some basic applications to different branches of operator equations theory. As a consequence, many extensions of it were proposed. From the perspective of this exposition, the following ones are of interest:

I) Contractive type extensions: the initial Banach contractive property is taken in a generalized way, as

(a02) $F(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(Tx,y)) \leq 0$,
for all $x,y \in X$;

where $F : R_+^6 \to R$ is a function. For the explicit case, some consistent lists of these may be found in the survey papers by Rhoades [20], Collaco and E Silva [6], Kincses and Totik [14], as well as the references therein. And, for the implicit case, certain particular aspects have been considered by Leader [16] and Turinici [25].

II) Structural extensions: the trivial relation $X \times X$ is replaced by a relation ∇ over X fulfilling or not certain regularity properties. For example, the case of ∇ being *reflexive, transitive* (hence, a *quasi-order*) was considered in the 1986 paper by Turinici [26]. Two decades later, this result was re-discovered – in a (partially) ordered context – by Ran and Reurings [19]; see also Nieto and Nodriguez-Lopez [17]. On the other hand, the "amorphous" case (∇ has no regularity properties at all) was discussed (via graph techniques) in Jachymski [11]; and (from a general perspective) by Samet and Turinici [22]. Some other aspects involving additional convergence structures may be found in Kasahara [13].

III) Nonself extensions: T is no longer a selfmap. In 1977, Bernfeld et al [3] introduced the concept of *PPF (past-present-future) dependent fixed point* for nonselfmappings (whose domain is distinct from their range). Furthermore, the quoted authors established – via iterative methods involving a certain Razumikhin class $\mathcal{R}_c$ – some PPF dependent fixed point theorems for contractive mappings of this type. As precise there, the obtained results are useful tools in the study of existence and uniqueness questions for solutions of nonlinear functional differential/integral equations which may depend upon the past history, present data and future evolution. As a consequence, this theory attracted a lot of contributors in the area; see, for instance, Dhage [7, 8], Hussain et al [10], Kaewcharoen [12], Kutbi and Sintunavarat [15], as well as the references therein. However, as proved in a recent paper by Cho, Rassias, Salimi and Turinici [4], the starting conditions [imposed by the problem setting] relative to the ambient Razumikhin class $\mathcal{R}_c$ may be converted into starting conditions relative to the constant class $\mathcal{K}$; so, ultimately, we may arrange for these PPF dependent fixed point results holding over $\mathcal{K}$. In this exposition we bring the discussion a step further, by establishing that

Fact-1) the algebraic closeness assumption [used in all these references] imposed upon the Razumikhin class $\mathcal{R}_c$ yields, in a direct way, $\mathcal{R}_c = \mathcal{K}$

Fact-2) the PPF dependent fixed point problem attached to the constant class $\mathcal{K}$ is reducible to a (standard) fixed point problem in the (complete) metrical structure induced by our initial Banach one, under no regularity assumption about the ambient Razumikhin class.

Finally, as an application of these conclusions, we show that a recent PPF dependent fixed point result in Agarwal et al [1] is reducible to a fixed point problem involving a class of contractions over standard metric structures [taken as before] introduced under the lines proposed by Samet et al [23]. Further aspects will be discussed elsewhere.

2. RAZUMIKHIN CLASSES

Let $(E, ||.||)$ be a Banach space; and $d(.,.)$ be the induced by norm metric on E:

$$d(x,y) = ||x - y||,\ x, y \in E;$$

hence, (E, d) is a complete metric space. Further, let $I = [a, b]$ be a closed real interval, and $E_0 := C(I, E)$ stand for the class of all continuous functions $\varphi : I \to E$, endowed with the supremum norm

$$||\varphi||_0 = \sup\{||\varphi(t)||; t \in I\},\ \varphi \in E_0.$$

As before, let $D(.,.)$ stand for the induced by norm metric on E_0

$$D(\varphi, \xi) = ||\varphi - \xi||_0,\ \varphi, \xi \in E_0;$$

clearly, (E_0, D) is a complete metric space.

Let $c \in I$ be fixed in the sequel. The *Razumikhin class* of functions in E_0 attached to c, is defined as

(b01) $\mathcal{R}_c = \{\varphi \in E_0; ||\varphi||_0 = ||\varphi(c)||\}$.

Note that $\mathcal{R}_c$ is nonvoid; because any constant function belongs to it. To substantiate this assertion, a lot of preliminary facts are needed.

(A) For each $u \in E$, let $H[u]$ denote the constant function of E_0, defined as

(b02) $H[u](t) = u,\ t \in I$.

Note that, by this definition,

$$||H[u]||_0 = ||u||,\ H[u](c) = u;$$

whence $H[u] \in \mathcal{R}_c$. Denote, for simplicity $\mathcal{K} = \{H[u]; u \in E\}$; this will be referred to as the *constant class* of E_0. The following properties of this class are almost immediate; so, we do not give details.

Proposition 2.1. *Under the above conventions,* *(cc1)* $H[u+v] = H[u] + H[v]$, $H[\lambda u] = \lambda H[u]$, $\forall u, v \in E$, $\forall \lambda \in R$; *hence,* $\mathcal{K}$ *is a linear subspace of* E_0

(cc2) $\|u\| = \|H[u]\|_0$, $\forall u \in E$

(cc3) the mapping $u \mapsto H[u]$ *is an algebraic and topological isomorphism between* $(E, \|.\|)$ *and* $(\mathcal{K}, \|.\|_0)$

(cc4) $\mathcal{K}$ *is D-complete (hence, D-closed) in* E_0.

(B) Returning to the general case, the following simple property holds.

Lema 2.1. *The Razumikhin class* $\mathcal{R}_c$ *is homogeneous, in the sense*

$$\lambda\mathcal{R}_c = \mathcal{R}_c, \ \forall \lambda \in R \setminus \{0\}; \ \textit{whence}, \ \mathcal{R}_c = -\mathcal{R}_c. \tag{1}$$

Proof. Given $[\lambda \in R, \varphi \in \mathcal{R}_c]$, denote $\xi = \lambda\varphi$. By definition,

$$\|\xi\|_0 = |\lambda| \cdot \|\varphi\|_0, \ \|\xi(c)\| = |\lambda| \cdot \|\varphi(c)\|.$$

This, along with the choice of φ, gives $\xi \in \mathcal{R}_c$; and completes the argument.

(C) Let $\mathcal{T} : E_0 \to E$ be a (nonself) mapping. We say that $\varphi \in E_0$ is a *PPF dependent fixed point* of $\mathcal{T}$, when $\mathcal{T}\varphi = \varphi(c)$. The class of all these will be denoted as PPF-Fix$(\mathcal{T}; E_0)$.

Concerning existence and uniqueness properties involving such points, the first contribution to this theory is the 1977 statement in Bernfeld et al [3]; referred to as: BLR theorem. The following concepts and constructions are necessary.

I) Given $k \geq 0$, call $\mathcal{T}$, *k-contractive*, provided

(b03) $d(\mathcal{T}\varphi, \mathcal{T}\xi) \leq kD(\varphi, \xi)$, for all $\varphi, \xi \in E_0$.

II) Let us introduce a relation (∇) over E_0, according to:

(b04) $\varphi \nabla \xi$ iff $\mathcal{T}\varphi = \xi(c)$ and $\varphi - \xi \in \mathcal{R}_c$.

III) Finally, for the starting element $\varphi_0 \in E_0$, let us say that the sequence $(\varphi_n; n \geq 0)$ in E_0 is (φ_0, ∇)-*iterative*, in case $\varphi_n \nabla \varphi_{n+1}$, $\forall n$.

Theorem 2.1. *Suppose that* $\mathcal{T}$ *is k-contractive, for some* $k \in [0, 1[$. *Then,*

i) *Given the starting point* $\varphi_0 \in E_0$, *any* (φ_0, ∇)*-iterative sequence* $(\varphi_n; n \geq 0)$ *in* E_0, *D-converges to an element of PPF*-Fix$(\mathcal{T}; E_0)$.

ii) *Given the couple of starting points* $\varphi_0, \xi_0 \in E_0$, *and letting* $(\varphi_n; n \geq 0)$, $(\xi_n; n \geq 0)$ *be a* (φ_0, ∇)*-iterative sequence and* (ξ_0, ∇)*-iterative sequence, respectively, we have the evaluation, for all* $n \geq 0$,

$$D(\varphi_n, \xi_n) \leq (1/(1-k))[D(\varphi_0, \varphi_1) + D(\xi_0, \xi_1)] + D(\varphi_0, \xi_0). \tag{2}$$

In particular, if $\varphi_0 = \xi_0$, *we have for all* $n \geq 0$,

$$D(\varphi_n, \xi_n) \leq (2/(1-k))D(\varphi_0, \varphi_1). \tag{3}$$

iii) *Let* $(\varphi_n; n \geq 0)$, $(\xi_n; n \geq 0)$ *be as above. If* $\varphi_n - \xi_n \in \mathcal{R}_c$, *for all* $n \geq 0$, *then, necessarily,* $\lim_n \varphi_n = \lim_n \xi_n$.

iv) *If* φ^*, ξ^* *are in PPF*-$\mathrm{Fix}(\mathcal{T}; E_0)$*, and* $\varphi^* - \xi^* \in \mathcal{R}_c$*, then* $\varphi^* = \xi^*$.

[For completeness reasons, we shall provide a proof of the above result, at the end of this exposition].

(D) Technically speaking, the BLR theorem is conditional in nature; because, for the starting $\varphi_0 \in \mathcal{R}_c$ [hence, all the more, for the starting $\varphi_0 \in E_0$], the set of all (φ_0, ∇)-iterative sequence $(\varphi_n; n \geq 0)$ in E_0 (taken as before) may be empty. To avoid this drawback, we have two possibilities:

Option-1) all considerations above are to be restricted to the constant class $\mathcal{K} \subseteq \mathcal{R}_c$; which, as a D-closed linear subspace of E_0, yields an appropriate setting for any algebraic and/or topological reasoning to be applied

Option-2) the initial Razzumikhin class $\mathcal{R}_c$ remains as it stands; but, with the price of imposing further (strong) restrictions upon it.

A discussion of these is to be sketched under the lines below.

Part 1. Concerning the first option, we note that, the imposed k-contractive condition upon $\mathcal{T}$ relates elements $\varphi, \xi \in E_0$ with elements $\mathcal{T}\varphi, \mathcal{T}\xi \in E$. On the other hand, at the level of constant class $\mathcal{K}$, the underlying condition writes

(b05) $\|\mathcal{T}\varphi - \mathcal{T}\xi\| \leq k\|\varphi(c) - \xi(c)\|$; for all $\varphi, \xi \in \mathcal{K}$;

so that, it relates elements $\varphi(c), \xi(c) \in E$ with elements $\mathcal{T}\varphi, \mathcal{T}\xi \in E$. Hence, as long as we have a selfmap of E that sends $\psi(c) \in E$ to $\mathcal{T}\psi \in E$ (for all $\psi \in \mathcal{K}$), the last condition is of selfmap type. The effectiveness of such a construction is illustrated by the considerations below. Let $T : E \to E$ be the selfmap of E introduced as

(b06) $Tu = \mathcal{T}(H[u]),\ u \in E$.

Proposition 2.2. *Under these conventions, the following are valid:*

i) *If* $x \in E$ *is a fixed point of* T*, then* $\xi := H[x] \in \mathcal{K}$ *is a PPF dependent fixed point of* $\mathcal{T}$

ii) *Conversely, if* $\zeta = H[z] \in \mathcal{K}$ *is a PPF dependent fixed point of* $\mathcal{T}$*, then* $z \in E$ *is a fixed point of* T.

Proof. i) If $x \in E$ is a fixed point of T, we have $x = Tx = \mathcal{T}(H[x])$; or, equivalently,

$$x = \mathcal{T}(\xi), \text{ where } \xi := H[x] \in \mathcal{K}.$$

This, by definition, gives $\xi(c) = \mathcal{T}(\xi)$; which tells us that $\zeta \in \mathcal{K}$ is a PPF dependent fixed point of $\mathcal{T}$.

ii) Suppose that $\zeta = H[z] \in \mathcal{K}$ is a PPF dependent fixed point of $\mathcal{T}$; that is: $\zeta(c) = \mathcal{T}(\zeta)$. This yields (by these notations)

$$z = \mathcal{T}(H[z]) = Tz;$$

so that, $z \in E$ is a fixed point of T.

Having these precise, we may now proceed to the formulation of announced result. Two basic concepts appear.

I) Given $k \geq 0$, call $\mathcal{T}$, *k-contractive*, provided

(b07) $d(\mathcal{T}\varphi, \mathcal{T}\xi) \leq kD(\varphi, \xi)$, for all $\varphi, \xi \in E_0$.

In particular, with $\varphi = H[x]$, $\xi = H[y]$ (where $x, y \in E$), this relation becomes

(b08) $d(Tx, Ty) \leq kD(H[x], H[y]) = kd(x, y)$, for all $x, y \in E$;

or, in other words: the associated selfmap $T : E \to E$ is $(d; k)$-contractive.

II) For the arbitrary fixed $\varphi_0 = H[x_0] \in \mathcal{K}$, we say that the sequence $(\varphi_n = H[x_n]; n \geq 0)$ in $\mathcal{K}$ is $(\varphi_0, \mathcal{T})$-*iterative*, provided $[\mathcal{T}\varphi_n = \varphi_{n+1}(c), \forall n \geq 0]$. Note that, the family of such sequences is nonempty. In fact, for the starting $x_0 \in E$, the (x_0, T)-iterative sequence $(x_n; n \geq 0)$ in E is well defined, according to the formula $(Tx_n = x_{n+1}; n \geq 0)$. But then, under the above notations,

$$\mathcal{T}\varphi_n = Tx_n = x_{n+1} = \varphi_{n+1}(c), \ \forall n \geq 0;$$

or, in other words: the sequence $(\varphi_n := H[x_n]; n \geq 0)$ in $\mathcal{K}$ is $(\varphi_0, \mathcal{T})$-iterative. Conversely, if the sequence $(\varphi_n := H[x_n]; n \geq 0)$ in $\mathcal{K}$ is $(\varphi_0, \mathcal{T})$-iterative, then (by the same formula), the sequence $(x_n; n \geq 0)$ of E is (x_0, T)-iterative; hence, for each $\varphi_0 \in \mathcal{K}$, the family of all $(\varphi_0, \mathcal{T})$-iterative sequences in $\mathcal{K}$ is a singleton.

Putting these together, it follows, from the Banach fixed point principle we just exposed, the following coincidence point result involving our data (referred to as: Constant BLR theorem):

Theorem 2.2. *Suppose that $\mathcal{T}$ is k-contractive, for some $k \in [0, 1[$. Then,*

i) *$\mathcal{T}$ has (in $\mathcal{K}$) a unique PPF dependent fixed point φ^* in $\mathcal{K}$*

ii) *for the arbitrary fixed $\varphi_0 \in \mathcal{K}$, the $(\varphi_0, \mathcal{T})$-iterative sequence $(\varphi_n; n \geq 0)$ in $\mathcal{K}$, D-converges to φ^*.*

Note that – unlike the situation encountered at BLR theorem – the iterative sequences above are constructible in a precise way [by means of the associated selfmap T]; so, this result is an effective one.

Part 2. The second option above starts from the fact that, the construction in $\mathcal{R}_c$ of iterative sequences given by BLR theorem requires the structural condition

(b09) $\mathcal{R}_c$ is *algebraically closed*: $\varphi, \xi \in \mathcal{R}_c \Longrightarrow \varphi - \xi \in \mathcal{R}_c$

(also referred to as: $\mathcal{R}_c$ is *a-closed*); this assertion seems to have been formulated, for the first time, in Dhage [8, Observation I]. On the other hand, the existence property above is retainable, at the level of $\mathcal{R}_c$, when

(b10) $\mathcal{R}_c$ is topologically closed: $\mathcal{R}_c$ is a D-closed part of E_0;

cf. Dhage [8, Observation II]. Summing up, the following "existential" version of Theorem 2.1 (referred to as: Existential BLR theorem) enters into our discussion:

Theorem 2.3. *Suppose that $\mathcal{T}$ is k-contractive, for some $k \in [0, 1[$. In addition, let us assume that $\mathcal{R}_c$ is algebraically and topologically closed. Then, $\mathcal{T}$ has a unique PPF dependent fixed point in $\mathcal{R}_c$.*

Note that, this result is *not* present in the 1977 paper by Bernfeld et al [3]. The above formulation is a quite recent "by-product" of BLR theorem, under the lines imposed by the above remarks; cf. Kutbi and Sintunavarat [15].

Concerning the structural requirements above, we stress that, from a methodological perspective, the algebraically closed condition is a very strong one. Before explaining our assertion, let us give a useful characterization of this concept. [Since the verification is almost immediate, we omit the details].

Lema 2.2. *The following conditions are equivalent:*

(b11) $\mathcal{R}_c$ *is algebraically closed*

(b12) $\mathcal{R}_c$ *is additive:* $\varphi, \psi \in \mathcal{R}_c \Longrightarrow \varphi + \psi \in \mathcal{R}_c$

(b13) $\mathcal{R}_c$ *is a linear subspace of* E_0.

Having these precise, we are in position to motivate our previous affirmation.

Proposition 2.3. *Suppose that $\mathcal{R}_c$ is algebraically closed; or, equivalently, additive. Then, necessarily, $\mathcal{R}_c = \mathcal{K}$; hence $\mathcal{R}_c$ is topologically closed as well.*

Proof. Suppose that $\mathcal{R}_c \setminus \mathcal{K} \neq \emptyset$; and take some function φ in this set difference; hence, in particular,

$$\varphi(r) \neq \varphi(c), \text{ for at least one } r \in I. \tag{4}$$

The function $\xi := H[\varphi(c)]$ belongs to the constant class $\mathcal{K}$; hence, to the Razumikhin class $\mathcal{R}_c$ as well. As $\mathcal{R}_c$ is algebraically closed, the difference function $\delta := \varphi - \xi$ is an element of $\mathcal{R}_c$; so that

$$||\delta||_0 = ||\delta(c)|| = 0;$$

or, equivalently,

$$\varphi(t) = \varphi(c), \ \forall t \in I;$$

in contradiction with the initial choice of φ. Hence, $\mathcal{R}_c = \mathcal{K}$, as claimed. The last affirmation is immediate, by the topological properties of $\mathcal{K}$ (see above).

Summing up, the following conclusions are to be noted:

Conc-1) If the Razumikhin class $\mathcal{R}_c$ is algebraically closed, we must have $\mathcal{R}_c = \mathcal{K}$; so that, Existential BLR theorem over $\mathcal{R}_c$ is identical – in a trivial way – with Constant BLR theorem over $\mathcal{K}$; which – as already shown – reduces to the Banach fixed point principle, without imposing any regularity condition upon $\mathcal{R}_c$. This means that, the algebraic (and/or topological) closeness condition upon $\mathcal{R}_c$ has a null generalizing effect, relative to constant BLR theorem.

Conc-2) If the Razumikhin class $\mathcal{R}_c$ is algebraically closed, we have [in view of $\mathcal{R}_c = \mathcal{K}$], that any PPF dependent fixed point result on $\mathcal{R}_c$ is identical – in a trivial way – with the corresponding PPF dependent fixed point statement over $\mathcal{K}$; which [by the associated selfmap construction] is reducible to (standard) fixed point theorems over the (complete) metric space (E, d), without imposing any regularity condition upon $\mathcal{R}_c$. So, as before, the algebraic (and/or topological) closeness condition upon $\mathcal{R}_c$ has a null generalizing effect relative to the (variant of considered result over) constant class $\mathcal{K}$.

In particular, this latter conclusion tells us that all recent PPF dependent fixed point results – based on such conditions upon $\mathcal{R}_c$ – obtained in Cirić et al [5]. Dhage [7, 8], Harjani et al [9], Hussain et al [10], Kaewcharoen [12], Kutbi and Sintunavarat [15], are in fact reducible to PPF dependent fixed point results over the constant class $\mathcal{K}$; and these, in turn, are deductible from corresponding fixed point theorems over the supporting metrical structure (E, d), without imposing any regularity condition to $\mathcal{R}_c$. A verification of this assertion for the PPF dependent fixed point result in Agarwal et al [1] is performed in the rest of our exposition. The remaining cases are to be treated in a similar way; we do not give details.

3. SVV TYPE CONTRACTIONS

Let X be a nonempty set. Take a metric $d : X \times X \to R_+$ over it; as well as a selfmap $T \in \mathcal{F}(X)$. The basic directions under which the question of determining the fixed points of T is to be solved were already sketched. As precise, a classical result in this direction is the 1922 one due to Banach [2]. In the following, a conditional version of the quoted statement is given, under the lines proposed in Samet et al [23]. Let the mapping $\alpha : X \times X \to R_+$ be fixed in the sequel.

I) We say that T is *α-admissible*, if

(c01) $(\forall x, y \in X)$: $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

II) Given $k \geq 0$, we say that T is *(SVV type) (α, k)-contractive*, provided

(c02) $\alpha(x, Tx)\alpha(y, Ty)d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$.

III) Further, let us say that α is *X-closed*, provided

(c03) whenever the sequence $(x_n; n \geq 0)$ in X and the element $x \in X$ fulfill $[\alpha(x_n, Tx_n) \geq 1, \forall n]$, and $x_n \xrightarrow{d} x$, then $\alpha(x, Tx) \geq 1$.

IV) Finally, let us say that T is *X-starting*, if

(c04) there exists $x_0 \in X$, such that $\alpha(x_0, Tx_0) \geq 1$.

The following conditional fixed point result (referred to as: SVV theorem) involving these data is then available:

Theorem 3.1. *Suppose that $\mathfrak{T}$ is (SVV type) (α, k)-contractive, for some $k \in [0, 1[$. In addition, suppose that (X, d) is complete, T is α-admissible, α is X-closed, and T is X-starting. Then,*

i) *For the arbitrary fixed $x_0 \in X$ with $\alpha(x_0, Tx_0) \geq 1$, the sequence $(x_n; n \geq 0)$ in X defined as $(x_{n+1} = Tx_n; n \geq 0)$ converges to a fixed point $x^* \in X$ of T, with $\alpha(x^*, Tx^*) \geq 1$*

ii) *T has exactly one fixed point $x^* \in X$, such that $\alpha(x^*, Tx^*) \geq 1$.*

Proof. There are two assertions to be clarified.

Step 1. Let us firstly verify that T cannot have more than one fixed point $x^* \in X$, such that $\alpha(x^*, Tx^*) \geq 1$. In fact, assume that, T would have another fixed point $y^* \in X$ such that $\alpha(y^*, Ty^*) \geq 1$. From the (SVV type) contractive condition, we have

$$\alpha(x^*, Tx^*)\alpha(y^*, Ty^*)d(Tx^*, Ty^*) \leq kd(x^*, y^*).$$

This, along with the imposed properties, yields $d(Tx^*, Ty^*) \leq kd(x^*, y^*)$; or, equivalently (as x^*, y^* are fixed points) $d(x^*, y^*) \leq kd(x^*, y^*)$; wherefrom (as $0 \leq k < 1$), $d(x^*, y^*) = 0$; hence, $x^* = y^*$.

Step 2. Let us now establish the existence part. Fix in the following some $x_0 \in X$ with $\alpha(x_0, Tx_0) \geq 1$; and let $(x_n := T^n x_0; n \geq 0)$ stand for the iterative sequence generated by it. From this very choice,

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1;$$

whence, by the admissible property of T,

$$\alpha(x_1, Tx_1) = \alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1;$$

and so on. By a finite induction procedure, one gets an evaluation like

$$\alpha(x_n, Tx_n) = \alpha(x_n, x_{n+1}) \geq 1, \ \forall n. \tag{1}$$

This tells us that the (SVV type) contractive condition applies to (x_n, x_{n+1}), for all $n \geq 0$. An effective application of it gives

$$\alpha(x_n, x_{n+1})\alpha(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1}), \ \forall n;$$

wherefrom

$$d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1}), \ \forall n.$$

This, again by a finite induction procedure, gives

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1), \ \forall n;$$

so that, as the series $\sum_n k^n$ converges, $(x_n; n \geq 0)$ is d-Cauchy. As (X, d) is complete, $x_n \xrightarrow{d} x^*$ for some $x^* \in X$; moreover, combining with (1) (and the closed property of

α), one derives $\alpha(x^*, Tx^*) \geq 1$. The (SVV type) contractive condition is applicable to each pair (x_n, x^*), where $n \geq 0$; and gives

$$\alpha(x_n, Tx_n)\alpha(x^*, Tx^*)d(Tx_n, Tx^*) \leq kd(x_n, x^*), \ \forall n;$$

wherefrom

$$d(x_{n+1}, Tx^*) = d(Tx_n, Tx^*) \leq kd(x_n, x^*), \ \forall n. \tag{2}$$

The sequence $(y_n := x_{n+1}; n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$; so that, $y_n \xrightarrow{d} x^*$ as $n \to \infty$. Passing to limit in (2), gives $d(x^*, Tx^*) = 0$; wherefrom (as d is sufficient), $x^* = Tx^*$. The proof is thereby complete.

Note that, the obtained result is not very general in the area; but, it will suffice for our purposes. Various extensions of it may be found in Samet et al [23].

4. AKS THEOREM

Under these preliminaries, we may now pass to the announced result concerning PPF dependent fixed points.

Let $(E, \|.\|)$ be a Banach space; and let d be the induced by norm metric on E ($d(x,y) = \|x-y\|$, $x, y \in E$); hence, (E, d) is complete. Further, let $I = [a, b]$ be a closed real interval, and $E_0 := C(I, E)$ stand for the class of all continuous functions $\varphi : I \to E$, endowed with the supremum norm ($\|\varphi\|_0 = \sup\{\|\varphi(t)\|; t \in I\}$, $\varphi \in E_0$). As before, let D stand for the induced by norm metric on E_0 ($D(\varphi, \xi) = \|\varphi - \xi\|_0$, $\varphi, \xi \in E_0$); clearly, (E_0, D) is complete.

Let $c \in I$ be fixed in the sequel. The *Razumikhin class* of functions in E_0 attached to c, is defined as

$$\mathcal{R}_c = \{\varphi \in E_0; \|\varphi\|_0 = \|\varphi(c)\|\}.$$

Note that $\mathcal{R}_c$ is nonvoid; because any constant function belongs to it. Precisely, for each $u \in E$, let $H[u]$ stand for the constant function of E_0, defined as: $H[u](t) = u$, $t \in I$. Note that, by this definition, $\|H[u]\|_0 = \|u\|$, $H[u](c) = u$; whence $H[u] \in \mathcal{R}_c$. Denote, for simplicity, $\mathcal{K} = \{H[u]; u \in E\}$; this will be referred to as the *constant class* of E_0.

Now, let $\mathcal{T} : E_0 \to E$ be a mapping. We say that $\varphi \in E_0$ is a *PPF dependent fixed point* of $\mathcal{T}$, when $\mathcal{T}\varphi = \varphi(c)$. As already noted, the natural way to determine such points is offered by the constant class $\mathcal{K}$ of E_0. Then, the points in question appear as fixed points of the selfmap $T : E \to E$, introduced as

$$Tu = \mathcal{T}(H[u]), \ \ u \in E.$$

Precisely (see above)

fp-1) If $z \in E$ is a fixed point of T, then $\zeta := H[z] \in \mathcal{K}$ is a PPF dependent fixed point of $\mathcal{T}$

fp-2) Conversely, if $\varphi = H[u] \in \mathcal{K}$ is a PPF dependent fixed point of $\mathcal{T}$, then $u \in E$ is a fixed point of T.

The fixed point result to be applied is SVV theorem; so, we have to clarify whether the required conditions in terms of T are obtainable via (nonself type) hypotheses in terms of $\mathcal{T}$. For an easy reference, we list the latter conditions. Let in the following $\alpha : E \times E \to R_+$ be a mapping.

I) We say that $\mathcal{T}$ is *α-admissible*, if

(d01) $(\forall \varphi, \xi \in E_0)$: $\alpha(\varphi(c), \xi(c)) \geq 1$ implies $\alpha(\mathcal{T}\varphi, \mathcal{T}\xi) \geq 1$.

In particular, take $\varphi = H[x]$, $\xi = H[y]$, where $x, y \in E$. Then, this condition yields

$(\forall x, y \in E)$: $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$;

or, in other words: the associated selfmap $T : E \to E$ is α-admissible.

II) Given $k \geq 0$, we say that $\mathcal{T}$ is *(α, k)-contractive*, provided

(d02) $\alpha(\varphi(c), \mathcal{T}\varphi)\alpha(\xi(c), \mathcal{T}\xi)d(\mathcal{T}\varphi, \mathcal{T}\xi) \leq kD(\varphi, \xi)$, $\forall \varphi, \xi \in E_0$.

As before, take $\varphi = H[x]$, $\xi = H[y]$, where $x, y \in E$. Then, by this condition,

$\alpha(x, Tx)\alpha(y, Ty)d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in E$;

i.e.: T is (SVV type) (α, k)-contractive.

III) Further, let us say that α is *E_0-closed*, provided

(d03) whenever the sequence $(\varphi_n; n \geq 0)$ in E_0 and the element $\varphi \in E_0$ fulfill $[\alpha(\varphi_n(c), \mathcal{T}\varphi_n) \geq 1, \forall n]$, and $\varphi_n \xrightarrow{D} \varphi$, then $\alpha(\varphi(c), \mathcal{T}\varphi) \geq 1$.

In particular, taking $(\varphi_n = H[x_n]; n \geq 0)$ and $\varphi = H[x]$, for some sequence $(x_n; n \geq 0)$ in E and some point $x \in E$, this requirement becomes:

whenever the sequence $(x_n; n \geq 0)$ in E and the element $x \in E$ fulfill $[\alpha(x_n, Tx_n) \geq 1, \forall n]$, and $x_n \xrightarrow{d} x$, then $\alpha(x, Tx) \geq 1$;

or, in other words: α is E-closed.

IV) Finally, let us say that $\mathcal{T}$ is *E_0-starting*, if

(d04) there exists $\varphi_0 \in E_0$, such that $\alpha(\varphi_0(c), \mathcal{T}\varphi_0) \geq 1$.

Likewise, let us say that $\mathcal{T}$ is *$\mathcal{K}$-starting*, if

(d05) there exists $\varphi_0 \in \mathcal{K}$, such that $\alpha(\varphi_0(c), \mathcal{T}\varphi_0) \geq 1$.

Clearly, if $\mathcal{T}$ is $\mathcal{K}$-starting, then $\mathcal{T}$ is E_0-starting as well. The reciprocal inclusion holds too, under an admissible property upon T.

Proposition 4.1. *Assume that the (nonself mapping) $\mathcal{T}$ is α-admissible and E_0-starting. Then, $\mathcal{T}$ is $\mathcal{K}$-starting.*

Proof. As $\mathcal{T}$ is E_0-starting, there exists $\varphi_0 \in E_0$ such that $\alpha(\varphi_0(c), \mathcal{T}\varphi_0) \geq 1$. On the other hand, for the point $\mathcal{T}\varphi_0 \in E$, we may consider the function $\xi_0 = H[\mathcal{T}\varphi_0]$ in the constant class $\mathcal{K}$. This, by definition, means

$$\xi_0(t) = \mathcal{T}\varphi_0,\ \forall t \in I;\ \text{whence},\ \xi_0(c) = \mathcal{T}\varphi_0.$$

The starting property of $\mathcal{T}$ becomes:

$$\alpha(\varphi_0(c), \xi_0(c)) \geq 1.$$

As $\mathcal{T}$ is α-admissible, this yields $\alpha(\mathcal{T}\varphi_0, \mathcal{T}\xi_0) \geq 1$. Combining with a preceding relation, we thus have

$$\alpha(\xi_0(c), \mathcal{T}\xi_0) \geq 1;$$

which tells us that $\mathcal{T}$ is $\mathcal{K}$-starting.

Now, as $\mathcal{T}$ is $\mathcal{K}$-starting, there exists $\varphi_0 = H[x_0]$ (where $x_0 \in E$) in $\mathcal{K}$, such that $\alpha(\varphi_0(c), \mathcal{T}\varphi_0) \geq 1$. This, by definition, means $\alpha(x_0, Tx_0) \geq 1$; and tells us that the associated selfmap T is E-starting.

Putting these together, it results, via SVV theorem (and our preliminary facts), the following PPF dependent fixed point result (referred to as: AKS theorem), involving these data:

Theorem 4.1. *Suppose that $\mathcal{T}$ is (α, k)-contractive, for some $k \in [0, 1[$. In addition, suppose that $\mathcal{T}$ is α-admissible, α is E_0-closed, and $\mathcal{T}$ is E_0-starting. Then*

i) *$\mathcal{T}$ is $\mathcal{K}$-starting (see above)*

ii) *For the arbitrary fixed $\varphi_0 \in \mathcal{K}$ with $\alpha(\varphi_0(c), \mathcal{T}\varphi_0) \geq 1$, the sequence $(\varphi_n; n \geq 0)$ in $\mathcal{K}$ defined as $(\varphi_{n+1}(c) = \mathcal{T}\varphi_n; n \geq 0)$ converges to a PPF dependent fixed point $\varphi^* \in \mathcal{K}$ of T, with $\alpha(\varphi^*(c), \mathcal{T}\varphi^*) \geq 1$*

iii) *$\mathcal{T}$ has a exactly one PPF dependent fixed point φ^* in the constant class $\mathcal{K}$, such that $\alpha(\varphi^*(c), \mathcal{T}\varphi^*) \geq 1$.*

In particular, when the Razumikhin class $\mathcal{R}_c$ is algebraically (as well as topologically) closed, and the starting condition is taken as

(d06) $\mathcal{T}$ is $\mathcal{R}_c$-*starting*: there exists $\varphi_0 \in \mathcal{R}_c$, with $\alpha(\varphi_0(c), \mathcal{T}\varphi_0) \geq 1$,

the obtained result is just the PPF dependent fixed point result in Agarwal et al [1]. However, as noted earlier, all these conditions have no generalizing effect upon our data; because, in view of $\mathcal{R}_c = \mathcal{K}$, the obtained statement is again AKS theorem; which is deductible without any regularity condition upon $\mathcal{R}_c$.

Finally, note that, by the same techniques, it follows that all PPF dependent fixed point results in Cirić et al [5], Dhage [7, 8], Harjani et al [9], Hussain et al [10], Kaewcharoen [12], Kutbi and Sintunavarat [15], are in fact reducible to (corresponding) fixed point statements over standard metric structures. Some other aspects will be delineated elsewhere.

5. PROOF OF BLR THEOREM

Let us now return to the BLR theorem. For completeness reasons, we shall provide a proof of this result which, in part, differs from the original one.

Let $(E, ||.||)$ be a Banach space; and let d be the induced by norm metric on E ($d(x, y) = ||x - y||$, $x, y \in E$); hence, (E, d) is complete. Further, let $I = [a, b]$ be a closed real interval, and $E_0 := C(I, E)$ stand for the class of all continuous functions $\varphi : I \to E$, endowed with the supremum norm ($||\varphi||_0 = \sup\{||\varphi(t)||; t \in I\}$, $\varphi \in E_0$). As before, let D stand for the induced metric ($D(\varphi, \xi) = ||\varphi - \xi||_0$, $\varphi, \xi \in E_0$); clearly, (E_0, D) is complete.

Let $c \in I$ be fixed in the sequel. The *Razumikhin class* of functions in E_0 attached to c, is defined as

$$\mathcal{R}_c = \{\varphi \in E_0; ||\varphi||_0 = ||\varphi(c)||\}.$$

Note that $\mathcal{R}_c$ is nonvoid; because any constant function belongs to it.

Finally, let $\mathcal{T} : E_0 \to E$ be a (nonself) mapping. We say that $\varphi \in E_0$ is a *PPF dependent fixed point* of $\mathcal{T}$, when $\mathcal{T}\varphi = \varphi(c)$. The class of all these will be denoted as PPF-Fix$(\mathcal{T}; E_0)$.

To establish the existence and uniqueness result in question, the following concepts and constructions are necessary.

I) Given $k \geq 0$, call $\mathcal{T}$, *k-contractive*, provided

$$d(\mathcal{T}\varphi, \mathcal{T}\xi) \leq kD(\varphi, \xi), \text{ for all } \varphi, \xi \in E_0.$$

Note that, as a consequence of this, $\mathcal{T}$ is (D, d) continuous: $\varphi_n \xrightarrow{D} \varphi$ implies $\mathcal{T}\varphi_n \xrightarrow{d} \mathcal{T}\varphi$.

II) Let us introduce a relation (∇) over E_0, according to:

$$\varphi \nabla \xi \text{ iff } \mathcal{T}\varphi = \xi(c) \text{ and } \varphi - \xi \in \mathcal{R}_c.$$

III) Finally, given the starting element $\varphi_0 \in E_0$, let us say that the sequence $(\varphi_n; n \geq 0)$ in E_0 is (φ_0, ∇)-*iterative*, in case $\varphi_n \nabla \varphi_{n+1}$, $\forall n$.

Having these precise, we may now proceed to the announced

Proof. **(BLR theorem)** There are several steps to be considered.

Part 1. Take a starting point $\varphi_0 \in E_0$, and let $(\varphi_n; n \geq 0)$ in E_0 be some (φ_0, ∇)-iterative sequence. By definition, $\varphi_0 \nabla \varphi_1$; which means

$$\mathcal{T}\varphi_0 = \varphi_1(c),\ d(\varphi_0(c), \varphi_1(c)) = D(\varphi_0, \varphi_1).$$

Further, $\varphi_1 \nabla \varphi_2$; which means

$$\mathcal{T}\varphi_1 = \varphi_2(c),\ d(\varphi_1(c), \varphi_2(c)) = D(\varphi_1, \varphi_2).$$

Note that, as a combination of these, we have (by the contractive property of $\mathcal{T}$)

$$D(\varphi_1, \varphi_2) = d(\varphi_1(c), \varphi_2(c)) = d(\mathcal{T}\varphi_0, \mathcal{T}\varphi_1) \leq kD(\varphi_0, \varphi_1).$$

This procedure may continue indefinitely; and gives the iterative relations

$$D(\varphi_{n+1}, \varphi_{n+2}) \le kD(\varphi_n, \varphi_{n+1}),\ \forall n \ge 0. \tag{1}$$

By a finite induction procedure, one gets

$$D(\varphi_n, \varphi_{n+1}) \le k^n D(\varphi_0, \varphi_1),\ \forall n \ge 0; \tag{2}$$

and since the series $\sum_n k^n$ converges, the sequence $(\varphi_n; n \ge 0)$ is D-convergent. As (E_0, D) is complete, $\varphi_n \xrightarrow{D} \varphi^*$ as $n \to \infty$, for some $\varphi^* \in E_0$. Passing to limit as $n \to \infty$ in the iterative relation that defines the sequence $(\varphi_n; n \ge 0)$, one gets (as $\mathcal{T}$ is (D, d)-continuous) $\mathcal{T}\varphi^* = \varphi^*(c)$; i.e.; φ^* is an element of PPF-Fix$(\mathcal{T}; E_0)$.

Part 2. Take a couple of starting points $\varphi_0, \xi_0 \in E_0$, and let $(\varphi_n; n \ge 0)$, $(\xi_n; n \ge 0)$ be their corresponding (φ_0, ∇)-iterative sequence and (ξ_0, ∇)-iterative sequence, respectively. By the preceding part, $(\varphi_n; n \ge 0)$ fulfills the iterative relations (1) and (2). Likewise, $(\xi_n; n \ge 0)$ fulfills the iterative relations

$$D(\xi_{n+1}, \xi_{n+2}) \le kD(\xi_n, \xi_{n+1}),\ \forall n \ge 0; \tag{3}$$

wherefrom (by a finite induction procedure)

$$D(\xi_n, \xi_{n+1}) \le k^n D(\xi_0, \xi_1),\ \forall n \ge 0. \tag{4}$$

From the triangle inequality, one gets, for all $n \ge 1$,

$$\begin{aligned}&D(\varphi_n, \xi_n) \le D(\varphi_{n-1}, \varphi_n) + D(\xi_{n-1}, \xi_n) + D(\varphi_{n-1}, \xi_{n-1}) \le \\ &k^{n-1}[D(\varphi_0, \varphi_1) + D(\xi_0, \xi_1)] + D(\varphi_{n-1}, \xi_{n-1}).\end{aligned}$$

By repeating the procedure, it follows that, after $(n-1)$ steps, the first part of conclusion **ii)** follows. Moreover, if $\varphi_0 = \xi_0$, then $\mathcal{T}\varphi_0 = \mathcal{T}\xi_0$ (that is: $\varphi_1(c) = \xi_1(c)$); wherefrom (by the choice of our iterative sequences)

$$\begin{aligned}&D(\varphi_0, \varphi_1) = d(\varphi_0(c), \varphi_1(c)),\\ &D(\xi_0, \xi_1) = d(\xi_0(c), \xi_1(c)) = d(\varphi_0(c), \varphi_1(c));\end{aligned}$$

and, this yields the second part of conclusion **ii)**.

Part 3. Let $(\varphi_n; n \ge 0)$, $(\xi_n; n \ge 0)$ be taken as in the premise above. By the very definition of these points and the working condition, we have, for all $n \ge 1$,

$$D(\varphi_n, \xi_n) = d(\varphi_n(c), \xi_n(c)) = d(\mathcal{T}\varphi_{n-1}, \mathcal{T}\xi_{n-1}) \le kD(\varphi_{n-1}, \xi_{n-1}).$$

This, by a finite induction procedure, gives

$$D(\varphi_n, \xi_n) \le k^n D(\varphi_0, \xi_0),\ \forall n.$$

Passing to limit as $n \to \infty$, and noting that both $(\varphi_n; n \ge 0)$ and $(\xi_n; n \ge 0)$ are D-convergent (see above), conclusion **iii)** follows.

Part 4. Let φ^*, ξ^* be two elements in PPF-Fix$(\mathcal{T}; E_0)$, taken as in the stated premise. By the working condition and the contractive property of T,

$$D(\varphi^*, \xi^*) = d(\varphi^*(c), \xi^*(c)) = d(\mathcal{T}\varphi^*, \mathcal{T}\xi^*) \leq kD(\varphi^*, \xi^*);$$

and, from this (as D is a metric), $\varphi^* = \xi^*$. The proof is thereby complete.

As above said, this result – as well as its extensions due to Pathak [18] and Som [24] – is just a conditional one; because the iterative sequences appearing there are not effective. The correction of this drawback by taking $\mathcal{R}_c$ as algebraically closed is, ultimately, without effect; for, as the preceding developments show, it brings the PPF dependent fixed point problem in question at the level of constant class, $\mathcal{K}$. From a theoretical viewpoint, this reduced problem may be of some avail in solving the (metrical) question we deal with. However, from a practical perspective, the resulting constant solutions for the nonlinear functional differential/integral equations based on these techniques – such as, the ones in Kutbi and Sintunavarat [15] – are not very promising. Further aspects will be discussed elsewhere.

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