ROMAI J., v.10, no.2(2014), 53–64

PROPERTIES OF NONLOCAL REACTION-DIFFUSION EQUATIONS FROM POPULATION DYNAMICS

Narcisa C. Apreutesei

Technical University of Iaşi, Department of Mathematics and Informatics, Iaşi, Romania,

napreut@gmail.com

Abstract Some models of integro-differential equations from Biology are analyzed. The integral term describes the nonlocal consumption of resources. Fredholm property of the corresponding linear operators are useful to prove the existence of travelling wave solutions. For some models, this can be done only when the support of the integral is sufficiently small. In this case, the integro-differential operator is close to the differential one. One uses a perturbation method which combines the Fredholm property of the linearized operators and the implicit function theorem. For some other models, Leray-Schauder method can be applied. This implies the construction of a topological degree for the corresponding operators and the establishment of a priori estimates for the solution.

Work presented at CAIM 2014, September 19-22, "Vasile Alecsandri" University of Bacău, Romania.

Keywords: nonlocal reaction-diffusion equations, Fredholm operator, proper operator, topological degree, travelling wave solutions.

2010 MSC: 35K57, 45K05,47A53, 47H11.

1. INTRODUCTION

In this survey paper we put together some results concerning travelling wave solutions for reaction-diffusion equations in population dynamics with an integral term. This integral term signifies the nonlocal consumption of resources. The results have mainly been proved in [2] − [6] , [13] . The general integro-differential equation we have under our attention is

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + F(u, J(u)), \text{ where } J(u) = \int_{-\infty}^{\infty} \varphi(x-y)\, u(y,t)\, dy. \tag{1.1}$$

Here $x, t \in \mathbb{R}$, $\alpha > 0$ is the diffusion coefficient, F is a nonlinear function, while $\varphi : \mathbb{R} \to \mathbb{R}$ is a given function with bounded support $[-N, N]$, $\varphi \geq 0$ on $\mathbb{R}$ and $\int_{-\infty}^{\infty} \varphi(x)\, dx = 1$. Function u represents the density of a population. Variable x can be interpreted as a spatial position or as a morphological feature of the population. The nonlinear term $F(u, J(u))$ includes in itself the reproduction and the mortality of the population.

Interpreting x as a spatial position, the integral $\int_{-\infty}^{\infty} \varphi(x-y)\, u(y,t)\, dy$ describes the consumption of resources at the point x by the individuals located at the space point y. Thus we deal with nonlocal consumption of resources. This can be interpreted as intra-specific competition (that is competition for resources of the individuals of the same species).

Interpreting x as a morphological characteristic, the model describes the microevolution of biological species. The individuals that have approximately the same value of x can be interpreted as a biological subspecies of the species under discussion. Individuals with different values of the feature x correspond to different subspecies. They are considered as populations separated in the morphological space.

If x is a morphological feature, for example the height of the individuals, then the integral shows that the animal eats leaves or herbs at the height from the interval $[x-N, x+N]$, where $[-N, N]$ is the support of φ.

If $\varphi(y)$ is the Dirac $\delta-$function, then the integral equals $u(x,t)$ and one arrives at the classical models in population dynamics,

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + F_0(u), \text{ where } F_0(u) = F(u,u). \tag{1.2}$$

If $F_0(u)$ has two zeros, one of them being stable and the other one unstable, we call it the monostable case. If $F_0(u)$ has three zeros $u_1 < u_2 < u_3$, where u_1, u_3 are stable and u_2 is unstable, then we call it the bistable case. For $F_0(u) = ku(1-u)$, $k > 0$ constant, problem (1.2) is the well-known KPP equation or Fisher equation (see [12], [15]). It was initially proposed to model the propagation of dominant genes.

There are many models from population dynamics with an integral term in the literature. We want to mention here papers [1] – [11], [13], [14], [16], [21] and the surveys [17] and [20].

The structure of the present paper is the following. Section 2 recalls notions and theoretical results that we need in the sequel. In the next section we present some models of nonlocal reaction-diffusion equations from population dynamics. Section 4 is devoted to properness and topological degree for the operators associated with the general problem (1.1). Fredholm property is also investigated. In the last section we discuss the existence of travelling wave solutions for the models introduced in Section 3. The method approached for each model is different.

2. Basic notions. *A travelling wave solution* of the parabolic system of equations

$$\frac{\partial u}{\partial t} = A \frac{\partial^2 u}{\partial x^2} + F(u), \; x \in \Omega, \; t \in \mathbb{R}$$

in the unbounded domain $\Omega \subseteq \mathbb{R}$, is a solution of the form $u(x,t) = w(x-ct)$, where c is a real constant, which designs the wave speed. Constant c is also unknown, together with function w. Observe that function $w(\xi)$, $\xi = x - ct$ is a solution of the ODE $Aw'' + cw' + F(w) = 0$ on $\mathbb{R}$. The solutions $w(\xi)$ of this equation such that

$(\exists)\lim_{\xi\to\pm\infty} w(\xi) = w_\pm$, $w_+ \neq w_-$, are called *wave fronts* connecting the points w_+ and w_-. Under some boundedness hypotheses on $w(\xi)$, it follows that $F(w_+) = 0$, $F(w_-) = 0$, thus w_+, w_- are stationary points of the ODE system $du/dt = F(u)$ corresponding to the above PDE. For more information, see [18], [19].

Consequently, the travelling wave solutions of problem (1.1) are functions $w(\xi)$, $\xi = x - ct$ that satisfy the problem

$$\alpha w'' + cw' + F(w, J(w)) = 0, \quad \lim_{\xi\to\pm\infty} w(\xi) = w_\pm, \tag{2.1}$$

where $w_\pm$ are zeros of the nonlinearity $F(w, w)$.

Now we recall the notions of Fredholm operators, properness and the definition of the topological degree.

Let E_1, E_2 be two Banach spaces and $L : E_1 \to E_2$ be a bounded linear operator. The operator L is *normally solvable* if its image $\operatorname{Im} L$ is closed. It is called a *Fredholm operator* if L is normally solvable, it has a finite dimensional kernel $\ker L$, and the codimension of its image is finite.

If L is a Fredholm operator and $\alpha(L) = \dim(\ker L)$, $\beta(L) = co\dim(\operatorname{Im} L)$ are the dimension of the kernel and the codimension of the image, respectively, then we define *the index* of the operator L as the difference $\kappa(L) = \alpha(L) - \beta(L)$. The index does not change under deformation in the class of Fredholm operators.

An operator $A : E_1 \to E_2$ (not necessarily linear) is called *proper* if the intersection of an inverse image of any compact set from E_2 with any closed ball $B \subset E_1$ is compact in E_1. In the linear case, properness means that the operator is normally solvable and has a finite dimensional kernel.

A topological degree can be defined for an operator acting between two Banach spaces E_1, E_2 and an open bounded set $D \subset E_1$ in the following way. Let E_1, E_2 be two Banach spaces, Φ be a class of operators included in $\{A : E_1 \to E_2\}$, and

$$H = \{A_\tau : E_1 \to E_2,\ \tau \in [0,1] \,/\, A_\tau \in \Phi,\ (\forall)\,\tau \in [0,1]\}$$

be a class of homotopies. Let $D \subset E_1$ be an open bounded set and $A \in \Phi$ be an operator such that $A(u) \neq 0$, for $u \in \partial D$. A *topological degree associated to A and to the set D* is an integer $\gamma(A, D)$ (if existing) with the following properties:

(*i*) *Homotopy invariance*. If $A_\tau \in H$ and $A_\tau(u) \neq 0$, $(\forall)\ u \in \partial D$, $\tau \in [0,1]$, then $\gamma(A_0, D) = \gamma(A_1, D)$.

(*ii*) *Additivity*. If $D_1,\ D_2 \subset D$ are open sets, $D_1 \cap D_2 = \Phi$ and $A \in \Phi$, $A(u) \neq 0$, $(\forall)$ $u \in \overline{D} \setminus (D_1 \cup D_2)$, then

$$\gamma(A, D) = \gamma(A, D_1) + \gamma(A, D_2).$$

(Here $\overline{D}$ denotes the closure of D.)

(*iii*) *Normalization*. There exists $J : E_1 \to E_2$ a bounded linear operator with a bounded inverse J^{-1} defined on all the space E_2 such that $\gamma(J, D) = 1$, for every bounded set $D \subset E_1$ with $0 \in D$.

3. Models from population dynamics with nonlocal consumption of resources. One presents below some important special cases for equation (1.1) .

3.1. We begin with a simple model taking into account the nonlocal consumption of resources:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + F(u, J(u)), \ F(u, J(u)) = ku(1 - J(u)), \ \alpha, k > 0, \tag{3.1}$$

for $x, t \in \mathbb{R}$. See for example [13] , [14] . We are here in the monostable case. The nonlinear term $F(u, J(u))$ describes the reproduction of the population. It is proportional to the density u of the population and to available resources, $1 - J(u) = 1 - \int_{-\infty}^{\infty} \varphi(x - y) u(y, t) dy$. In the above model, one takes into account three properties: random mutations (the diffusion), intra-specific competition (the integral), and self-reproduction with the same phenotype (the nonlinear term).

3.2. An important model that we have under our study was proposed in the paper [3]:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + F(u, J(u)), \text{ with } F(u, J(u)) = ku^2(1 - J(u)) - bu. \tag{3.2}$$

Here the term $ku^2(1 - J(u))$ describes the reproduction of the population, while $-bu$ describes its mortality. Unlike the previous model, where the reproduction is proportional to the density u, in the model (3.2) the reproduction is proportional to the square of the density. In the first case we deal with asexual reproduction, while in the second case we have sexual reproduction. Mathematically, function F is in the bistable case.

3.3. Another model with nonlocal consumption of resources is given by

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + F(u, J(u)), \ F(u, J(u)) = f(u) J(u) - g(u), \tag{3.3}$$

where f and g are functions with specific properties ([5]). As a particular situation, in [9] the authors have taken $F(u, J(u)) = u(1 - u) J(u) - bu$. This is also a bistable case. The term $u(1 - u) J(u)$ signifies again the sexual reproduction, while $-bu$ describes the mortality. This integro-differential equation can model plants which distribute their pollen in some area around their location. Another possible interpretation refers to biological cells which can send signalling molecules stimulating other cells to proliferate.

3.4. One can extend the above models to ecosystems composed by two species:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \alpha_1 \dfrac{\partial^2 u}{\partial x^2} + F(u, v, J(u), J(v)) \\ \dfrac{\partial v}{\partial t} = \alpha_2 \dfrac{\partial^2 v}{\partial x^2} + G(u, v, J(u), J(v)) \end{cases}. \tag{3.4}$$

where F and G are given by

$$F := k_1 u^m (1 - a_1 J(u) - b_1 J(v)) - p_1 u, \tag{3.5}$$

$$G := k_2 v^m (1 - a_2 J(u) - b_2 J(v)) - p_2 v. \tag{3.6}$$

Here one takes $m = 1$ for the asexual reproduction and $m = 2$ for sexual reproduction. The model was introduced and studied by N. Apreutesei, A. Ducrot and V. Volpert ([2]). For both F and G, the first term represents the natality and the terms $-p_1 u$ and $-p_2 v$ describe the mortality. Some of the properties of this system have been deduced for a general bistable F.

4. Fredholm property of the operators, properness and topological degree. In this section we are concerned with the properties of the operators associated to problem (1.1). The results for a single equations can be found mainly in [4] and secondary in [3], while for systems of equations they are contained in [2] and [6].

Assume that F is in the bistable case. If one works in the usual Holder spaces $E = C^{2+\alpha}(\mathbb{R})$, $E^0 = C^{\alpha}(\mathbb{R})$, the topological degree may not exist. Though one works in weighted spaces $E_\mu = C_\mu^{2+\alpha}(\mathbb{R})$, $E_\mu^0 = C_\mu^{\alpha}(\mathbb{R})$ for $\mu(x) = 1 + x^2$. The norm in the weighted space E_μ is $\|u\|_\mu = \|\mu u\|_E$.

We are looking for the solutions w of (2.1) under the form $w = u + \psi$, where $\psi \in C^\infty(\mathbb{R})$, such that $\psi(x) = w_+$ for $x \geq 1$ and $\psi(x) = w_-$ for $x \leq -1$. Thus equation (2.1) becomes

$$\alpha(u+\psi)'' + c(u+\psi)' + F(u+\psi, J(u+\psi)) = 0.$$

Let A be the operator in the left-hand side, that is $A : E \to E^0$,

$$Au = \alpha(u+\psi)'' + c(u+\psi)' + F(u+\psi, J(u+\psi)). \tag{4.1}$$

Suppose that $F(u, U)$ is differentiable with respect to both variables u and U. The linearization of A about a function $u_1 \in E$ is the operator $L : E \to E^0$,

$$Lu = \alpha u'' + cu' + \frac{\partial F}{\partial u}(u_1 + \psi, J(u_1+\psi))\, u + \frac{\partial F}{\partial U}(u_1+\psi, J(u_1+\psi))\, J(u),$$

where $\partial F/\partial u$ and $\partial F/\partial U$ are the derivatives of $F(u, U)$ with respect to u and U.

For the linearized operator L, we introduce the limiting operators.

Recall here what it means. If we consider the simple case of the linear operator

$$Lu = a(x)u'' + b(x)u' + c(x)u,\ x \in \mathbb{R}$$

and suppose that the coefficients a, b, c have limits at $\pm\infty$, say

$$a^\pm = \lim_{x\to\pm\infty} a(x),\ b^\pm = \lim_{x\to\pm\infty} b(x),\ c^\pm = \lim_{x\to\pm\infty} c(x),$$

then the limiting operators associated to L are

$$L^{\pm}u = a^{\pm}u'' + b^{\pm}u' + c^{\pm}u.$$

If these limits do not exist, but a, b, c are bounded functions, we take the shifted coefficients $a(x + x_k)$, $b(x + x_k)$, $c(x + x_k)$, where $x_k \to \infty$. Next we choose subsequences that converge to some limiting functions $\widehat{a}(x)$, $\widehat{b}(x)$, $\widehat{c}(x)$, uniformly on every bounded set. Then, the limiting operators are the operators with the limiting coefficients, that is

$$\widehat{L}u = \widehat{a}(x)\, u'' + \widehat{b}(x)\, u' + \widehat{c}(x)\, u.$$

It is clear that L can have a family of limiting operators, depending on the choice of the sequence x_k and of the convergent subsequences of the coefficients.

Since for $w_1 = u_1 + \psi$ there exist the limits $\lim_{x\to\pm\infty} w_1(x) = w_{\pm}$, it follows that $J(w_1) = J(u_1 + \psi) \to w_{\pm}$ as $x \to \pm\infty$ and thus the limiting operators are

$$L^{\pm}u = \alpha u'' + cu' + \frac{\partial F}{\partial u}(w_{\pm}, w_{\pm})\, u + \frac{\partial F}{\partial U}(w_{\pm}, w_{\pm})\, J(u).$$

In order to construct a topological degree, we prove the properness of A in the more general case when the coefficient c and function F depend also on a parameter $\tau \in [0, 1]$. Let $A_\tau : E_\mu \to E_\mu^0$, $\tau \in [0, 1]$ be the operator defined through

$$A_\tau u = \alpha(u + \psi)'' + c(\tau)(u + \psi)' + F_\tau(u + \psi, J(u + \psi)). \tag{4.2}$$

The linearization L_τ of A_τ about a function $u_1 \in E_\mu$ is

$$L_\tau u = \alpha u'' + c(\tau)\, u' + \frac{\partial F_\tau}{\partial u}(u_1 + \psi, J(u_1 + \psi))\, u + \frac{\partial F_\tau}{\partial U}(u_1 + \psi, J(u_1 + \psi))\, J(u),$$

while its limiting operators are given by

$$L_\tau^{\pm}u = \alpha u'' + c(\tau)\, u' + \frac{\partial F_\tau}{\partial u}(w_{\pm}, w_{\pm})\, u + \frac{\partial F_\tau}{\partial U}(w_{\pm}, w_{\pm})\, J(u).$$

For our problem assume the following hypotheses are satisfied:

$(H1)$ For any $\tau \in [0, 1]$, function $F_\tau(u, U)$ satisfies the Lipschitz condition with a constant $K > 0$. Similarly for its derivatives with respect to u and U, $\partial F_\tau/\partial u$ and $\partial F_\tau/\partial U$.

$(H2)$ $c(\tau)$, $F_\tau(u, U)$ and the derivatives of $F_\tau(u, U)$ are Lipschitz continuous in τ, i. e. there exists a constant $c > 0$ such that

$$|c(\tau) - c(\tau_0)| \le c\,|\tau - \tau_0|, \ |F_\tau(u, U) - F_{\tau_0}(u, U)| \le c\,|\tau - \tau_0|,$$

$$\left|\frac{\partial F_\tau(u, U)}{\partial u} - \frac{\partial F_{\tau_0}(u, U)}{\partial u}\right| \le c|\tau - \tau_0|, \ \left|\frac{\partial F_\tau(u, U)}{\partial U} - \frac{\partial F_{\tau_0}(u, U)}{\partial U}\right| \le c|\tau - \tau_0|,$$

$(\forall)\, \tau, \tau_0 \in [0, 1]$, for all (u, U) from any bounded set in $\mathbb{R}^2$.

$(H3)$ (**Condition** NS) For any $\tau \in [0,1]$, the limiting equations

$$\alpha u'' + c(\tau) u' + \frac{\partial F_\tau}{\partial u}(w_\pm, w_\pm) u + \frac{\partial F_\tau}{\partial U}(w_\pm, w_\pm) J(u) = 0$$

do not have nonzero solutions in E.

Now we can formulate some of the main results of this section about Fredholm propert of L_τ and the properness of A_τ.

Theorem 4.1. *Under assumptions* $(H1)-(H3)$, *the operator* $A_\tau(u) : E_\mu \times [0,1] \to E_\mu^0$ *is proper with respect to* (u,τ) *on* $E_\mu \times [0,1]$.

Theorem 4.2. *If condition NS is satisfied, then the operator* $L_\tau : E_\mu = C_\mu^{2+\alpha} \to E_\mu^0 = C_\mu^\alpha$ *is normally solvable with a finite dimensional kernel.*

Condition NS(λ). *For each* $\tau \in [0,1]$, *the limiting equations* $L_\tau^\pm u - \lambda u = 0$ *associated to the operator* $L_\tau - \lambda I$ *do not have nonzero solutions in* E_μ, *for any* $\lambda \geq 0$.

Theorem 4.3. *If Condition NS*(λ) *is satisfied, then* L_τ, *regarded as an operator from* E_μ *to* E_μ^0, *has the Fredholm property and its index is zero.*

Suppose in addition that:

$(H4)$ $F(u,U)$ and its derivatives with respect to u and U are Lipschitz continuous in (u,U);

$(H5)$ The limiting equations

$$\alpha u'' + cu' + \frac{\partial F}{\partial u}(w_\pm, w_\pm) u + \frac{\partial F}{\partial U}(w_\pm, w_\pm) J(u) - \lambda u = 0$$

do not have nonzero solutions in E, $(\forall)\, \lambda \geq 0$.

Consider $\mathcal{F}$ the class of operators A defined through (4.1), such that $(H4)-(H5)$ are satisfied. Consider also the class $\mathcal{H}$ of homotopies $A_\tau : E_\mu \to E_\mu^0$, $\tau \in [0,1]$, of the form (4.2), satisfying $(H1)-(H2)$ and

$(H6)$ For every $\tau \in [0,1]$, the equations

$$\alpha u'' + c(\tau) u' + \frac{\partial F_\tau}{\partial u}(w_\pm, w_\pm) u + \frac{\partial F_\tau}{\partial U}(w_\pm, w_\pm) J(u) - \lambda u = 0$$

do not have nonzero solutions in E, $(\forall)\, \lambda \geq 0$.

Theorem 4.4. *Suppose that functions* F_τ *and* $c(\tau)$ *satisfy conditions* $(H1)-(H2)$ *and* $(H4)-(H6)$. *Then a topological degree exists for the class* $\mathcal{F}$ *of operators and the class* $\mathcal{H}$ *of homotopies.*

5. Existence of travelling waves. In this section, one analyzes the existence of travelling wave solutions for each equation exposed in Section 3.

5.1. Travelling wave solutions for problem (3.1). We begin with a short presentation of integro-differential equations in the monostable case. It includes equation (3.1) or a more general monostable equation. This means that $F_0(u)$ has only two zeros, u_1, u_2, where u_1 is stable, u_2 is unstable, i. e. $F_0'(u_1) < 0$, $F_0'(u_2) > 0$. Many papers are concerned with the study of travelling wave front solutions of nonlocal Fisher equations. See for example [1], [7], [8], [10], [13], [14], [16], [17], [21]. The main results of these papers show that the travelling waves exist for all values of the speed greater or equal to a minimal speed.

5.2. Travelling wave solutions for problem (3.2). We treat now the bistable case given of problem (3.2). In this subsection and in the next one function $F_0(u)$ has three zeros, $u_1 < u_2 < u_3$ with $F_0'(u_1) < 0$, $F_0'(u_2) > 0$, $F_0'(u_3) < 0$. This means that u_1 and u_3 are stable, while u_2 is unstable.

If the support of φ is small, we use a perturbation method, based on the Fredholm property of the linearized integro-differential operators and on the implicit function theorem. If the support of φ is large, the implicit function theorem is not applicable any more and one uses numerical simulations. The results below are mainly included in [3].

Travelling wave solutions of problem (3.2) have the form $u(x,t) = w(x-ct)$, where c is a constant which should be found together with the unknown function w. Then, function w satisfies the equation $\alpha u'' + cu' + F(u, J(u)) = 0$. Denote by A the operator corresponding to the left-hand side of this equation and consider it in Holder spaces with $0 < \alpha < 1$:

$$A : C^{2+\alpha}(\mathbb{R}) \to C^{\alpha}(\mathbb{R}),\ Au = \alpha u'' + cu' + F(u, J(u)),\ (\forall)\, u \in C^{2+\alpha}(\mathbb{R}).$$

The operator linearized about function $u_1 \in C^{2+\alpha}(\mathbb{R})$ associated to A is

$$L : C^{2+\alpha}(\mathbb{R}) \to C^{\alpha}(\mathbb{R}),\ Lu = \alpha u'' + cu' + d(x)\, u - k u_1^2 \int_{-\infty}^{\infty} \varphi(x-y)\, u(y)\, dy,$$

with $d(x) = 2ku_1(x) - b - 2ku_1(x) \int_{-\infty}^{\infty} \varphi(x-y)\, u_1(y)\, dy$. To study the normal solvability of the operator L, we introduce the limiting operators $\widehat{L}$ associated to L. If $x_m \in \mathbb{R}$, $|x_m| \to \infty$, then we choose a subsequence of $u_1(x + x_m)$ converging to a limiting function $\widehat{u_1}$: $u_1(x + x_m) \to \widehat{u_1}(x)$ as $m \to \infty$. Then the limiting operators are defined by

$$\widehat{L}u = \alpha u'' + cu' + \widehat{d}(x)\, u - k(\widehat{u_1})^2 \int_{-\infty}^{\infty} \varphi(x-y)\, u(y)\, dy,$$

where

$$\widehat{d}(x) = 2k\widehat{u_1}(x) - b - 2k\widehat{u_1}(x) \int_{-\infty}^{\infty} \varphi(x-y)\, \widehat{u_1}(y)\, dy.$$

Condition NS'(λ). The limiting equations associated to the operator $L - \lambda I$, $\widehat{L}u - \lambda u = 0$, do not have nonzero solutions in $C^{2+\alpha}(\mathbb{R})$, for any $\lambda \geq 0$.

Under this hypothesis, L satisfies the Fredholm property and its index is zero ([3]).

Now we study the existence of travelling waves in the case when the support of φ is narrow. For the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + F_0(u), F_0(u) = F(u,u) = ku^2(1-u) - bu,$$

the equation $F_0(u) = 0$ has three zeros, $u_+ = 0 < u_\# < u_-$. It is well-known ([18]) that the above PDE has a travelling wave solution $u(x,t) = w(x - ct)$ with the limits $u \to u_\pm$ as $x \to \pm\infty$.

Assume that F_0 is an arbitrary function with the properties below:

$$F_0(0) = F_0(u_-) = 0,\ F_0'(0) < 0,\ F_0'(u_-) < 0.$$

We are in the bistable case. Suppose also that the equation

$$\alpha u'' + cu' + F_0(u) = 0,\ u(-\infty) = 0,\ u(+\infty) = u_-$$

has a solution $(u_0, c_0) \in C^{2+\alpha}(\mathbb{R}) \times \mathbb{R}$ with $u_0' > 0$. In particular we may take $F_0(u) = ku^2(1-u) - bu$ (see [18]).

Now we perturb this ODE with an integral term. Let $\varphi \in L^1(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} \varphi(y)\, dy = 1,\ \int_{-\infty}^{\infty} |y|\, \varphi(y)\, dy < \infty,\ \int_{-\infty}^{\infty} y^2 \varphi(y)\, dy < \infty.$$

We put $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$, $\varepsilon \neq 0$ and $J_\varepsilon(u) = \varphi_\varepsilon * u = \int_{-\infty}^{\infty} \varphi_\varepsilon(x-y)\, u(y)\, dy$.

One investigates the existence of solutions $(u_\varepsilon, c_\varepsilon)$ of the perturbed problem

$$\alpha u'' + cu' + F(u, J_\varepsilon(u)) = 0,\ u(-\infty) = 0,\ u(+\infty) = u_-. \tag{5.1}$$

We can formulate now the main result from [3].

Theorem 5.1. *Under the above assumptions, if the support of φ is narrow, there exists $\varepsilon_0 > 0$ such that equation (5.1) has a solution $(u_\varepsilon, c_\varepsilon) \in C^{2+\alpha}(\mathbb{R}) \times \mathbb{R}$, for any ε with $|\varepsilon| < \varepsilon_0$. Moreover, the solution is of the class C^1 with respect to ε.*

Therefore, if the support of φ is small, the behavior of solution to the nonlocal equation is similar to that of the reaction -diffusion equation.

If the support of φ is not small, numerical simulations show that the behavior of solutions is different in comparison with the usual reaction-diffusion equation. The constant solution u_- can lose its stability. A periodic in space stationary solution appears. It is possible that the wave with the limit $u(-\infty) = u_-$ still exists, but becomes unstable.

There are various biological interpretations of these results. One of them is related to the microevolution of biological species. If we consider the space variable x as a morphological parameter, then the individuals that have approximately the same value of this parameter can be interpreted as a biological subspecies, while two populations that have different values of morphological parameter correspond to two different subspecies. Initially we have a unique species with approximately the same value of the morphological parameter. After some time, it splits into two different subspecies. Some time later, new subspecies appear and so on, until the whole morphological space is completely filled. These results confirm Darwin's principle about emergence of biological subspecies due to variability and struggle for life.

The difference between the monostable case from subsection 5.1 and the bistable cases is the following: in the monostable case, the population develops for any positive initial condition, while for the bistable case, there is a threshold. If the density of the population becomes less than a certain value or if the population is strongly localized with respect to the morphological parameter, then it will disappear.

5.3. Travelling wave solutions for problem (3.3). N. Apreutesei and V. Volpert ([5]) study the existence of travelling wave solutions for (3.4), i. e. solutions of the problem

$$\alpha w'' + cw' + F(u, J(u)) = 0, \ \lim_{x\to\pm\infty} w(x) = w_\pm, \tag{5.2}$$

where $w_+ < w^* < w_-$ are the zeros of the nonlinearity $F(u, J(u)) = f(u) J(u) - g(u)$. In [9] the authors took the particular case $F(u, J(u)) = u(1-u) J(u) - bu$.

The proof of the existence of travelling waves is based on the Leray- Schauder method, which requires the existence of a topological degree for the corresponding operator and a priori estimates of the solutions. Assume that $f \geq 0$ on $[w_+, w_-]$ and that $F_0'(w_\pm) < 0$, $F_0'(w^*) > 0$ (i. e. we deal with a bistable case):

$$\alpha w'' + cw' + f(w) J(w) - g(w) = 0, \ \lim_{x\to\pm\infty} w(x) = w_\pm. \tag{5.3}$$

First we obtain a priori estimates of monotone solutions. Next one separates the monotone and nonmonotone solutions:

Proposition 5.2. *a) There exists $R > 0$ such that for all monotone solutions w of* (5.3), *we have $\|w - \psi\|_\mu \leq R$.*

b) There exists $r > 0$ such that for any monotone w_M and nonmonotone w_N solutions of (5.3), *inequality $\|w_M - w_N\|_\mu \geq r$ is satisfied.*

Theorem 5.3. *There exists a monotone travelling wave solution to problem* (5.3), *i. e. a constant c and a twice continuously differentiable monotone function $w(x)$ satisfying* (5.3).

5.4. Travelling wave solutions for problem (3.4). Competition reaction-diffusion systems are well known models in population dynamics. If we add in the system an integral term, one obtains integro-differential systems of competitive type. System (3.4) appears as a particular case. The integral term signifies intra-specific competition, i. e. the competition of the individuals of the same species. In [2] one studied how intra-specific competition can influence the competition between species. We analyzed the bistable case. One proves the existence of waves for reaction-diffusion systems in the case where the support of the kernel of the integral is sufficiently narrow. When the support is large, numerical simulations have been performed. In the former case the behavior of solutions is similar to that of the reaction-diffusion system without integral. In the latter, the homogeneous in space solutions can lose their stability. Instead of the usual travelling waves, we will observe propagation of periodic waves. The results are similar with those from subsection 5.2 for a single equation.

Some biological interpretations of the results can be derived. Intra-specific competition can result in the emergence of new biological subspecies. The new subspecies continue to compete for resources in both ways, intra-specifically and with other subspecies.

Acknowledgement. This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS - UEFISCDI, project number PN-II-ID-PCE-2011-3-0563, contract no. 343/5.10.2011.

References

[1] S. Ai, *Traveling wave fronts for generalized Fisher equations with spatio-temporal delays*, J. Differential Equations **232**, no. 1, 104–133 (2007).

[2] N. Apreutesei, A. Ducrot, V. Volpert, *Competition of species with intra-specific competition*, Math. Model. Nat. Phenom., **3**, no. 4, 1-27 (2008).

[3] N. Apreutesei, A. Ducrot, V. Volpert, *Travelling waves for integro-differential equations in population dynamics*, Discrete Contin. Dyn. Syst., Ser. B **11**, no. 3, 541–561 (2009).

[4] N. Apreutesei, V. Volpert, *Properness and topological degree for nonlocal reaction-diffusion operators*, Abstr. Appl. Anal., **2011**, Article ID 629692, 21 pages (2011).

[5] N. Apreutesei, V. Volpert, *Existence of travelling waves for a class of integro-differential equations from population dynamics*, Int. Electron. J. Pure Appl. Math., **5**, no. 2, 53-67 (2012).

[6] N. Apreutesei, V. Volpert, *Properness and topological degree for nonlocal integro-differential systems*, Topological Methods in Nonlinear Analysis, **43**, no. 1, 215-229 (2014).

[7] P. Ashwin, M. Bartuccelli, T. Bridges, S. Gourley, *Travelling fronts for the KPP equation with spatio-temporal delay*, Z. Angew. Math. Phys., **53**, 103-122 (2002).

[8] J. B. Burie, A. Calonnec, A. Ducrot, *Singular perturbation analysis of travelling waves for a model in phytopathology*, Math. Model. Nat. Phenom., **1**, no. 1, 49–63 (electronic)(2006).

[9] I. Demin, V. Volpert, *Existence of waves for a nonlocal reaction-diffusion equation*, Math. Model. Nat. Phenom., **5**, no. 5, 80-101 (2010).

[10] F. Dkhil, A. Stevens, *Traveling wave speeds of nonlocally perturbed reaction-diffusion equations*, Asymptot. Anal., **46**, 1(2006), 81–91.

[11] A. Ducrot, *Travelling wave solutions for a scalar age-structured equation*, Discrete Contin. Dyn. Syst. Ser. B, **7**, 2(2007), 251–273, (electronic).

[12] R.A. Fisher, The wave of advance of advantageous genes, *Ann.Eugenics* **7**(1937), 355-369.

[13] S. Genieys, V. Volpert, P. Auger, *Pattern and waves for a model in population dynamics with nonlocal consumption of resources*, *Math. Model. Nat. Phenom.* **1**, 1(2006), 63-80.

[14] S. A. Gourley, *Travelling front solutions of a nonlocal Fisher equation*, J. Math. Biol., **41**(2000), 272-284.

[15] A.N. Kolmogorov, I.G. Petrovsky, N.S. Piskunov, Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *B. Univ. d'Etat à Moscou, Sér. Intern. A* **1**(1937), 1-26.

[16] R. Lefever, O. Lejeune, On the origin of tiger bush, *Bull. Math. Biology* **59**, no. 2(1997), 163-294.

[17] S. Ruan, Spatial-temporal dynamics in nonlocal epidemiological models, in *Mathematics for Life Science and Medicine*, edited by Y. Iwasa, K. Sato and Y. Takeuchi, Springer-Verlag, New York, 2007, pp. 99-122.

[18] A. Volpert, Vit. Volpert, Vl. Volpert, *Traveling Wave Solutions of Parabolic Systems, Translation of Mathematical Monographs*, vol. **140**, Amer. Math. Society, Providence, 1994.

[19] V. Volpert, *Elliptic Partial Differential Equations, Volume 1. Fredholm Theory of Elliptic Problems in Unbounded Domains*, Birkhauser, 2011.

[20] V. Volpert, S. Petrovskii, Reaction-diffusion waves in biology, *Physics of Life Reviews* **6**(2009), 267-310.

[21] J. Wu, D. Wei, M. Mei, Analysis on the critical speed of traveling waves, *Appl. Math. Lett.* **20**, 6(2007), 712–718.