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LQG HOMING FOR JUMP-DIFFUSION PROCESSES

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Abstract The problem of optimally controlling one-dimensional diffusion processes until they enter a given stopping set is extended to the case of jump-diffusion processes. We assume that the jump size is small. By making a logarithmic transformation, the optimal control problem is reduced to a first-passage problem for the corresponding uncontrolled process. A particular example is solved explicitly.

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1. INTRODUCTION

We consider a controlled jump-diffusion process $X_u(t)$ defined by the stochastic differential equation

$$dX_u(t) = \mu dt + b_0 u(t) dt + \sigma dW(t) + \epsilon dN(t)$$

starting from $X_u(t_0) = x$, where μ, $b_0 \neq 0$ and $\sigma > 0$ are constants, $W(t)$ is a standard Brownian motion and $N(t)$ is a time-homogeneous Poisson process with rate $\lambda \geq 0$. The positive constant ϵ is the jump size. These processes are often used in mathematical finance to model the evolution of stock prices. If $\lambda = 0$, then $X_u(t)$ is a controlled Wiener process with drift μ and dispersion parameter σ (see [2], for instance).

Our aim is to minimize the expected value of the cost criterion

$$J(x, t_0) = \int_{t_0}^{T(x)} \frac{1}{2} q_0 u^2(t) dt + K[X_u(T(x)), T(x)],$$

where q_0 is a positive constant, K is a general termination cost function and $T(x)$ is a first-passage time random variable defined by

$$T(x) = \inf\{t \geq t_0 : X_u(t) = a \text{ or } b \mid X_u(t_0) = x \in [a, b]\}.$$

Whittle [4] has considered this type of problem, which he called "LQG homing", for diffusion processes.

If $\lambda = 0$, we can transform the stochastic control problem into a purely probabilistic problem for the uncontrolled process $X_0(t)$. We will try to extend this result to the case of jump-diffusion processes. That is, the case when $\lambda > 0$.

2. OPTIMAL CONTROL

To solve our problem, we define the value function

$$F(x, t_0) = \inf_{u(t),\ t_0 \le t \le T(x)} E[J(x, t_0)].$$

By making use of Bellman's principle of optimality, we can write that

$$\begin{aligned} F(x, t_0) &= \inf_{u(t),\ t_0 \le t \le t_0+\Delta t} E\Big[\int_{t_0}^{t_0+\Delta t} \frac{1}{2} q_0 u^2(t)\, dt \\ &\quad + F(x + [\mu + b_0 u(t_0)]\Delta t + \sigma W(t_0 + \Delta t) \\ &\quad + \epsilon N(t_0 + \Delta t), t_0 + \Delta t)\Big] \\ &= \inf_{u(t),\ t_0 \le t \le t_0+\Delta t} E\Big[\frac{1}{2} q_0 u^2(t_0) \Delta t \\ &\quad + F(x + [\mu + b_0 u(t_0)]\Delta t + \sigma W(t_0 + \Delta t) \\ &\quad + \epsilon N(t_0 + \Delta t), t_0 + \Delta t) + o(\Delta t)\Big]. \end{aligned}$$

Next, for a Poisson process with rate λ (and $N(t_0) = 0$),

$$P[N(t_0 + \Delta t) = 0] = e^{-\lambda \Delta t} = 1 - \lambda \Delta t + o(\Delta t)$$

and

$$P[N(t_0 + \Delta t) = 1] = \lambda \Delta t\, e^{-\lambda \Delta t} = \lambda \Delta t + o(\Delta t).$$

Moreover, with $W(t_0) = 0$, we can write that

$$E[W(t_0 + \Delta t)] = 0$$

and then

$$E[W^2(t_0 + \Delta t)] = V[W(t_0 + \Delta t)] = \sigma^2 \Delta t.$$

Hence, assuming that F is differentiable with respect to t_0 and twice differentiable with respect to x, we deduce from Taylor's formula that

$$\begin{aligned} & E[F(x + [\mu + b_0 u(t_0)]\Delta t + \sigma W(t_0 + \Delta t) + \epsilon, t_0 + \Delta t)] \\ &\quad = F(x + \epsilon, t_0) + [\mu + b_0 u(t_0)]\Delta t \frac{\partial F(x + \epsilon, t_0)}{\partial x} \\ &\quad + \frac{1}{2}\sigma^2 \Delta t \frac{\partial^2 F(x + \epsilon, t_0)}{\partial x^2} + \Delta t \frac{\partial F(x + \epsilon, t_0)}{\partial t_0} + o(\Delta t). \end{aligned}$$

This formula remains valid if we replace ϵ by 0.

From what precedes, we obtain, after simplification, that

$$
\begin{aligned}
0 \;=\; & \inf_{u(t),\, t_0 \le t \le t_0+\Delta t} \left\{ \frac{1}{2} q_0 u^2(t_0) \Delta t + \Delta t \frac{\partial F(x,t_0)}{\partial t_0} \right. \\
& + [\mu + b_0 u(t_0)] \Delta t \frac{\partial F(x,t_0)}{\partial x} + \frac{1}{2} \sigma^2 \Delta t \frac{\partial^2 F(x,t_0)}{\partial x^2} \\
& \left. + [F(x+\epsilon,t_0) - F(x,t_0)] \, \lambda \Delta t + o(\Delta t) \right\}.
\end{aligned}
$$

Dividing each side of the above equation by Δt, and letting Δt decrease to 0, we find that

$$
\begin{aligned}
0 \;=\; & \inf_{u(t_0)} \left\{ \frac{1}{2} q_0 u^2(t_0) + \frac{\partial F(x,t_0)}{\partial t_0} + [\mu + b_0 u(t_0)] \frac{\partial F(x,t_0)}{\partial x} \right. \\
& \left. + \frac{1}{2} \sigma^2 \frac{\partial^2 F(x,t_0)}{\partial x^2} + \lambda \, [F(x+\epsilon,t_0) - F(x,t_0)] \right\},
\end{aligned}
$$

which is the dynamic programming equation (DPE) for our problem.

Next, differentiation with respect to $u(t_0)$ yields that the optimal control $u^*(t_0)$ is given by

$$u^*(t_0) = -\frac{b_0}{q_0} \frac{\partial F(x,t_0)}{\partial x}.$$

Substituting this value into the DPE, we find that we must solve the second-order, non-linear differential-difference equation

$$
\begin{aligned}
0 \;=\; & \frac{\partial F(x,t_0)}{\partial t_0} + \mu \frac{\partial F(x,t_0)}{\partial x} - \frac{1}{2} \frac{b_0^2}{q_0} \left[\frac{\partial F(x,t_0)}{\partial x} \right]^2 + \frac{1}{2} \sigma^2 \frac{\partial^2 F(x,t_0)}{\partial x^2} \\
& + \lambda \, [F(x+\epsilon,t_0) - F(x,t_0)] .
\end{aligned}
\tag{1}
$$

3. LINEARISATION

Now, let

$$\alpha = \frac{b_0^2}{q_0 \sigma^2}$$

and define

$$\Phi(x,t_0) = e^{-\alpha F(x,t_0)}.$$

Then, we find that Eq. (1) becomes

$$
\begin{aligned}
0 \;=\; & \frac{\partial \Phi(x,t_0)}{\partial t_0} + \mu \frac{\partial \Phi(x,t_0)}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 \Phi(x,t_0)}{\partial x^2} \\
& - \alpha \lambda \Phi(x,t_0) \, [F(x+\epsilon,t_0) - F(x,t_0)] .
\end{aligned}
$$

Set (see [3])

$$R(x,\epsilon) = \frac{F(x+\epsilon,t_0) - F(x,t_0)}{\Phi(x+\epsilon,t_0) - \Phi(x,t_0)} = \frac{[F(x+\epsilon,t_0) - F(x,t_0)]/\epsilon}{[\Phi(x+\epsilon,t_0) - \Phi(x,t_0)]/\epsilon}.$$

If ϵ is small, we can write that

$$R(x,\epsilon) \simeq \frac{\frac{\partial F(x,t_0)}{\partial x}}{\frac{\partial \Phi(x,t_0)}{\partial x}} = \frac{dF(x,t_0)}{d\Phi(x,t_0)} = -\frac{1}{\alpha\Phi(x,t_0)}.$$

It follows that

$$F(x+\epsilon,t_0) - F(x,t_0) \simeq -\frac{1}{\alpha\Phi(x,t_0)}[\Phi(x+\epsilon,t_0) - \Phi(x,t_0)],$$

so that the non-linear equation (1) that we must solve is linearised to

$$\begin{aligned} 0 \quad \simeq \quad & \frac{\partial\Phi(x,t_0)}{\partial t_0} + \mu\frac{\partial\Phi(x,t_0)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2\Phi(x,t_0)}{\partial x^2} \\ & +\lambda[\Phi(x+\epsilon,t_0) - \Phi(x,t_0)]. \end{aligned} \tag{2}$$

The previous equation is actually the Kolmogorov backward equation satisfied by

$$M(x,t_0) := E\left[e^{-\alpha K[X_0(T_0(x)),T_0(x)]}\right], \tag{3}$$

where $T_0(x)$ is the same as $T(x)$, but for the uncontrolled process $X_0(t)$. Furthermore, the boundary condition

$$\Phi(x,t_0) = e^{-\alpha K(x,t_0)} \quad \text{if } x = a \text{ or } b$$

is the appropriate one for the function $M(x,t_0)$.

Finally, because ultimate absorption of X_0 into the stopping set $\{a,b\}$ is certain, the solution to the first-passage problem is unique and we can write that $M(x,t_0) = \Phi(x,t_0)$.

Proposition 3.1. *If the jump size ϵ is small, then the optimal control $u^*(t_0)$ can be obtained (approximately) from the mathematical expectation $M(x,t_0)$ [see (3)] defined in terms of the uncontrolled process $X_0(t)$ that corresponds to $X_u(t)$.*

Remarks.

1) This result generalises the theorem proved by Whittle [4].

2) We have transformed the optimal control problem into a purely probabilistic problem. However, solving this probabilistic problem is generally quite difficult. In the next section, we will solve a particular problem.

4. AN EXAMPLE

Assume that $t_0 = \mu = 0$, $b_0 = q_0 = \sigma = \lambda = 1$, $a = 0$ and $b = 2\epsilon$. We then have $\alpha = 1$. Moreover, we choose the termination cost function $K[X_u(T(x)), T(x)] = -\ln[T(x) + 1]$, so that the function $\Phi(x, t_0)$ becomes

$$\Phi(x, t_0) = E\,[T_0(x) + 1] := m(x) + 1$$

and

$$u^*(t_0) = \frac{m'(x)}{m(x) + 1}.$$

Abundo [1] has shown that the function $m(x)$ is the unique solution of the system

$$\begin{array}{rcll} \frac{1}{2} m''(x) + m(x + \epsilon) - m(x) & = & -1 & \text{for } x \in (0, 2\epsilon), \\ m(x) & = & 0 & \text{for } x \notin (0, 2\epsilon) \end{array} \tag{4}$$

and he computed it explicitly. Making use of the expression that he derived (which is quite involved), we can compute the optimal control $u^*(0)$. This function is shown in the figure below, in the case when $\epsilon = 1$, together with the optimal control obtained when $\lambda = 0$, namely

$$u^*(0) \overset{\lambda=0}{=} \frac{-2x + 2}{-x^2 + 2x + 1} \quad \text{for } 0 \leq x \leq 2.$$

Remark. The value $\epsilon = 1$ is probably too large to obtain a very good approximation. However, for $\epsilon \leq 0{,}2$, the two curves are almost identical.

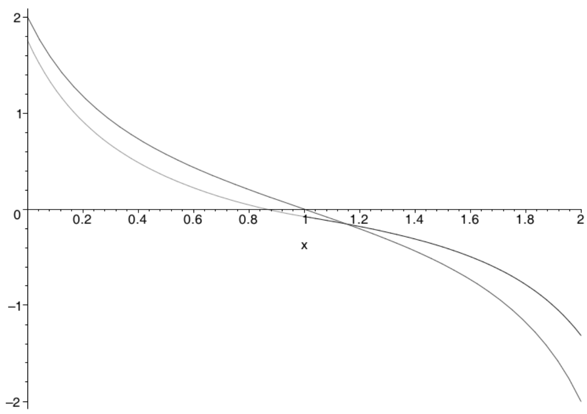

Fig. 1.: Optimal control $u^*(0)$ when $\epsilon = 1$ and $\lambda = 1$, together with $u^*(0)$ when $\epsilon = 1$ and $\lambda = 0$ (starting above).

5. CONCLUSION

In this note, we have extended the result proved by Whittle [4] to the case of jump-diffusion processes. Here, for the sake of simplicity, we considered a controlled one-dimensional jump-diffusion process which, when the parameter λ is equal to zero, reduces to a controlled Wiener process. We could theoretically further extend our result to general n-dimensional jump-diffusion processes. However, in practice, it will be very difficult to obtain explicit solutions to problems in two or more dimensions.

In Section 4, we presented an example that was as simple as possible. Even then, the exact solution to the system (4) is quite involved and very tedious to compute. Furthermore, we saw that from this solution, we deduced an approximate expression for the optimal control, which is valid when the size ϵ of the jumps is small. It would be interesting to obtain the exact solution to Eq. (1), using numerical methods, and compare this exact solution to the approximate one, and this, for various values of ϵ. In general, even for the linearised equation (2), it would probably be necessary to appeal to numerical methods to obtain its solution.

Finally, we could try to find another way of linearising the differential-difference equation (1), which would be valid for any ϵ (at least approximately). We could also apply the technique used in this note to find approximate solutions to stochastic differential games, rather than optimal control problems.

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