Homage to Professor Ilie Burdujan
Professor Ilie Burdujan is an eminent representative of the Romanian culture and spirit, which brought an important contribution to modern algebra, to the theory of differential equations, and to the education of new generations of highly-qualified specialists. In the autumn of 2014 Professor Ilie Burdujan turned 70. This was an excellent opportunity for us to stop on his own mathematical and personal path and to bring once more into light his life and activity up to this moment.

Ilie Burdujan was born at November 10th, 1944 in village Pirscov, County Buzău, Romania. His native village is located in the Sub-Carpathians Curvature, in Buzău valley on the left bank of the River Buzău at its confluence with the River Bâlânăsea. He attends the courses of the primary school in his native village. Then he finished in 1962 his high school studies at the theoretical high school in Pătărlagele, County Buzău. In 1962 he enrolled at the Faculty of Mathematics and Mechanics of the "Al. I. Cuza" University of Iași.

During the student days he was influenced by the famous and exigent teachers and scholars as Academicians M. Haimovici and R. Miron as well as Professors I. Creangă, I. Popa, A. Haimovici, Gh. Gheorghiev, N.Negoescu, V.Crucianu and other. Soon he became one of the best students of the Faculty of Mathematics and Mechanics of the University. In 1968, after his graduation from the Faculty, he has begun preparing his Doctor’s Degree - the chosen specialization being Differential Geometry of quasi-groups and loops - at the "Al. I. Cuza" University of Iași, his main adviser being Professor Gheorghe Gheorghiev. Professor Gheorghe Gheorghiev was one of the best specialists in the Differential Geometry and its Applications. In Geometry School from Iași, geometrical research harmoniously correlates with actual problems of Differential Equations, Algebra, Analytical Mechanics, Theoretical Physics, Optimal Control, Biology, etc. This fact and the scientific interest of the supervisor determined the direction of Burdujan’s further mathematical investigations.

The scholarly activity and didactic career of Professor Ilie Burdujan could be schematically splitted as below:

- 1968 - 1974: Researcher at the Institute of Mathematics of the Romanian Academy, Iași Branch;
- 1974 - 1975: Researcher at the Institute "Petru Poni" of the Romanian Academy;
- 1975 - 1996: assistant, lecturer, associate professor at the Department of Mathematics, University "Gh. Asachi" Iași, Romania;
- 1996 - 2010: Associate Professor and Professor of Department of Mathematics at the University of Agricultural Sciences and Veterinary Medicine, Iași, Romania.
2010 - present: Professor of the University of Agricultural Sciences and Veterinary Medicine.

In over one hundred research papers, two monographs and ten textbooks, I. Burdujan has contributed to fundamental knowledge of various areas of mathematics and its applications. Areas of interests of the Professor Burdujan are: Differential geometry of quasi-groups and loops, Differential geometry of Cliffordian manifolds, non-associative algebras, Homogeneous Systems in Yamaguti’s sense, systems of quadratic differential equations, Lie semigroups, theory of relativity, relativistic magneto-hydrodynamics, rheology, physics of plasma, applications of mathematics in biology.

His thesis ”Continuous and Lie quasigroups and loops and their applications in differential geometry” for a doctor’s degree was defended in 1977 at the ”Al. I. Cuza” University of Iași. There he obtained several results concerning the topology on the multiplication group of a connected and locally connected topological quasigroup, results that tried to give some answers to the next main problem: is or is not a topological quasigroup/loop a homogeneous space in Cartan’s meaning (this problem is just a reformulation of the Fifth Hilbert’s Problem for the multiplication group of a topological quasigroup)? To this end he extended the well-known algebraic construction due to L.V.Sabinin and M.Kikkawa for loops with right identity element. This construction suggests the way to organize the tangent space at the identity element of an analytic loop as a homogeneous system in Yamaguti’s meaning. Actually, the main result of his thesis was to prove that the tangent algebra at the identity element of an analytic loop is necessarily a homogeneous system in Yamaguti’s sense. Certainly, by applying this result to a Lie group the Lie algebra structure of tangent space at the identity is again found exactly in terms used by Lie himself. Unfortunately, the result assuring the local characterization of a Lie group by its Lie algebra couldn’t be extended to the pair Lie loop-homogeneous system even if a formula of type Campbell-Hausdorff were obtained in this thesis. Consequently, there issued the problem of characterizing, at least locally, a Lie loop by its tangent homogeneous system. It seems to be a positive answer only for monoassociative Lie loop. Similar results were later obtained by M.A.Akivis and L.V.Sabinin. Another local characterization of an analytic loop is given by means of the so-called structural operators that contains the generators of the isotopy subgroup of identity in the multiplication group of Lie loop.

Another contribution of I.Burdujan in differential geometry concerns the geometry of the so-called Cliffordian manifolds. This mathematical structure was introduced by I.Burdujan as a direct generalization of quaternionic/quaternionic manifolds. For such manifolds were generalized several results for the geometry of quaternionic manifolds, especially that obtained by V.Oproiu, D.V. Alekseevsky, S. Marchiafava, etc.. The main result obtained here is the construction of the natural linear connection as a specific object that open the way for construct the Pontriagin classes for Cliffordian manifolds. These results of I.Burdujan have been of interest for A.Moroianu, U.Semmelmann, J.Hrdina, P.Vašk, e.al.

The almost Cliffordian and almost Clifford manifolds are similarly defined as the almost quaternionic and almost quaternionic manifolds. The problem of finding out all almost Cliffordian connections is solved. Some results concerning the 1-integrability of a Clifford structure are obtained.
Moreover, on an almost Clifford manifold \((M_{8n}; Q)\) he studied the integrability of an almost complex structure which is compatible with the almost Clifford structure \(Q\). The obtained results extend the results of Alekseevsky, Marchiafava and Pontecorvo, Tricerri. More exactly, he proved that the existence of three compatible complex structures \(I_1, I_2, I_3\) such that \(I_1 \neq \pm I_2, I_1 \neq \pm I_3, I_j \neq I_k \circ I_\ell\) for all \((j, k, \ell) \in S_3\) where \(S_3\) denote the symmetric group with three elements, forces \((M_{8n}; Q)\) to be a Clifford manifold.

The next results was proved for Clifford-Kähler manifolds:

**Theorem 1.1.** Any Clifford-Kähler manifold is an Einstein space.

2. The restricted holonomy group of a Clifford-Kähler \(8m\)-dimensional manifold is a subgroup of \(O(\mathfrak{p}(m))\) if and only if the Ricci tensor vanishes identically.

3. For any Clifford-Kähler manifold \((M, V, g)\) the bundle \(V\) is locally parallelizable if and only if the Ricci tensor vanishes identically.

In order to extent the results of Bonan concerning quaternionic manifolds for the study of Cliffordian manifolds, I.Burdujan characterized the abstract algebraic model of tangent structure of any Cliffordian manifold, the Clifford algebras of type \(C\ell_{0,3}\), by a specific group of automorphisms and anti-automorphisms. More exactly he proved the next result:

any associative algebra \(A(\cdot)\), without second order nilpotents, on which a commutative group \(G(\cdot)\) of order 16 acts faithfully as a group of 8 involutive automorphisms of \(A(\cdot)\) and 8 anti-automorphisms of its, is necessarily a \(C\ell_{0,3}\)-Clifford algebra.

The idea for this last result comes from the paper of Bonan where the quaternion algebra was characterized by means of a set of three involutive automorphisms and the usual conjugation what allows to organize the tangent manifold of a quaternionic manifold as a quaternionic manifold. Consequently, the tangent manifold of a Clifford manifold can be organized as a Clifford manifold.

It must be remarked that the group used in our characterization is not isomorphic with that used by Chernov allowing to consider any Clifford algebra as a projection of a group algebra.

The real difficulties he had met in studying the Lie quasigroups and loops led I.Burdujan to start an intensive study of nonassociative algebras that could be viewed as the linear models for the geometry of quasigroups and loops as well as for Lie semigroups. As it was natural he tried to give an answer to the problem of characterizing a binary nonassociative and/or noncommutative algebra by a complete family of deviations from associativity and/or commutativity. To this end, I.Burdujan find a set of deviations from associativity and commutativity naturally associated with any algebra that allows to organize the ground space of algebra as a homogeneous system (shortly, HS) in Yamaguti’s meaning. More exactly, he build a covariant functor \(Y_2\) from the category of all binary \(K\)-algebras \(\mathcal{ALG}\) (with the algebra homomorphisms as morphisms) to the category \(\mathbb{HS}\) of all homogeneous systems over \(K\) (with HS-homomorphisms as morphisms). This functor extends the well known covariant functor from the category of all associative algebras \(\mathcal{ALG}_{\text{Ass}}\) to the category of all Lie algebras \(\mathcal{ALG}_{\text{Lie}}\) which associates, to each associative algebra, the Lie algebra defined by means of its commutators. Moreover, it is strongly connected with Yamaguti’s functor \(Y_1\) from the category \(\mathbb{HS}\) to the category \(\mathbb{Lie}_{\text{al}}\).

As the Lie algebra of a Lie group can be realized as the tangent algebra at the unity of group as well as the algebra of the left/right invariant vector fields on the group, I.Burdujan has pointed out some Lie algebras of vector fields
associated to any homogeneous system, that he named their infinitesimal groups. In the particular case when the homogeneous system is the one associated to an algebra $A(\cdot)$ this Lie algebra is isomorphic to a Lie algebra of the form $A \oplus \mathcal{L}$, i.e. it is an object of category $\text{Lie}_\oplus$ (here $\mathcal{L}$ is a Lie algebra). The main result in this framework is below presented.

**Theorem 2.** Let $L = V \oplus A$ be a Lie algebra in $\text{Lie}_\oplus$, $B$ be the Lie subalgebra generated by $V$ and $A_1 = A \cap B$. If $L_1 = V \oplus D_V = (Y_2 \circ Y_1)(L)$, then $D_V$ is in one of the following three mutually distinct situations:

1° $D_V = 0$ (this is the case when the HS induced on $V$ by $L$ is just a Lie subalgebra of $L$),

2° $D_V$ is isomorphic to $A_1$,

3° $D_V$ is isomorphic to $A_1/I$, where $I$ is the proper and nontrivial ideal of $A_1$ that contains all elements inducing on $V$ (via ad-representation) the 0-transformation.

One of the most important problems of the non-associative algebra theory is: does characterize or does not characterize an algebra, up to an isomorphism, its associated homogeneous system? This problem is closely connected with the problem: is it or is it not possible to (uniquely) determine the structure constants of an algebra knowing the structure constants of its associated homogeneous system? This last problem is equivalent to the consistency problem for an algebraic system. Moreover, for any such a compatible system, the problem of uniqueness of its solution arises. An answer to this problem could be obtained by means of the next proposition.

**Proposition 1.** If the algebras $A(\cdot)$ and $A(\ast)$, defined on the vector space $A$ by means of the binary operations $\cdot$ and $\ast$, induce the same homogeneous system on $A$ then the deformation algebra $A(\circ)$ defined on $A$ by the binary operation

$$x \circ y = x \cdot y - x \ast y$$

for all $x, y \in A$

satisfies the identities

$$x \circ (y \cdot z) - y \circ (x \cdot z) + x \cdot (y \circ z) - y \cdot (x \circ z) -
- x \circ (y \circ z) + y \circ (x \circ z) = 0,$$

for all $x, y, z \in A$,

and

$$[x_1, x_2, \ldots, x_n, x_{n+1} \circ x_{n+2}] - x_{n+1} \circ [x_1, x_2, \ldots, x_n, x_{n+2}] -
- [x_1, x_2, \ldots, x_n, x_{n+1}] \circ x_{n+2} = 0,$$

for all $x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2} \in A, n \geq 2$.

Consequently, the Lie algebra of homogeneous system must be a Lie algebra of derivations for this deformation algebra. This result allow to prove that not any homogeneous system is the one associated to a binary algebra. More exactly, it was proved that the mesonic Lie triple system (a particular kind of homogeneous system) cannot be the homogeneous system associated to a binary algebra. On the other hand, this result allows to prove that the homogeneous system associated to mathematical models of SIR-like epidemiological models characterize them up to an isomorphism.

Another important part of scientific interest of I.Burdujan was the classification, up to an isomorphism, of real 3-dimensional commutative algebras. To this end he realized that it is important to analyze separately the family of such algebras having at least of a (nonzero) derivation. Moreover, I. Burdujan stressed that algebra is characterized by the lattice of its subalgebras and used
this natural remark for obtaining the classification of algebras having at least a derivation. For them he obtained the complete classification of isomorphic classes. Namely he proved the next result.

**Theorem 3.** The following results are true:

1. The class of all real 3-dimensional commutative algebras that contains a derivation with proper complex eigenvalues contains 16 classes of isomorphism. In particular, the subclass of NN-algebras (where \(x^2 = 0\) implies \(x = 0\)) contains 9 classes of isomorphism.

2. The class of all real 3-dimensional commutative algebras that contains a nilpotent derivation of order 3 contains 9 classes of isomorphism.

3. The class of all real 3-dimensional commutative algebras that contains a nilpotent derivation of second order contains 53 isomorphism classes.

4. The class of all real 3-dimensional commutative algebras that contains a semi-simple derivation with 2-dimensional kernel contains 153 classes of isomorphism.

5. The class of all real 3-dimensional commutative algebras that contains a semi-simple derivation with 1-dimensional kernel contains 35 classes of isomorphism.

6. The class of all real 3-dimensional commutative algebras that contains a semi-simple non-degenerate derivation contains 4 classes of isomorphism.

The problem of classification up to an isomorphism of real commutative 3-dimensional algebras having no derivation is a very difficult one. Some easiness arose for algebras having no nilpotent of order two, the so called NN-algebras. For NN-algebras was proved the next result.

**Theorem 4.** Let \((A, \cdot)\) be an NN-finite-dimensional real algebra and \(R\) be the radical Lie of the algebra \(\text{Der}A\). The following results hold:

1. If \(\dim A \leq 2\), then \(\text{Der}A = \{0\}\).

2. If \(\dim A = 3\), then \(\dim \text{Der}A \in \{0, 1\}\).

3. If \(\dim A = 4\), then either \(\text{Der}A\) is isomorphic to \(\text{su}(2)\), or \(\dim \text{Der}A \in \{0, 1\}\).

4. If \(\dim A \in \{5, 6\}\), then every subalgebra of \(\text{Der} A\) is isomorphic to one of the following Lie algebras:
   (i) \(\text{su}(2) \oplus \text{su}(2)\);
   (ii) \(\text{su}(2) \oplus R\) and \(\dim R \in \{0, 1\}\);
   (iii) \(R\), where \(R\) is an Abelian Lie subalgebra and \(\dim R \in \{0, 1, 2\}\).

The results presented in Theorems 3 and 4 are the foundation for getting the classification of homogeneous quadratic dynamical systems on \(\mathbb{R}^3\). In fact, Professor I. Burdujan has developed an algebraic theory of the qualitative theory of polynomial dynamical systems.

Recall that the modern theory of dynamical systems has its origins in the works of H. Poincaré, dedicated to celestial mechanics and to the problem of three bodies. The study of quadratic dynamical systems and algebraical aspects of that theory has its origins in the works of H. Poincaré, S.Lie, G. Darboux, A. M. Lyapunov, P. Painlevé, G. D. Birkhoff, A.N. Kolmogorov, L. Autonne, I. Bendixson, A.A.Andronov etc. The Poincaré’s problem of center was the first problem of the theory of quadratic equations, being systematically studied using techniques initiated by Poincaré and Lyapunov. This problem was solved completely for quadratic systems of the real plane by W. Kapteyn (1911, 1912),

The importance of this problem was highlighted by D. Hilbert, who formulated in 1900 his famous Problem 16, which asks to determine an upper bound of the number of limit cycles of a polynomial system. Famous investigations related to the Hibert’s Problem 16, were realized by H. Dulac (1908, 1923), Chin Yuan-shun (1958), Yu. Ilyashenko (1981), W. A. Coppel (1966), N. N. Bautin (1954), J. Ecalle (1981), H. Zoladek (1994) and other. This huge amount of results concerns mainly the planar dynamical systems. The method developed by I. Burdujan can be used in studying HQDSs on finite-dimensional spaces even on Banach spaces. As long as this was possible, he preferred to work in the framework of Banach spaces in order to extract the invariant meaning of some definitions and results.

Let $E$ be a Banach space, $T : E \to E$ be a linear transformation, $C$ be a constant vector function on $E$ and $F : E \to E$ be a homogeneous function of degree 2. Any equation of the form $(E)$: $dX/dt = C + TX + F(X)$ is called a quadratic differential equation (briefly, QDE) on $E$ and equation of the form $(E_0)$: $dX/dt = F(X)$ is called a homogeneous quadratic differential equation (briefly, HQDE) on $E$. In case $E = \mathbb{R}^n$ the equation $(E_0)$ becomes $(S)$: $dx^i/dt = a^i_{jk}x^jx^k$ and it is called a quadratic homogeneous system of differential equations (briefly, SHQDE) on $\mathbb{R}^n$. Since any QDE can be homogenized by a standard construction, it was sufficient to restrict the study only on HQDEs.

One of the goals of the theory of HQDE is to get a center-affine classification, using only algebraic tools. L. Markus (1960) suggested the idea of associating to each class of affine-equivalent SHQDE a class of isomorphic commutative algebras associated to the same $(1,2)$-tensor. This association is a concrete realization of the universal property of the tensor product. Concretely, the polar form $G$ of the quadratic form $F$ from the equation $(E_0)$ defines a symmetric covariant $(1,2)$-tensor on $E$, which endows $E$ with a binary operation $(\cdot)$. Hence, $G$ is a technical tool that allows to associate to any HQDE a binary algebra $(E, \cdot)$ on $E$. I. Burdujan establish that two HQDEs are affine-equivalent if and only if the respective binary algebraic structures on $E$ are isomorphic. Thus he has proved the following result:

**Theorem 5.** There is a bijective correspondence between the set of all affine-equivalent classes of HQDEs on a real Banach $E$ and the set of all isomorphic binary algebras on $E$.

Similar results for some concrete infinite dimensional Banach spaces and concrete HQDE were already obtained by M. Gavurin (1952). For finite dimensional Banach spaces this result, in the form of more or less explicit, were found in the papers due to L. Markus (1960), H. Röhrl (1977), S. Walker (1991), M. Kinyon and A. A. Sagle (1994, 1995).

Theorem 5 reflects the existence of a correlation between properties of commutative algebra $(E, \cdot)$ and qualitative properties of respective HQDE. Some correlations are the next:

1.1. The set of the second order nilpotent elements of the algebra of $(E, \cdot)$ is in bijective correspondence with the set of the stationary solutions of the respective HQDE $(E_0)$.
1.2. The set of the second order idempotent elements of the algebra of \((E, \cdot)\) is in bijective correspondence with the set of the ray-solutions of the respective HQDE \((E_o)\).

1.3. \((E, \cdot)\) is a nil-algebra if and only if all solutions of HQDE \((E_o)\) are polynomials.

1.4. \(E^2 = E \cdot E\) is a nonzero proper ideal of \((E, \cdot)\) if and only if at least one first integral of \((E_o)\) is linear.

1.5. If \((E, \cdot)\) is an algebra with the associative power, then all solutions of \((E_o)\) are rational functions.

1.6. If the algebra \((E, \cdot)\) has an idempotent, then the null solution of \((E_o)\) is not asymptotically stable.

1.7. If the system \((E)\) has a Lyapunov function, then the algebra \((E, \cdot)\) has no idempotents.

1.8. If the finite-dimensional commutative algebra \((E, \cdot)\) has a non-zero proper ideal, then the system \((S)\) contains some uncoupled subsystem.

Now we mention the following general result.

**Theorem 6.** Two homogeneous quadratic equations \((E)\) and \((E')\) defined on the Banach spaces \(E\) and \(E'\), respectively, are affine-equivalent if and only if the associated algebras \((E, \cdot)\) and \((E', \cdot)\) are continuous (topologically) isomorphic.

A continuous linear operator on \(E\) is an automorphism of the equation \((E)\) if and only if it is a continuous automorphism of the associated algebra.

By I. Burdujan were analyze the \(n\)-homogeneous dynamical systems \((n > 2)\). It is remarkable that such systems can turn (not uniquely) in SHQDE.

Classification up to isomorphism of non-associative algebras (commutative or not) requires a refinement of the general theory of these algebras. A very useful tool of research of non-associative algebras consists of the homogeneous systems in Yamaguti sense (briefly, homogeneous system or HS). These algebras were introduced by K. Yamaguti in 1958 as tangent algebras in unit class of the homogeneous spaces in Cartan sense. As it was already remarked, I.Burdujan has built a covariant functor from the category of all algebras to the category of homogeneous systems, which extend the usual functor from the category of associative algebras to category of Lie algebras.

So, he has opened the road of using of the theory of algebra representations in study of quadratic homogeneous dynamical systems.

Let us remark that all classification results listed in Theorem 3 give classification results up to an affinity for the corresponding HQDSs.

In the works of Professor I. Burdujan were indicated distinct applications of the theory of differential equations in biology, epidemiology, medicine, economics, engineering, social sciences, etc.

Consistent papers on inhomogeneous fully ionized and magnetized plasma interacting with an alternative electric field has obtained in his join papers with D.Zoler, published in Beiträge aus der Plasmaphysik, Wave-wave interaction in a CGL-plasma - in Doklady Bolgarskoj Akademij Nauk, the diffraction of fast magnetoacoustic waves by a plasma layer perturbed by the wave passing - in Magnitnaja Gidrodinamika, the diffraction of fast magnetoacoustic waves by a plasma layer of a periodically varying density published in Physica.

Moreover he has contributed with some results in general relativity, relativistic magneto-hydrodynamics as well as in rheology.
I. Burdujan has given an enormous number of invited lectures in the Romania and throughout the world, and he has helped to organize many mathematical conferences. He was organizer of five editions (2002-2006) of the Annual Symposium on Applied Mathematics in Biology and Biophysics, Iași, and of five editions of the International Conference on Applied and Industrial Mathematics.

Professor I. Burdujan is the President of Romanian Society of Industrial and Applied Mathematics (ROMAI, 2009 - at present), member of Romanian Society of Mathematical Sciences, Romanian Society of Biophysics, American Mathematical Society, is the Editor-in-Chief of ROMAI Mathematical Journal (2009 - at present) and of Proceedings of the Annual Symposium on Applied Mathematics in Biology and Biophysics (2002-2006) (under Scientific Papers of University of Agricultural Sciences and Veterinary Medicine, Iași).

Professor Ilie Burdujan was awarded the prize “Academician Constantin Sibirschi” (2014) for his contribution in studying the finite-dimensional HQDSs.

At the age of 70, full of vigor and optimism, the Professor Ilie Burdujan is a prominent personality and continues an active presence in the academic community of the Romania and Republic of Moldova. We wish him a good health, prosperity and new accomplishments in his prodigious scientific and didactic activities! La mulți ani!

Mitrofan M. Choban, Anca Veronica Ion, Mihail Popa

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