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COMBINATORIAL OPTIMIZATION ALGORITHMS FOR P_4 -SPARSE GRAPHS

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Abstract We give a characterization of P_4-sparse graphs using weak decomposition. We also give recognition algorithms for P_4-sparse graphs and we determine the combinatorial optimization numbers in efficient time.

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1. INTRODUCTION

The class of P_4-sparse graphs was introduced by Hoàng ([6]) as the class of graphs for which every set of five vertices induces at most one P_4 and Hoàng also gave a number of characterizations for these graphs. The class of P_4-sparse graphs generalizes both the P_4-free and the P_4-reducible graphs. Class of cographs (P_4-free) was introduced by Lerchs ([9]) and P_4-reducible graphs were introduced by Jamison and Olariu ([7]) as those in which no vertex belongs to more than one induced P_4. Both cographs and P_4-reducible graphs can be recognized in linear time ([2],[3],[7]). In ([8]), Jamison and Olariu gave a constructive characterization asserting that P_4-sparse graphs are exactly the graphs constructible from single-vertex graphs by three graph operations. This result leads to a linear time recognition algorithm for this class. The classes of P_4-sparse graphs, cographs and P_4-reducible graphs have applications in many areas of applied mathematics, computer science and engineering, mainly because of their good algorithmic and structural properties.

An equivalent class of P_4-sparse graphs ([12]) is (C_5, P, P_5, $\overline{P}$, *co-fork, fork, house*)-free.

2. PRELIMINARIES

Throughout this paper, $G = (V, E)$ is a connected, finite and undirected graph ([1]), without loops and multiple edges, having $V = V(G)$ as the vertex set and $E = E(G)$ as the set of edges. $\overline{G}$ is the complement of G. If $U \subseteq V$, by $G(U)$ we denote the subgraph of G induced by U. By $G - X$ we mean the subgraph $G(V - X)$, whenever

$X \subseteq V$, but we simply write $G - v$, when $X = \{v\}$. If $e = xy$ is an edge of a graph G, then x and y are adjacent, while x and e are incident, as are y and e. If $xy \in E$, we also use $x \sim y$, and $x \nsim y$ whenever x, y are not adjacent in G. If $A, B \subset V$ are disjoint and $ab \in E$ for every $a \in A$ and $b \in B$, we say that A, B are *totally adjacent* and we denote by $A \sim B$, while by $A \nsim B$ we mean that no edge of G joins some vertex of A to a vertex from B and, in this case, we say A and B are *totally non-adjacent*.

The *neighborhood* of the vertex $v \in V$ is the set $N_G(v) = \{u \in V : uv \in E\}$, while $N_G[v] = N_G(v) \cup \{v\}$; we denote $N(v)$ and $N[v]$, when G appears clearly from the context. The *degree* of v in G is $d_G(v) = |N_G(v)|$. The neighborhood of the vertex v in the complement of G will be denoted by $\overline{N}(v)$.

The neighborhood of $S \subset V$ is the set $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = S \cup N(S)$. A graph is complete if every pair of distinct vertices is adjacent.

By P_n, C_n, K_n we mean a chordless path on $n \geq 3$ vertices, a chordless cycle on $n \geq 3$ vertices, and a complete graph on $n \geq 1$ vertices, respectively.

The $fork$ (or *chair*) graph is the graph with vertices a, b, c, d, e and edges ab, bc, cd, be. The $co-fork$ graph is the graph with vertices a, b, c, d, e and edges ab, bc, cd, bd, ca, ae. The *bull* is the graph consisting of a triangle and two disjoint pendant edges. A *house* graph is isomorphic to $\overline{P_5}$. The P graph is the graph with vertices a, b, c, d, e and edges ab, bc, cd, da, ae.

A *dominating set* for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D. The *domination number* $\nu(G)$ is the number of vertices in a smallest dominating set for G. The *stability number* $\alpha(G)$ of a graph G is the cardinality of the largest stable set. Recall that a stable set of G is a subset of the vertices such that no two of them are connected by an edge. The *clique number* of a graph G is the number of vertices in a maximum clique of G, denoted $\omega(G)$.

Let $\mathcal{F}$ denote a family of graphs. A graph G is called $\mathcal{F}$*-free* if none of its subgraphs are in $\mathcal{F}$.

The *Zykov sum* of the graphs G_1, G_2 is the graph $G = G_1 + G_2$ having:

$$V(G) = V(G_1) \cup V(G_2),$$
$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

A *gem* graph is isomorphic to $K_1 + P_4$.

3. A NEW CHARACTERIZATION OF P_4-SPARSE GRAPHS USING THE WEAK DECOMPOSITION

The notion of weak decomposition (a partition of the set of vertices in three classes A, B, C such that A induces a connected graph and C is totally adjacent to B and totally non-adjacent to A) and the study of its properties allow us to obtain several important results such as: characterization of cographs, $K_{1,3}$-free graphs, $\{P_4, C_4\}$-free and paw-free graphs.

Definition 3.1. *([10], [11]) A set $A \subset V(G)$ is called a weak set of the graph G if $N_G(A) \neq V(G) - A$ and $G(A)$ is connected. If A is a weak set, maximal with respect to set inclusion, then $G(A)$ is called a weak component. For simplicity, the weak component $G(A)$ will be denoted with A.*

Definition 3.2. *([10], [11]) Let $G = (V, E)$ be a connected and non-complete graph. If A is a weak set, then the partition $\{A, N(A), V - A \cup N(A)\}$ is called a weak decomposition of G with respect to A.*

Below we recall a characterization of the weak decomposition of a graph.
The name of "*weak component*" is justified by the following result.

Theorem 3.1. *([10], [11]) Every connected and non-complete graph $G = (V, E)$ admits a weak component A such that $G(V - A) = G(N(A)) + G(\overline{N}(A))$.*

Theorem 3.2. *([4], [5]) Let $G = (V, E)$ be a connected and non-complete graph and $A \subset V$. Then A is a weak component of G if and only if $G(A)$ is connected and $N(A) \sim \overline{N}(A)$.*

The next result, that follows from Theorem 3.2, ensures the existence of a weak decomposition in a connected and non-complete graph.

Corollary 3.1. *If $G = (V, E)$ is a connected and non-complete graph, then V admits a weak decomposition (A, B, C), such that $G(A)$ is a weak component and $G(V - A) = G(B) + G(C)$.*

Theorem 3.2 provides an $O(n + m)$ algorithm for building a weak decomposition for a non-complete and connected graph.

Algorithm for the weak decomposition of a graph ([10])
Input: A connected graph with at least two nonadjacent vertices, $G = (V, E)$.
Output: A partition $V = (A, N, R)$ such that $G(A)$ is connected, $N = N(A)$, $A \nsim R = \overline{N}(A)$.

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begin
    A := any set of vertices such that A ∪ N(A) ≠ V
    N := N(A)
    R := V − A ∪ N(A)
    while (∃n ∈ N, ∃r ∈ R such that nr ∉ E ) do
        begin
            A := A ∪ {n}
            N := (N − {n}) ∪ (N(n) ∩ R)
            R := R − (N(n) ∩ R)
        end
end
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A new characterization of P_4-sparse graphs, using weak decomposition, is given below.

Theorem 3.3. *Let $G = (V, E)$ be a connected and non-complete graph, $\{gem, bull\}$-free. Let (A, N, R) be a weak decomposition with $G(A)$ as the weak component. $G = (V, E)$ is P_4-sparse if and only if:*
i) $A \sim N \sim R$;
ii) $G(V - R)$, $G(V - A)$ are P_4-sparse.

Proof. I) Let conditions i) and ii) be fulfilled. We show that G is P_4-sparse. From ii) it follows that $G(A)$, $G(N)$, $G(R)$, $G(A \cup N)$ and $G(N \cup R)$ are P_4-sparse graphs. Because $G(A \cup R)$ is not connected it follows that $G(A \cup R)$ is P_4-sparse.

However, we suppose that there is $X \subseteq V$ with $|X| = 5$ so that either $G(X) = C_5$ or $G(X) = P$ or $G(X) = P_5$ or $G(X) = fork$ or $G(X) = \overline{P}$ or $G(X) = \overline{P_5}$ or $G(X) = co - fork$.

If $|X \cap R| = 1$ then $A \sim N$ does not hold.

If $|X \cap R|$=2 ($r_1, r_2 \in R$) then either ($|X \cap N|$=1 and $|X \cap A|$=2) or ($|X \cap N|$=2 and $|X \cap A|$=1). For $|X \cap N|$=1 and $|X \cap A|$=2, $\exists n \in X \cap N$ and $\exists a_1, a_2 \in A \cap X$ with $r_1n, r_2n, a_1n, a_2n \in E$. For $|X \cap N|$=2 and $|X \cap A|$=1, $\exists n_1, n_2 \in N \cap X$, $\exists a \in A \cap X$ with n_1r_1, n_1r_2, n_2r_1, $n_2r_2, n_1a, n_2a \in E$. The above statements are not possible for $G(X) = C_5$, $G(X) = P_5$, $G(X) = \overline{P_5}$, while the above statements are possible for $G(X) = P$, $G(X) = fork$, $G(X) = \overline{P}$, $G(X) = co - fork$, and $A \sim N$ does not hold.

If $|X \cap R|$=3 then, because $R \sim N$, $\exists r_1, r_2, r_3 \in X \cap R$ and either $\exists n \in X \cap N$ with $r_1n, r_2n, r_3n \in E$ (for $G(X) = P$, $G(X) = fork$, $G(X) = \overline{P}$ or $G(X) = \overline{P_5}$ or $G(X) = co - fork$, and $A \sim N$ does not hold) or $\nexists n \in X \cap N$ with $r_1n, r_2n, r_3n \in E$ (for $G(X) = C_5$, $G(X) = P_5$).

$|X \cap R| \in \{4, 5\}$ it is not possible, because $X \cap A \neq \emptyset$ and $X \cap N \neq \emptyset$.

Other situations do not exist.

II) Let G be P_4-sparse. We show that i) and ii) hold. From the propriety of heredity of P_4-sparse graphs, it follows ii). From the weak decomposition with $G(A)$ as weak component, $N \sim R$ holds. We show that $A \sim N$ holds.

We assume that $\exists n \in N$, $\exists b \in A$, such that $nb \notin E$. From $N = N_G(A)$, for $n \in N$, $\exists a \in A$ such that $na \in E$. Because G(A) is connected, for $a, b \in A$, $\exists P_{ab} \subseteq G(A)$. There exists a first vertex, from a to b, $w \in V(P_{ab})$, with $nw \notin E$. Let v be on P_{ab}, the vertex before it w. Then $nv \in E$, $wv \in E$. Consider an arbitrary $r \in R$.

*Case*1. We suppose that $a = v$ and $b = w$. Because $|V(G)| \geq 5$ there is at least a vertex either in R ($r' \in R$) (and then there exists either $\overline{P}$ (if $rr' \in E$) or $fork$ (if $rr' \notin E$) as induced subgraph in G) or in N ($n' \in N$) (and then there exists either (P_5 or C_5 or P or $\overline{P}$, if $nn' \notin E$) or ($\overline{P}$ or $house$ or $co - fork$ or gem, if $nn' \in E$) as induced subgraph in G) or in A ($a' \in A$) (and then there exists either (P_5 or $fork$ or $\overline{P}$) (if $a'n \notin E$) or ($fork$ or P or $bull$ or $co - fork$) (if $a'n \in E$)).

*Case*2. We suppose that $a = v$ and $b \neq w$. Because $G(A)$ is connected, $\exists P_{ab} \subseteq G(A)$, then there is P_5 as induced subgraph in G.

*Case*3. We suppose that $a \neq v$ and $b \neq w$. Then there is (either P_5 or $fork$ or $\overline{P}$ or $bull$) as induced subgraph in G.

*Case*4. We suppose that $a \neq v$ and $b = w$. Then there is (either *bull* or *fork*) as induced subgraph in G. Other situations do not exist. ∎

Algorithm 1
The recognition algorithm for P_4-sparse, $\{gem, bull\}$-free graphs
Input: A connected, non-complete graph $G = (V, E)$.
Output: An answer to the question: Is G P_4-sparse ?
begin
1. $L_G \leftarrow \{G\}$
2. *while* $L_G \neq \emptyset$ *do*
3. extracts an element H from L_G
4. determine the weak decomposition (A, N, R) with $[A]_H$ weak component
5. *if* $(\exists a \in A, \exists n \in N$ such that $an \notin E)$ *then*
   G is not P_4-sparse *else*
6. introduce in L_G subgraphs [V-R], [V-A] incomplete and of at least order 4
7. Return: G is P_4-sparse
8. end
EndRecognition

The complexity of the algorithm 1
Because step 4 takes $O(n + m)$ time, and the other steps of the cycle *while* take less time, it results that the algorithm is executed in an overall time of $O(n(n + m))$.

Corollary 3.2. *Let $G = (V, E)$ be a connected and non-complete graph. Let (A, N, R) be a weak decomposition, with $G(A)$ as weak component. If $G = (V, E)$ is P_4-sparse, $\{gem, bull\}$-free then:*
$\nu(G) = min\{|N|, |A| + |R|\}$;
$\omega(G) = \omega(G(N)) + max\{\omega(G(A)), \omega(G(R))\}$;
$\alpha(G) = max\{\alpha(G(A)) + \alpha(G(R)), \alpha(G(N))\}$.

Algorithm 2
The determination of domination number for P_4-sparse, $\{gem, bull\}$-free graphs
Input: A connected, non-complete graph $G = (V, E)$.
Output: The determination of $\nu(G)$
begin
1. $L_G \leftarrow \{G\}$
2. extract an element H from L_G
3. determine the weak decomposition (A, N, R) with $[A]_H$ weak component
4. *if* $(|N| < |A| + |R|)$ *then*
5. $\nu(G)$=$|N|$ *else*
6. $\nu(G)$=$|A| + |R|$
7. end
EndAlgorithm 2

The complexity of the algorithm 2

Because step 3 takes $O(n + m)$ time, it results that the algorithm is executed in an overall time of $O(n + m)$.

We notice that the determination of α and ω, takes $O(n(n + m))$ time.

4. CONCLUSIONS AND FUTURE WORK

We give a characterization of P_4-sparse, $\{gem, bull\}$-free graphs using weak decomposition. We also give recognition algorithms for P_4-sparse, $\{gem, bull\}$-free graphs. Finally, we determine the combinatorial optimization numbers of these graphs in efficient time. Our future work concerns some applications of P_4-sparse graphs including the medicine.

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