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FUNCTION CYCLICAL CONTRACTIONS IN METRIC SPACES

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Abstract A dimension type variant is established for the fixed point result – involving function cyclical contractions over metric spaces – due to Kirk et al [Fixed Point Th., 4 (2003), 79-89].

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1. INTRODUCTION

Let X be a nonempty set. Call the subset $Y \in 2^X$, *almost singleton* (in short: *asingleton*), provided $[y_1, y_2 \in Y$ implies $y_1 = y_2]$; and *singleton* if, in addition, $Y \in (2)^X$; note that in this case $Y = \{y\}$, for some $y \in X$. [As usually, 2^X denotes the class of all subsets in X; and $(2)^X$ stands for the subclass of all nonempty members in 2^X]. Let also $d : X \times X \to R_+ := [0, \infty[$ be a *metric* over X; the couple (X, d) will be then referred to as a *metric space*. Finally, let $T \in \mathcal{F}(X)$ be a selfmap of X. [Here, for each couple (U, V) of nonempty sets, $\mathcal{F}(U, V)$ stands for the class of all functions from U to V; if $U = V$, one writes $\mathcal{F}(U, U)$ as $\mathcal{F}(U)$]. Denote $\mathrm{Fix}(T) = \{x \in X; x = Tx\}$ (the class of all *fixed points* of T in X). In the following, existence of such points is to be determined, under partition type regularity conditions. Given the natural number $p \geq 1$, let us say that the family $\mathcal{A} = \{A_0, ..., A_{p-1}\} \subseteq (2)^X$ is a *closed semi-partition* of X, when

(csp-1) $\mathcal{A}$ is a *semi-partition*: $X = \cup\mathcal{A} = A_0 \cup ... \cup A_{p-1}$

(csp-2) A_i is d-closed (in the usual sense), for all $i \in \{0, ..., p-1\}$.

Suppose that we fixed such an object $\mathcal{A}$ in the sequel. Clearly, $B := \cap\mathcal{A}$ is d-closed too; but the alternative $B = \emptyset$ cannot be avoided. Assume in the following that

(a01) $\mathcal{A}$ is T-*cyclically invariant*:
$T(A_i) \subseteq A_{i+1}, \forall i \in \{0, ..., p-1\}$ [where, $A_p = A_0$].

Note that, in such a case, $\mathrm{Fix}(T) \subseteq B$ and the restriction $S := T|_B$ is a selfmap of B. Concerning the fixed points of such maps, an appropriate answer was provided in the 2003 paper due to Kirk et al [9]; to state it, some conventions are needed. Given $\varphi \in \mathcal{F}(R_+)$, let us say that $T \in \mathcal{F}(X)$ is $(\mathcal{A}, \varphi)$-*cyclically contractive*, provided

(a02) $d(Tx, Ty) \leq \varphi(d(x, y))$,
for all $x \in A_i$, $y \in A_{i+1}$, and all $i \in \{0, ..., p-1\}$.

The regularity conditions upon $\varphi \in \mathcal{F}(R_+)$ to be considered may be described as below. Call $\varphi \in \mathcal{F}(R_+)$, *regressive* provided [$\varphi(0) = 0$ and $\varphi(t) < t$, $\forall t > 0$]; the class of all these will be denoted as $\mathcal{F}(re)(R_+)$. Further, let us say that $\varphi \in \mathcal{F}(R_+)$ is *right-usc* on $R_+^0 :=]0, \infty[$, provided

(a03) $\limsup_{t \to s+} \varphi(t) \leq \varphi(s)$, $\forall s \in R_+^0$.

We are now in position to state the quoted result (referred to as: Theorem KSV).

Theorem 1.1. *Assume that $\mathcal{A}$ is T-cyclically invariant and T is $(\mathcal{A}, \varphi)$-cyclically contractive, for some right-usc (on R_+^0) function $\varphi \in \mathcal{F}(re)(R_+)$. In addition, let X be d-complete. Then, $B := \cap\mathcal{A}$ is nonempty d-closed; and the restriction $S := T|_B$ has a unique fixed point in B.*

In particular, when $\mathcal{A} = \{X\}$, Theorem KSV is directly comparable with the related fixed point statements in Boyd and Wong [3] or Matkowski [11]; which, in turn, extend – from a functional perspective– the well known Banach's contraction principle [1]. Consequently, Theorem KSV was considered as interesting enough to be generalized in various directions; see Karapinar and Sadarangani [6], Nashine and Kadelburg [12], Păcurar and Rus [13], Chen [4], and the references therein. It is the main aim of this exposition to get (in Section 3) a dimension type functional extension of Theorem KSV, based on contractive conditions like in Turinici [16]. Then, in Section 4, some particular versions of our main result are given; which, in particular, cover in a direct way Theorem KSV above. Finally, Section 2 has a preliminary character. Further aspects will be discussed elsewhere.

2. PRELIMINARIES

Let X be a nonempty set. By a *sequence* in X, we mean any mapping $x : N \to X$, where $N = \{0, 1, ...\}$ is the set of *natural numbers*. For simplicity reasons, it will be useful to denote it as $(x(n); n \geq 0)$, or $(x_n; n \geq 0)$; moreover, when no confusion can arise, we further simplify this notation as $(x(n))$ or (x_n), respectively. Also, any sequence $(y_n := x_{i(n)}; n \geq 0)$ with $i(n) \to \infty$ as $n \to \infty$ will be referred to as a *subsequence* of $(x_n; n \geq 0)$. Given the (nonempty) subset Y of X, let us say that a sequence $(x_n; n \geq 0)$ in X is *h-nearly* in Y (where $h \geq 0$ is some rank), provided

(b01) $x_n \in Y$, for all $n \geq h$.

When $h = 0$, this convention means: $(x_n; n \geq 0)$ is in Y; and, if $h \geq 0$ is generic, the resulting property will be referred to as: $(x_n; n \geq 0)$ is *nearly* in Y.

(A) Further, let $d : X \times X \to R_+$ be a metric over X; remember that the couple (X, d) is then referred to as a *metric space*. We introduce a d-convergence and a d-Cauchy structure on X as follows. Given the sequence (x_n) in X and the point $x \in X$, we say that (x_n), *d-converges* to x (written as: $x_n \xrightarrow{d} x$), provided $d(x_n, x) \to 0$ as $n \to \infty$; i.e.,

$$\forall \varepsilon > 0, \exists i = i(\varepsilon)\colon\ n \geq i \Longrightarrow d(x_n, x) < \varepsilon;$$

or, equivalently:

$$\forall \varepsilon > 0, \exists i = i(\varepsilon)\colon\ n \geq i \Longrightarrow d(x_n, x) \leq \varepsilon;$$

The set of all such points $x \in X$ will be denoted $\lim_n(x_n)$; it is an asingleton, by the properties of $d(.,.)$. If $\lim_n(x_n)$ is nonempty (hence, a singleton), then (x_n) is called *d-convergent*; in this case, $\{z\} = \lim_n(x_n)$ will be written as $z = \lim_n(x_n)$.

By this very definition, we have the *hereditary* and *reflexive* properties:

(conv-1) $x_n \xrightarrow{d} x$ implies $y_n \xrightarrow{d} x$,
for each subsequence $(y_n = x_{i(n)}; n \geq 0)$ of $(x_n; n \geq 0)$

(conv-2) $[x_n = u,\ \forall n \geq 0]$ implies $x_n \xrightarrow{d} u$.

As a consequence, the convergence structure $(\xrightarrow{d})$ has all regularity properties required in Kasahara [7].

Further, call the sequence (x_n), *d-Cauchy* when $d(x_m, x_n) \to 0$ as $m, n \to \infty$ with $m < n$; i.e.,

$$\forall \varepsilon > 0, \exists j = j(\varepsilon)\colon\ j \leq m < n \Longrightarrow d(x_m, x_n) < \varepsilon;$$

or, equivalently,

$$\forall \varepsilon > 0, \exists j = j(\varepsilon)\colon\ j < m < n \Longrightarrow d(x_m, x_n) \leq \varepsilon.$$

As before, we have the hereditary and reflexive properties

(Cauchy-1) (x_n) is d-Cauchy implies (y_n) is d-Cauchy,
for each subsequence $(y_n = x_{i(n)}; n \geq 0)$ of $(x_n; n \geq 0)$

(Cauchy-2) $[x_n = u,\ \forall n \geq 0]$ implies (x_n) is d-Cauchy.

Finally, call $(x_n; n \geq 0)$, *d-semi-Cauchy*, when $d(x_n, x_{n+1}) \to 0$; and *strong d-semi-Cauchy*, provided $(d(x_n, x_{n+i}) \to 0$, as $n \to \infty$, for each $i \geq 1)$. Clearly, the metrical properties of $d(.,.)$ give

(∀ sequence): d-Cauchy $\Longrightarrow$ strong d-semi-Cauchy $\Longleftrightarrow$ d-semi-Cauchy.

In addition, each d-convergent sequence is d-Cauchy, as it can be directly seen.

(B) In the following, two auxiliary statements concerning these data are given. The former of these involves the Lipschitz property of $d(.,.)$.

Proposition 2.1. *The mapping $(x, y) \mapsto d(x, y)$ is d-Lipschitz, in the sense*

$$|d(x, y) - d(u, v)| \leq d(x, u) + d(y, v), \ \forall (x, y), (u, v) \in X \times X. \tag{1}$$

As a consequence, this map is d-continuous; i.e.,

$$x_n \xrightarrow{d} x, \ y_n \xrightarrow{d} y \text{ imply } d(x_n, y_n) \to d(x, y). \tag{2}$$

The proof is immediate, by the properties of $d(.,.)$; we do not give details.

Our next statement is devoted to the d-semi-Cauchy sequences in X which are not d-Cauchy. Let us say that the subset Θ of R_+^0 is $(>)$-*cofinal* in R_+^0, when: for each $\varepsilon > 0$, there exists $\theta \in \Theta$ with $\varepsilon > \theta$. Further, given the sequence $(r_n; n \geq 0)$ in R_+ and the point $r \in R_+$, let us write

$r_n \to r+$, if $r_n \to r$ and $r_n > r$, for all $n \geq 0$ large enough.

Proposition 2.2. *Suppose that $(x_n; n \geq 0)$ is a sequence in X with*

(b02) $(x_n; n \geq 0)$ is d-semi-Cauchy but not d-Cauchy.

Further, let Θ be a $(>)$-cofinal part of R_+^0, and $p \geq 1$ be a natural number. There exist then a number $\theta \in \Theta$, and a couple of natural number sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, with

$$\begin{aligned} &(\forall j \geq 0)\text{: } j < m(j) < n(j), \text{ and} \\ &d(x_{m(j)}, x_{n(j)}) > \theta, \ d(x_{m(j)}, x_{n(j)-1}) \leq \theta \end{aligned} \tag{3}$$

$$\begin{aligned} &(\alpha_j := d(x_{m(j)}, x_{n(j)}); j \geq 0) \text{ is a sequence in } R_+^0 \\ &\text{with } \alpha_j \to \theta+ \text{ as } j \to \infty \end{aligned} \tag{4}$$

$$\begin{aligned} &\text{for each couple of maps } H, K \in \mathcal{F}(N, \{0, ..., p\}), \\ &(\beta_j := d(x_{m(j)+H(j)}, x_{n(j)+K(j)}); j \geq 0) \text{ is a} \\ &\text{nearly in } R_+^0 \text{ sequence in } R_+ \text{ with } \beta_j \to \theta \text{ as } j \to \infty. \end{aligned} \tag{5}$$

Proof. By definition, the d-Cauchy property of our sequence writes:

$\forall \varepsilon \in R_+^0, \exists k = k(\varepsilon)$: $k < m < n \Longrightarrow d(x_m, x_n) \leq \varepsilon$.

As Θ is a $(>)$-cofinal part in R_+^0, this property may be also written as

$\forall \theta \in \Theta, \exists k = k(\theta), \forall (m, n)$: $k < m < n \Longrightarrow d(x_m, x_n) \leq \theta$.

The negation of this property means: there exists $\theta \in \Theta$ such that

$$(\forall j \geq 0) : \ A(j) := \{(m, n) \in N \times N; j < m < n, d(x_m, x_n) > \theta\} \neq \emptyset.$$

Having this precise, denote, for each $j \geq 0$,

$$m(j) = \min \mathrm{Dom}(A(j)),\ n(j) = \min A(j)(m(j)).$$

The couple of rank-sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$ fulfills (3), as it can be directly seen. Moreover, by the d-semi-Cauchy condition,

$$C_n := d(x_n, x_{n+1}) + ... + d(x_{n+p-1}, x_{n+p}) \to 0 \text{ as } n \to \infty;$$

hence, in particular (as $m(j) \to \infty$, $n(j) \to \infty$, when $j \to \infty$)

$$\lim_j C_{m(j)} = \lim_j C_{n(j)-1} = \lim_j C_{n(j)} = 0.$$

This yields (by the triangular inequality), for all $j \geq 0$:

$$\theta < d(x_{m(j)}, x_{n(j)}) \leq d(x_{m(j)}, x_{n(j)-1}) + C_{n(j)-1} \leq \theta + C_{n(j)-1};$$

so, passing to limit as $j \to \infty$ gives (4). Finally, by a previous auxiliary fact,

$$|\beta_j - \alpha_j| \leq C_{m(j)} + C_{n(j)}, \text{ for all } j \geq 0.$$

Taking the limit as $j \to \infty$, one derives (5). The proof is thereby complete. ∎

In particular, when $\Theta = R_+^0$, this statement is, essentially, the one due to Khan et al [8]; but, the line of argument goes back to Boyd and Wong [3].

3. MAIN RESULT

Let X be a nonempty set, and $d : X \times X \to R_+$ be a metric over it; the couple (X, d) will be then referred to as a *metric space*. Further, given the selfmap T of X, remember that we denoted $\mathrm{Fix}(T) = \{x \in X; x = Tx\}$; each point of this set is referred to as *fixed* under T. Such elements are to be determined according to the context below, comparable with the one in Rus [14, Ch 2, Sect 2.2]:

pic-1) We say that T is a *Picard operator* (modulo d) if, for each $x \in X$, the iterative sequence $(T^n x; n \geq 0)$ is d-convergent; and a *globally Picard operator* (modulo d) if, in addition, $\mathrm{Fix}(T)$ is an asingleton

pic-2) We say that T is a *strong Picard operator* (modulo d) if, for each $x \in X$, $(T^n x; n \geq 0)$ is d-convergent with $\lim_n(T^n x) \in \mathrm{Fix}(T)$; and a *globally strong Picard operator* (modulo d) if, in addition, $\mathrm{Fix}(T)$ is an asingleton (hence, a singleton).

I) As precise, the basic regularity conditions to be imposed here involve closed semi-partitions. Given the natural number $p \geq 1$, let $\mathcal{A} = \{A_0, ..., A_{p-1}\} \subseteq (2)^X$ be such an object; i.e. (according to a previous convention)

$$X = \cup\mathcal{A} := A_0 \cup ... \cup A_{p-1}, \text{ and } A_i \text{ is } d\text{-closed}, \forall i \in \{0, ..., p-1\}.$$

For technical reasons, it will be useful to write this finite family $\mathcal{A}$ as a sequence $(A_i; i \geq 0)$ in $(2)^X$, as:

$(i, j \in N)$: $A_i = A_j$ iff $i \equiv j$ (modulo p); whence:
$(\forall k \geq 0)$: $A_k = A_i$, whenever $k = np + i$, with $n \geq 0$, $i \in \{0, ..., p-1\}$.

Clearly, $B := \cap\mathcal{A}$ is d-closed too; but the alternative $B = \emptyset$ cannot be avoided. The following condition is to be considered (see above)

(c01) $\mathcal{A}$ is *T-cyclically invariant*: $T(A_i) \subseteq A_{i+1}$, $\forall i \geq 0$.

Note that, in this case, $\mathrm{Fix}(T) \subseteq B$; moreover, $S := T|_B$ is a selfmap of B.

II) The next condition upon our data is of (dimensional) function contractive type. Given $F \in \mathcal{F}(R_+^3, R_+)$, call T, *$(\mathcal{A}, F)$-cyclically contractive*, when

(c02) $d(Tx, Ty) \leq F(d(x, y), d(x, Tx), d(y, Ty))$,
for all $x \in A_i$, $y \in A_{i+1}$ with $x \neq y$, and for all $i \geq 0$.

The properties of F to be used are described as follows. Let us say that $F \in \mathcal{F}(R_+^3, R_+)$ is *admissible*, when it satisfies the global conditions

(adm-1) $w > 0 \Longrightarrow F(w, 0, 0) < w$

(adm-2) $u, v > 0$, $v \leq F(u, u, v) \Longrightarrow v < u$

and the local conditions: $\forall t > 0$, $\exists a(t), b(t) \in]0, t[$, such that:

(adm-3) $t < v < u < t + a(t) \Longrightarrow F(u, u, v) \leq b(t)$

(adm-4) $0 < v < u < a(t)$, $|w - t| < a(t) \Longrightarrow F(w, u, v) \leq b(t)$

(adm-5) $0 < u, w < a(t) \Longrightarrow F(w, u, t) \leq b(t)$;

the class of all these functions will be denoted as $\mathcal{F}(adm)(R_+^3, R_+)$. [Note that, in the local conditions above, one may arrange for $a(t) < b(t)$, $\forall t > 0$; we do not give further details].

The main result of this exposition is

Theorem 3.1. *Suppose that $\mathcal{A}$ is T-cyclically invariant and T is $(\mathcal{A}, F)$-cyclically contractive, where $F \in \mathcal{F}(adm)(R_+^3, R_+)$. In addition, let X be d-complete. Then,*
j) *T is a globally strong Picard operator (modulo d)*
jj) *B is nonempty closed and $S := T|_B$ is globally strong Picard (modulo d).*

Proof. Let us establish the uniqueness (in B) of the fixed point of T. Assume that $z_1, z_2 \in B$ are such that $z_1 = Tz_1$, $z_2 = Tz_2$, and $z_1 \neq z_2$; hence $\delta := d(z_1, z_2) > 0$. By the contractive condition and (adm-1), we have

$$\delta = d(z_1, z_2) = d(Tz_1, Tz_2) \leq F(\delta, 0, 0) < \delta; \tag{1}$$

a contradiction. Therefore, $z_1 = z_2$, which establishes our claim. Now we prove the Picard property. Take $x = x_0 \in X$; note that, by definition,

$$x_0 \in A_i, \text{ for some } i = i(x_0) \in \{0, ..., p-1\}.$$

Denote $(x_n = T^n x_0; n \geq 0)$; by the cyclical invariance property, we then have

$$x_n \in A_{i+n}, \text{ for all } n \geq 0. \tag{2}$$

If $x_n = x_{n+1}$ for some $n \geq 0$, we are done; so, without loss, assume

(c03) $x_n \neq x_{n+1}$ (hence, $\rho_n := d(x_n, x_{n+1}) > 0$), for all $n \geq 0$.

Part 1. For each $n \geq 0$, the contractive condition applies to the couple (x_n, x_{n+1}); and yields

$$\rho_{n+1} \leq F(\rho_n, \rho_n, \rho_{n+1}), \ \forall n \geq 0. \tag{3}$$

So, if one takes (adm-2) into account,

$$\rho_{n+1} < \rho_n, \text{ for all } n \geq 0; \tag{4}$$

i.e.: the sequence $(\rho_n; n \geq 0)$ is strictly descending in R_+. As a consequence, $\rho := \lim_n \rho_n$ exists in R_+; with, in addition: $\rho < \rho_n$, $\forall n$. Suppose that $\rho > 0$; and let $a(\rho), b(\rho) \in]0, \rho[$ be given by (adm-3). From this convergence property, there exists a rank $n(\rho) \geq 0$, such that

$$\rho < \rho_{n+1} < \rho_n < \rho + a(\rho), \text{ for all } n \geq n(\rho). \tag{5}$$

This, along with (adm-3), gives

$$F(\rho_n, \rho_n, \rho_{n+1}) \leq b(\rho), \ \forall n \geq n(\rho).$$

Combining with (3), we then get

$$(\rho <)\rho_{n+1} \leq b(\rho) < \rho, \ \ \forall n \geq n(\rho);$$

a contradiction. This tells us that $\rho = 0$; i.e.

$$\rho_n := d(x_n, x_{n+1}) \to 0, \text{ as } n \to \infty. \tag{6}$$

Part 2. Suppose that

(c04) there exist $i, j \in N$ such that $i < j$, $x_i = x_j$.

Denoting $q = j - i$, we thus have $q > 0$ and $x_i = x_{i+q}$; so that

$$x_i = x_{i+nq}, x_{i+1} = x_{i+nq+1}, \text{ for all } n \geq 0.$$

By the introduced notations,

$$\rho_i = \rho_{i+nq}, \text{ for all } n \geq 0.$$

This, along with $\rho_{i+np} \to 0$ as $n \to \infty$, yields $\rho_i = 0$; contradiction. Hence, our working hypothesis cannot hold; wherefrom

$$\text{for all } i, j \in N\text{: } i \neq j \text{ implies } x_i \neq x_j; \tag{7}$$

or, in other words: the mapping $n \mapsto x_n$ is injective.

Part 3. We now show that $(x_n; n \geq 0)$ is d-Cauchy. Suppose that this is not true. By a preceding auxiliary fact (with $\Theta := R^0_+$), there exist a number $\theta \in R^0_+$, and a couple of rank-sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, with

$$\begin{aligned}&(\forall j \geq 0)\colon\ j < m(j) < n(j), \text{ and}\\ &d(x_{m(j)}, x_{n(j)}) > \theta,\ d(x_{m(j)}, x_{n(j)-1}) \leq \theta\end{aligned} \tag{8}$$

$$\begin{aligned}&(\alpha_j := d(x_{m(j)}, x_{n(j)}); j \geq 0) \text{ is a sequence in } R^0_+\\ &\text{with } \alpha_j \to \theta+ \text{ as } j \to \infty\end{aligned} \tag{9}$$

$$\begin{aligned}&\text{for each couple of maps } H, K \in \mathcal{F}(N, \{0, ..., p\}),\\ &(\beta_j := d(x_{m(j)+H(j)}, x_{n(j)+K(j)}); j \geq 0) \text{ is a}\\ &\text{nearly in } R^0_+ \text{ sequence in } R_+ \text{ with } \beta_j \to \theta \text{ as } j \to \infty.\end{aligned} \tag{10}$$

Let $L : N \to \{0, ..., p-1\}$ be the mapping introduced as

(c05) $q(j) := n(j) + L(j) \equiv m(j) + 1$ (modulo p), $j \geq 0$.

Note that $L(.)$ is uniquely determined in this way; we do not give details. Moreover, by a previous relation,

$$(\forall j \geq 0):\ x_{m(j)} \in A_{i+m(j)},\ x_{q(j)} \in A_{i+m(j)+1}.$$

This, along with the injective property of the mapping $n \mapsto x_n$ (see above), tells us that the contractive condition applies to $(x_{m(j)}, x_{q(j)})$, for each $j \geq 0$; and gives (for the same ranks)

$$d(x_{m(j)+1}, x_{q(j)+1}) \leq F(d(x_{m(j)}, x_{q(j)}), \rho_{m(j)}, \rho_{q(j)}). \tag{11}$$

From (10), we must have

$$\lim_j d(x_{m(j)+1}, x_{q(j)+1}) = \lim_j d(x_{m(j)}, x_{q(j)}) = \theta;$$

just take $(H(j) = 1, K(j) = L(j) + 1; j \geq 0)$ in the former case, and $(H(j) = 0, K(j) = L(j); j \geq 0)$ in the latter case. Let $a(\theta), b(\theta) \in]0, \theta[$ be the constants assured by (adm-4). From (6) and this limit relation, there must be some rank $j(\theta) \geq 0$ such that, for all $j \geq j(\theta)$:

$$0 < \rho_{q(j)} < \rho_{m(j)} < a(\theta),\ |d(x_{m(j)}, x_{q(j)}) - \theta| \leq a(\theta).$$

From the conclusion of (adm-4), we therefore get

$$F(d(x_{m(j)}, x_{q(j)}), \rho_{m(j)}, \rho_{q(j)}) \leq b(\theta), \text{ for all } j \geq j(\theta);$$

so that, combining with (11), one gets an evaluation like

$$d(x_{m(j)+1}, x_{q(j)+1}) \leq b(\theta), \text{ for all } j \geq j(\theta).$$

Passing to limit as $j \to \infty$, yields (see above) $\theta \leq b(\theta) < \theta$; a contradiction. Hence, the working assumption cannot be true; and our claim follows.

Part 4. As X is d-complete, there exists $z \in X$ such that

$$x_n \xrightarrow{d} z; \text{ i.e., } \lambda_n := d(x_n, z) \to 0 \text{ as } n \to \infty. \tag{12}$$

Note that, as $\mathcal{A}$ is T-cyclically invariant, we have $z \in B$; hence, in particular, B is nonempty closed. Two assumptions are open before us :

i) For each n, there exists $m > n$ with $x_m = z$. There exists then a strictly ascending rank sequence $(i(n); n \geq 0)$ (hence, $i(n) \to \infty$ as $n \to \infty$), such that $x_{i(n)} = z$ (hence, $x_{i(n)+1} = Tz$), $\forall n$. Letting n tends to infinity and using the fact that $(y_n := x_{i(n)+1}; n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$, we get $z = Tz$; i.e., z is a fixed point of T.

ii) There exists some rank $h \geq 0$, such that

(c06) $n \geq h \Longrightarrow x_n \neq z$; whence, $\lambda_n = d(x_n, z) > 0$.

We show that the alternative $\sigma := d(z, Tz) > 0$ gives a contradiction. In fact, by the previous relation, the contractive condition is applicable to (x_n, z), for each $n \geq h$; and gives

$$d(x_{n+1}, Tz) \leq F(\lambda_n, \rho_n, \sigma), \text{ for all } n \geq h.$$

Further, by the triangular property,

$$\sigma \leq \lambda_{n+1} + d(x_{n+1}, Tz), \ \forall n;$$

so, by simply combining these,

$$\sigma \leq \lambda_{n+1} + F(\lambda_n, \rho_n, \sigma), \text{ for all } n \geq h. \tag{13}$$

Let $a(\sigma), b(\sigma) \in]0, \sigma[$ be given by (adm-5). By (6) and the convergence property (12), there exists $k(\sigma) \geq h$ in such a way that

$$0 < \rho_n, \lambda_n < a(\sigma), \ \forall n \geq k(\sigma).$$

This, along with (adm-5), gives us

$$F(\lambda_n, \rho_n, \sigma) \leq b(\sigma), \ \forall n \geq k(\sigma);$$

wherefrom, combining with (13),

$$\sigma \leq \lambda_{n+1} + b(\sigma), \ \forall n \geq k(\sigma).$$

Passing to limit as $n \to \infty$ gives $\sigma \leq b(\sigma) < \sigma$; contradiction. So, the underlying working assumption cannot be accepted; whence, $z = Tz$. ∎

Note that, when $A_0 = ... = A_{p-1} = X$, Theorem 3.1 is just the main result in Turinici [16], proved via similar methods.

4. PARTICULAR ASPECTS

Let $\varphi \in \mathcal{F}(R_+)$ be a function; remember that it is *regressive*, if $\varphi(0) = 0$ and $[\varphi(t) < t, \forall t > 0]$; the class of all these functions will be denoted as $\mathcal{F}(re)(R_+)$. For any such function φ, put

(d01) $\Lambda\varphi(s) = \inf_{\varepsilon>0} \sup\{\varphi(t); s - \varepsilon < t < s + \varepsilon\}$, $s \in R_+^0$;

note that, by the regressive property,

$$\varphi(s) \le \Lambda\varphi(s) \le s, \quad \forall s \in R_+^0. \tag{14}$$

Now, call $\varphi \in \mathcal{F}(re)(R_+)$, *bilaterally BW-admissible*, provided

(d02) $\Lambda\varphi(s) < s$, for all $s \in R_+^0$.

(The convention is motivated by the developments in Boyd and Wong [3]; we do not give details). Denote in this case, for each $s > 0$,

(d03) $b(s) = (1/2)(\Lambda\varphi(s) + s)$; hence, $\varphi(s) \le \Lambda\varphi(s) < b(s) < s$.

By the very definition above, it follows that, for each $s > 0$, there must be some $a(s) \in]0, b(s)/2[$ in such a way that

$$s - a(s) < t < s + a(s) \Longrightarrow \varphi(t) < b(s). \tag{15}$$

Having these precise, two basic examples of admissible functions will be given. The former of them is contained in

Proposition 4.1. *Suppose that the (regressive) function $\varphi \in \mathcal{F}(re)(R_+)$ is bilaterally BW-admissible. Then, the function $F \in \mathcal{F}(R_+^3, R_+)$ given as*

(d04) $F(w, u, v) = \varphi(w)$, $u, v, w \in R_+$

is admissible (see above).

Proof. The global properties (adm-1) and (adm-2) are clear, by the regressiveness of φ. On the other hand, for each $t > 0$, let the quantities $a(t)$ and $b(t)$ be introduced as before. It is then clear that, in such a case, the local properties (adm-3)-(adm-5) hold too. ∎

As a direct consequence of this, the following version of Theorem 3.1 is available. Let (X, d) be a metric space; and $\mathcal{A} = \{A_0, ..., A_{p-1}\}$ be a closed semi-partition of X. Further, let $T \in \mathcal{F}(X)$ be a selfmap of X; call it *KSV-type* $(\mathcal{A}, \varphi)$*-cyclically contractive* (where $\varphi \in \mathcal{F}(R_+)$), provided

(d05) $d(Tx, Ty) \le \varphi(d(x, y))$,
for all $x \in A_i$, $y \in A_{i+1}$ with $x \ne y$, and for all $i \ge 0$.

(This convention may be viewed as a "strong" variant of the one introduced by Kirk et al [9]).

Theorem 4.1. *Suppose that, $\mathcal{A}$ is T-cyclically invariant and T is KSV-type $(\mathcal{A}, \varphi)$-cyclically contractive, where $\varphi \in \mathcal{F}(re)(R_+)$ is bilaterally BW-admissible. In addition, let X be d-complete. Then, conclusions of Theorem 3.1 are retainable.*

This result may be viewed as a corrected extended form of Theorem 1.1. It also extends the related statement in Karapinar and Sadarangani [6]. Further aspects may be found in Păcurar and Rus [13]; see also Turinici [17].

Let us now return to the general framework. Another example of admissible functions is given by

Proposition 4.2. *Suppose that the (regressive) function $\varphi \in \mathcal{F}(re)(R_+)$ is bilaterally BW-admissible. Then, the function $F \in \mathcal{F}(R_+^3, R_+)$ given as*

(d06) $F(w,u,v) = \varphi(\max\{u,v,w\})$, $u,v,w \in R_+$

is admissible (in the described sense).

The argument is similar with the one in our previous auxiliary statement; so, we omit the details.

As a direct consequence of this, the following counterpart of Theorem 4.1 may be stated. Let again (X,d) be a metric space; and $\mathcal{A} = \{A_0, ..., A_{p-1}\}$ be a closed semi-partition of X. Further, let $T \in \mathcal{F}(X)$ be a selfmap of X; call it *L-type* $(\mathcal{A}, \varphi)$*-cyclically contractive* (where $\varphi \in \mathcal{F}(R_+)$), provided

(d07) $d(Tx,Ty) \leq \varphi(\max\{d(x,y), d(x,Tx), d(y,Ty)\})$,
for all $x \in A_i$, $y \in A_{i+1}$ with $x \neq y$, and for all $i \geq 0$.

(This concept may be viewed as a "cyclic" variant of the one introduced by Leader [10]; we do not give details).

Theorem 4.2. *Suppose that, $\mathcal{A}$ is T-cyclically invariant and T is L-type $(\mathcal{A}, \varphi)$-cyclically contractive, where $\varphi \in \mathcal{F}(re)(R_+)$ is bilaterally BW-admissible. In addition, let X be d-complete. Then, conclusions of Theorem 3.1 are retainable.*

It is to be stressed that the obtained statements are independent. In addition, Theorem 4.2 is comparable with some results in Samet and Turinici [15]; but, as before, no direct relationship between these is available. Further aspects may be found in Berzig [2]; see also Choudhury et al [5].

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