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A HYPERBOLIC THREE-PHASE RELATIVISTIC FLOW MODEL

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Abstract A hyperbolic fluid model describing relativistic single-component three–phase flow is considered. Two different regimes are compared: one in which the pressure non–equilibrium effects are maintained and phase equilibrium is attained through a relaxation procedure, the other being a fully relaxed system in which equal pressures are initially imposed. The consistency of the relaxation procedure with the laws of thermodynamics is discussed. The wave velocities for both models are obtained and compared, and the influence of phase equilibrium on the propagation of weak discontinuity waves is studied: the highest hydrodynamical velocity of the relaxed system can never exceed than of the relaxation system. **Key words**: Three–phase flows, hyperbolic relativistic model, interfacial interaction terms, entropy inequality, characteristic velocities.

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1. INTRODUCTION

Compressible multi–material flows and multi-phase mixture arise in a very large variety of scientific and technological problems.

There are several approaches to multi–fluid flow processes in classical [1, 34, 28, 2, 12, 3, 4, 47, 35, 43, 55, 51, 48, 56, 13, 32, 33, 49, 40, 45, 46, 50, 41, 44, 42, 58, 39, 17, 26, 25, 57, 29] and relativistic [36, 6, 37, 15, 16, 31, 59, 10, 24, 9, 22, 20, 21, 14, 54, 8, 30, 18, 53, 52, 27, 23, 19, 11] framework, but none of these models has been universally accepted as a complete formulation for modelling multi–phase flow.

The mathematical modelling has certain inherent difficulties originating from the existence of interfaces between the phases.

These interfaces separate different pure media, as well as different mixtures of materials in which wave dynamics plays an important role. Such situations appear frequently in contexts like astrophysics, physics of explosives, nuclear physics, power engineering, among others.

The key point in the modelling is the description of mass, momentum and energy transfer across the interfaces. The different approaches can be casted into two classes: a) models considering the interface as a sharp discontinuity; b) models considering the interface as a diffusive zone, like contact discontinuities in gas dynamics.

Within the context of multi–phase flow theory, the determination of the thermodynamic flow variables at the interfaces is achieved, together with the derivation of phisically and mathematically consistent thermodynamic laws for the mixture.

Here, a new multi–phase diffusive approach, based on the concept of thermodynamically compatible hyperbolic system, is outlined for the evolution of a single–component three–phase flow in presence of pressure non–equilibrium effects, in which phase equilibrium is reached through a relaxation procedure.

Following the ideas developed by Hérard [26, 25] and Lund and Flåtten [39], the relaxation model for single–component three–phase relativistic flow is determined in pressure disequilibrium.

The limit of instantaneous pressure relaxation is then performed and discussed.

In this paper, the non equilibrium model consists in eleven equations exhibiting, among the variables, the pressures of the three phases and the associated relaxation term, but a single four–velocity. The equations reduces to nine in the limit of stiff pressure. Basic properties of these models are presented: entropy inequality and hyperbolicity.

Each fluid phase has its particle number density, r_k, its specific internal energy, ε_k, and its energy density, ρ_k [38], such that:

$$\rho_k = r_k(1 + \varepsilon_k)\,, \qquad k = 1, 2, 3\,. \tag{1}$$

The units are such that the velocity of light is unitary: $c = 1$.

The four–velocity, u^α, is the future pointing unitary four-vector such that

$$g_{\alpha\beta} u^\alpha u^\beta = 1\,, \tag{2}$$

where $g_{\alpha\beta}$ are the covariant components of Lorentz metric tensor with signature $+, -, -, -$.

2. THE BASIC RELATIVISTIC MODEL

The formulation of model under consideration in this paper consists of one balance law for the particle number density of each phase:

$$\nabla_\alpha (X_k r_k u^\alpha) = 0\,, \qquad k = 1, 2, 3\,, \tag{3}$$

as well as the energy–momentum conservation of the mixture

$$\nabla_\alpha T^{\alpha\beta} = 0\,, \tag{4}$$

where $T^{\alpha\beta}$ is energy–momentum tensor given by

$$T^{\alpha\beta} = r f u^\alpha u^\beta - p g^{\alpha\beta}\,. \tag{5}$$

Here, r is the particle number density of the mixture, p the total pressure, X_k the volume fraction of component k ($X_1 + X_2 + X_3 = 1$), f "relativistic" total specific

enthalpy

$$f = 1 + h = 1 + \varepsilon + \frac{p}{r} = \frac{\rho + p}{r}\,, \tag{6}$$

where $h = \varepsilon + p/r$ is the "classical" specific enthalpy of the mixture, ε the total specific internal energy and $\rho = r(1+\varepsilon)$ the total energy density.

The spatial projection of equation (5) and its projection along u^α are, respectively,

$$rfu^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta}\partial_\alpha p = 0\,, \tag{7}$$

$$u^\alpha \left(\partial_\alpha \varepsilon + p\partial_\alpha \frac{1}{r}\right) = 0\,, \tag{8}$$

where $\gamma^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ is the spatial projection tensor orthogonal to u^α.

However, we have to determine suitable expressions for the bulk quantities r, ε and f. First, we introduce mass fraction Y_k of phase k:

$$Y_k = \frac{X_k r_k}{r}\,, \qquad \text{with} \quad Y_1 + Y_2 + Y_3 = 1\,. \tag{9}$$

The particle number density r and the specific internal energy ε are defined as:

$$\begin{aligned} r &= X_1 r_1 + X_2 r_2 + X_3 r_3 = X_1 r_1 + X_2 r_2 + (1 - X_1 - X_2) r_3\,, \\ \varepsilon &= Y_1 \varepsilon_1 + Y_2 \varepsilon_2 + Y_3 \varepsilon_3 = Y_1 \varepsilon_1 + Y_2 \varepsilon_2 + (1 - Y_1 - Y_2)\varepsilon_3\,, \end{aligned} \tag{10}$$

with

$$\begin{aligned} X_1 &= X_1\,, \quad X_2 = X_2\,, \quad X_3 = 1 - X_1 - X_2 \\ Y_1 &= Y_1\,, \quad Y_2 = Y_2\,, \quad Y_3 = 1 - Y_1 - Y_2\,. \end{aligned} \tag{11}$$

From (11) and (12) and the expression of total energy density, we deduce

$$\rho = X_1 \rho_1 + X_2 \rho_2 + X_3 \rho_3\,. \tag{12}$$

Furthermore, we suppose that the following Dalton's law for pressure is valid

$$p = X_1 p_1 + X_2 p_2 + X_3 p_3\,, \tag{13}$$

where p_k $(k = 1, 2, 3)$ is the pressure of k-th phase, so we have that

$$\begin{aligned} f &= Y_1 f_1 + Y_2 f_2 + Y_3 f_3\,, \\ rf &= X_1 r_1 f_1 + X_2 r_2 f_2 + X_3 r_3 f_3\,. \end{aligned} \tag{14}$$

Taking into account equations (4) and (10), we obtain the balance law for total particle number density:

$$\nabla_\alpha (r u^\alpha) = 0\,. \tag{15}$$

By virtue of (4), (8) and (15), the evolution equation for mass fraction Y_k can be written as:

$$u^\alpha \partial_\alpha Y_k = 0\,, \qquad k = 1, 2, 3\,. \tag{16}$$

Using (16) and (13), energy-momentum tensor (6) can be written as:

$$T^{\alpha\beta} = \sum_{k=1}^{3} X_k (r_k f_k u^\alpha u^\beta - p_k g^{\alpha\beta}) \,. \tag{17}$$

Therefore, the following energy–momentum evolution equation is valid for each phase k:

$$\nabla_\alpha (X_k r_k f_k u^\alpha u^\beta - X_k p_k g^{\alpha\beta}) = F_k^\beta \,, \quad k = 1, 2, 3 \,, \tag{18}$$

with

$$F_1^\beta + F_2^\beta + F_3^\beta = 0 \,, \tag{19}$$

where F_k^β represent the loss and source terms in the separate balance.

We assume that the interfacial transfer terms F_k^β, also called "nozzling terms" [7], are defined [25, 7] as:

$$\begin{aligned} F_{1\alpha} &= p_{12}\partial_\alpha X_2 + p_{13}\partial_\alpha X_3 \,, \\ F_{2\alpha} &= p_{21}\partial_\alpha X_1 + p_{23}\partial_\alpha X_3 \,, \\ F_{3\alpha} &= p_{31}\partial_\alpha X_1 + p_{32}\partial_\alpha X_2 \,, \end{aligned} \tag{20}$$

where the 6 unknown interfacial pressure p_{ik} $(i \neq k, i, k = 1, 2, 3)$ must be such as to verify (19) and also be consistent with the laws of thermodynamics.

In order to ensure that (19) is valid, the first constrains arising on the 6 pressures are

$$p_{21} + p_{31} = p_{12} + p_{32} = p_{23} + p_{13} \,. \tag{21}$$

Thus, there exists three independent unknown interfacial pressure, owing the last constrains.

After that, the projection along u^α (energy evolution equation) and the spatial projection of equation (18) for k-th phase, $k = 1, 2, 3$, are, respectively,

$$X_1 r_1 \left(D\varepsilon_1 + D\frac{1}{r_1} \right) - (p_1 - p_{13}) DX_1 - (p_{12} - p_{13}) DX_2 = 0 \,, \tag{22}$$

$$X_2 r_2 \left(D\varepsilon_2 + D\frac{1}{r_2} \right) - (p_{21} - p_{23}) DX_1 - (p_2 - p_{23}) DX_2 = 0 \,, \tag{23}$$

$$X_3 r_3 \left(D\varepsilon_3 + D\frac{1}{r_3} \right) - (p_{31} - p_3) DX_1 - (p_{32} - p_3) DX_2 = 0 \,, \tag{24}$$

$$\begin{aligned} X_1 \left(r_1 f_1 u^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta} \partial_\alpha p_1 \right) - (p_1 - p_{13}) \gamma^{\alpha\beta} \partial_\alpha X_1 - \\ -(p_{12} - p_{13}) \gamma^{\alpha\beta} \partial_\alpha X_2 = 0 \,, \end{aligned} \tag{25}$$

$$\begin{aligned} X_2 \left(r_2 f_2 u^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta} \partial_\alpha p_2 \right) - (p_{21} - p_{23}) \gamma^{\alpha\beta} \partial_\alpha X_1 - \\ -(p_2 - p_{23}) \gamma^{\alpha\beta} \partial_\alpha X_2 = 0 \,, \end{aligned} \tag{26}$$

$$X_3\left(r_3 f_3 u^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta}\partial_\alpha p_3\right) - (p_{31} - p_3)\gamma^{\alpha\beta}\partial_\alpha X_1 - \\ -(p_{32} - p_3)\gamma^{\alpha\beta}\partial_\alpha X_2 = 0\,, \tag{27}$$

where $D = u^\alpha \partial_\alpha$.

3. RELAXATION SYSTEM

In this section we derive separate energy evolution equations for each phase. We assume that the fundamental thermodynamic differential relations

$$T_k dS_k = d\varepsilon_k + p_k d\frac{1}{r_k} \tag{28}$$

is valid, where $S_k(\varepsilon_k, r_k)$ is specific entropy and T_k temperature of fluid component k. Hence, we have:

$$T_k u^\alpha \partial_\alpha S_k = u^\alpha\left(\partial_\alpha \varepsilon_k + p_k \partial_\alpha \frac{1}{r_k}\right)\,. \tag{29}$$

After that, by virtue of (22)-(24) we can write:

$$X_1 r_1 T_1 DS_1 = (p_1 - p_{13})DX_1 + (p_{12} - p_{13})DX_2\,, \tag{30}$$

$$X_2 r_2 T_2 DS_2 = (p_{21} - p_{23})DX_1 + (p_2 - p_{23})DX_2\,, \tag{31}$$

$$X_3 r_3 T_3 DS_3 = (p_{31} - p_3)DX_1 + (p_{32} - p_3)DX_2\,. \tag{32}$$

Taking into account total specific entropy of the mixture, S, given by

$$S = Y_1 S_1 + Y_2 S_2 + Y_3 S_3\,, \tag{33}$$

and by virtue of (22)-(24), (8) and (16), we have that:

$$\begin{aligned} \nabla_\alpha(rSu^\alpha) &= \textstyle\sum_{k=1}^3 \nabla_\alpha(rY_k S_k u^\alpha) = \sum_{k=1}^3 rY_k DS_k = \sum_{k=1}^3 X_k r_k DS_k \\ &= \tfrac{1}{T_1}\{(p_1 - p_{13})DX_1 + (p_{12} - p_{13})DX_2\} \\ &\quad + \tfrac{1}{T_2}\{(p_{21} - p_{23})DX_1 + (p_2 - p_{23})DX_2\} \\ &\quad + \tfrac{1}{T_3}\{(p_{31} - p_3)DX_1 + (p_{32} - p_3)DX_2\}\,. \end{aligned} \tag{34}$$

To be consistent with second law of thermodynamics we consider the following solution for the six unknown p_{ik}

$$\begin{aligned} p_{12} = p_{21} = p_{23} = p_2\,, \\ p_{13} = p_{31} = p_{32} = p_3\,, \end{aligned} \tag{35}$$

with the following evolution equations for volume fractions X_1 and X_2:

$$DX_1 = X_1\left[X_2\pi_{12}(p_1 - p_2) + X_3\pi_{13}(p_1 - p_3)\right]\,, \tag{36}$$

$$DX_2 = X_2\left[X_1\pi_{21}(p_2 - p_1) + X_3\pi_{23}(p_2 - p_3)\right]\,, \tag{37}$$

in which π_{12}, π_{21}, π_{13} and π_{23} are the relaxation coefficients, with

$$\pi_{12} = \pi_{21} \geq 0\,, \qquad \pi_{13} \geq 0\,, \qquad \pi_{23} \geq 0\,. \tag{38}$$

In this way, equations (30)-(32) give, respectively:

$$\begin{aligned} X_1 r_1 T_1 DS_1 &= (p_1 - p_3)DX_1 + (p_2 - p_3)DX_2 \\ &= X_1 X_2 \pi_{12}(p_1 - p_2)^2 + X_1 X_3 \pi_{13}(p_1 - p_3)^2 \\ &\quad + X_2 X_3 \pi_{23}(p_2 - p_3)^2\,, \end{aligned} \tag{39}$$

$$DS_2 = 0\,, \tag{40}$$

$$DS_3 = 0\,, \tag{41}$$

thus, equation (3.2) becomes

$$\begin{aligned} \nabla_\alpha(rSu^\alpha) \quad = \quad & \tfrac{1}{T_1}\Big\{X_1 X_2 \pi_{12}(p_1 - p_2)^2 + X_1 X_3 \pi_{13}(p_1 - p_3)^2 \\ & + X_2 X_3 \pi_{23}(p_2 - p_3)^2\Big\}\,, \end{aligned} \tag{42}$$

and it expresses, according to the second law of thermodynamics, that the mixture entropy S always evolves with positive or null variations.

Therefore, complete relaxation system describing the evolution model under investigation is composed of eleven governing differential equations in terms of eleven variables u^α, r_1, r_2, r_3, S_1, S_2, S_3, X_1 and X_2:

$$\begin{cases} \nabla_\alpha(X_1 r_1 u^\alpha) = 0\,, \\[2ex] \nabla_\alpha(X_2 r_2 u^\alpha) = 0\,, \\[2ex] \nabla_\alpha[(1 - X_1 - X_2) r_3 u^\alpha] = 0\,, \\[2ex] u^\alpha \partial_\alpha X_1 = X_1\left[X_2 \pi_{12}(p_1 - p_2) + X_3 \pi_{13}(p_1 - p_3)\right]\,, \\[2ex] u^\alpha \partial_\alpha X_2 = X_2\left[X_1 \pi_{21}(p_2 - p_1) + X_3 \pi_{23}(p_2 - p_3)\right]\,, \\[2ex] r f u^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta}\partial_\alpha\left[X_1 p_1 + X_2 p_2 + (1 - X_1 - X_2)p_3\right] = 0\,, \\[2ex] u^\alpha \partial_\alpha S_1 = \dfrac{1}{r_1 X_1 T_1}\Big\{X_1 X_2 \pi_{12}(p_1 - p_2)^2 + X_1 X_3 \pi_{13}(p_1 - p_3)^2 \\ \qquad\qquad\qquad\qquad + X_2 X_3 \pi_{23}(p_2 - p_3)^2\Big\}\,, \\[2ex] u^\alpha \partial_\alpha S_2 = 0\,, \\[2ex] u^\alpha \partial_\alpha S_3 = 0\,, \end{cases} \tag{43}$$

with $p_k = p_k(r_k, S_k)$.

4. DISCONTINUITIES

In a domain Ω of the space-time V_4, let Σ be a regular time–like hypersurface, being $\varphi(x^\alpha) = 0$ its local equation.

In the following N_α will denote the normalized four–vector

$$N_\alpha = \frac{L_\alpha}{\sqrt{-L^\beta L_\beta}}\,, \qquad L_\alpha = \partial_\alpha \varphi\,, \qquad N_\alpha N^\alpha = -1\,. \tag{44}$$

We consider weak discontinuity waves Σ on which field variables u^α, r_1, r_2, r_3, S_1, S_2, S_3, X_1 and X_2 are continuous, but jump discontinuities may occur in their normal derivatives. In this case, if Q denotes any of these fields, then there exists [38, 5] the distribution δQ, with support Σ, such that

$$\bar{\delta}\,[\nabla_\alpha Q] = N_\alpha \delta Q\,, \tag{45}$$

where $\bar{\delta}$ is the Dirac measure defined by φ with Σ as support, square brackets denote the jump across Σ and δ is an operator of infinitesimal discontinuity.

Then, from system (35), we obtain the following linear homogeneous system in the distributions δu^α, δr_1, δr_2, δr_3, δS_1, δS_2, δS_3, δX_1 and δX_2:

$$\begin{cases} X_k r_k N_\alpha \delta u^\alpha + X_k L \delta r_k + r_k L \delta X_k = 0\,, \qquad k = 1,2\,, \\[2ex] X_3 r_3 N_\alpha \delta u^\alpha + X_3 L \delta r_3 - r_3 L(\delta X_1 + \delta X_2) = 0\,, \\[2ex] L\delta X_k = 0\,, \qquad k = 1,2\,, \\[2ex] rfL\delta u^\alpha - \gamma^{\alpha\beta} N_\beta \displaystyle\sum_{k=1}^{3} \left\{ X_k \left[\left(\frac{\partial p_k}{\partial r_k}\right) \delta r_k + \left(\frac{\partial p_k}{\partial S_k}\right) \delta S_k \right] \right. \\ \qquad\qquad \left. + (p_1 - p_3)\delta X_1 + (p_2 - p_3)\delta X_2 \right\} = 0\,, \\[2ex] L\delta S_k = 0\,, \qquad k = 1,2,3\,, \end{cases} \tag{46}$$

where $L = u^\alpha N_\alpha$. Moreover, from the unitary character of u^α, we have

$$u_\alpha \delta u^\alpha = 0\,. \tag{47}$$

Now, we focus on the normal speeds of propagation of the various waves with respect to an observer moving with the mixture velocity u^α. The normal speed λ_Σ of propagation of the wave front Σ, described by a time–like world line having tangent vector field u^α, that is with respect to the time direction u^α, is given by [38, 5]:

$$\begin{aligned} &\lambda_\Sigma^2 = \tfrac{L^2}{\ell^2}\,, \\ &\ell^2 = -\left(g^{\alpha\beta - u^\alpha u^\beta}\right) N_\alpha N_\beta = -N^\alpha N_\beta + L^2 = 1 + L^2\,. \end{aligned} \tag{48}$$

The local causality condition, *i.e.* the requirement that the characteristic hypersurface Σ has to be time–like or null (or, equivalently, that the normal N_α has to be space–like or null, that is $g^{\alpha\beta}N_\alpha N_\beta \leq 0$), is equivalent to the condition

$$0 \leq \lambda_\Sigma^2 \leq 1\,. \tag{49}$$

From the above equations (38), we obtain as first a wave moving with the mixture corresponding to $L = 0$, whose associated discontinuities have to satisfy the conditions

$$N_\alpha \delta u^\alpha = 0\,, \qquad \delta p = 0\,. \tag{50}$$

Since the coefficients characterizing the discontinuities exhibit nine degrees of freedom, system (38) admits nine independent eigenvectors corresponding to $L = 0$ in the space of field variables.

For $L \neq 0$, we have that

$$\delta S_1 = \delta S_2 = \delta S_3 = \delta X_1 = \delta X_2 = 0\,. \tag{51}$$

Moreover, equation $(38)_4$, multiplied by N_α, gives us:

$$rfLN_\alpha \delta u^\alpha + \ell^2 \left[X_1 \left(\frac{\partial p_1}{\partial r_1} \right)_{S_1} \delta r_1 + X_2 \left(\frac{\partial p_2}{\partial r_2} \right)_{S_2} \delta r_2 + X_3 \left(\frac{\partial p_3}{\partial r_3} \right)_{S_3} \delta r_3 \right] = 0\,. \tag{52}$$

Equation (3.3) can be written also as

$$rfLN_\alpha \delta u^\alpha + \ell^2 \left(X_1 f_1 \lambda_1^2 \delta r_1 + X_2 f_2 \lambda_2^2 \delta r_2 + X_3 f_3 \lambda_3^2 \delta r_3 \right) = 0\,, \tag{53}$$

where we denoted by

$$\lambda_k^2 = \left(\frac{\partial p_k}{\partial \rho_k} \right)_{S_k}\,, \qquad k = 1, 2, 3\,, \tag{54}$$

the hydrodynamical waves in each phase k. In fact, from

$$p_k(r_k, S_k) = p_k \left[\rho_k(r_k, S_k), S_k \right] \tag{55}$$

we have that

$$\begin{cases} \left(\dfrac{\partial p_k}{\partial r_k} \right)_{S_k} = \left(\dfrac{\partial p_k}{\partial \rho_k} \right)_{S_k} \left(\dfrac{\partial \rho_k}{\partial r_k} \right)_{S_k}\,, \\[2ex] \left(\dfrac{\partial \rho_k}{\partial r_k} \right)_{S_k} = f_k\,, \end{cases} \tag{56}$$

and, therefore

$$\left(\frac{\partial p_k}{\partial r_k} \right)_{S_k} = f_k \left(\frac{\partial p_k}{\partial \rho_k} \right)_{S_k} = f_k \lambda_k^2\,. \tag{57}$$

As a result of (43), equations $(38)_{1,2}$ become

$$r_k N_\alpha \delta u^\alpha + L\delta r_k = 0\,, \qquad k = 1, 2, 3. \tag{58}$$

Equations (3.3) and (58) represent a linear homogeneous system in four scalar distributions $N_\alpha \delta u^\alpha$, δr_1, δr_2 and δr_3, which have nontrivial solutions only if the determinant of the coefficient vanishes. Therefore, we find the equation

$$\Sigma_1 \equiv rfL^2 - \left(X_1 r_1 f_1 \lambda_1^2 + X_2 r_2 f_2 \lambda_2^2 + X_3 r_3 f_3 \lambda_3^2\right)\ell^2 = 0\,, \tag{59}$$

which corresponds to two hydrodynamical waves propagating in such a three–phase fluid system with speeds of propagation λ_{Σ_1} given by

$$rf\lambda_{\Sigma_1}^2 = X_1 r_1 f_1 \lambda_1^2 + X_2 r_2 f_2 \lambda_2^2 + X_3 r_3 f_3 \lambda_3^2\,. \tag{60}$$

The associated discontinuities can be written in terms of $\psi_1 = N_\alpha \delta u^\alpha$ as follows

$$\begin{cases} \delta u^\alpha = -\dfrac{1}{\ell}\psi_1 n^\alpha\,, \\[2ex] \delta r_k = -\dfrac{r_k}{L}\psi_1\,, \qquad k = 1, 2, 3, \\[2ex] \delta S_1 = \delta S_2 = \delta S_3 = \delta X_1 = \delta X_2 = 0\,, \end{cases} \tag{61}$$

where n^α is the unitary space–like four-vector defined by

$$n^\alpha = \frac{1}{\ell}(N^\alpha - Lu^\alpha)\,. \tag{62}$$

So all velocities (the eigenvalues of system (35)) are real and there is a complete set of eigenvectors in the space of field variables, *i.e.* eleven independent eigenvectors (nine from $L = 0$ and 2 from $\Sigma_1 = 0$) for the eight independent field variables u^α, r_1, r_2, r_3, S_1, S_2, S_3, X_1 and X_2. Therefore, system of governing equations (35) is hyperbolic.

5. RELAXED SYSTEM

Now, we consider system obtained by letting the relaxation coefficients π_{ik} tend to infinity, *i.e.* pressure equilibrium (equality of pressures, $p_1 = p_2 = p_3$) is instantaneously achieved, and replace (36) and (37) with the assumption

$$p_1 = p_2 = p_3 = p\,. \tag{63}$$

In this limit, the energy equations (39)-(41) can be written as

$$u^\alpha \partial_\alpha S_k = 0\,, \qquad k = 1, 2, 3\,. \tag{64}$$

At this point, the following volume fractions evolution equations are derived

$$u^\alpha \partial_\alpha X_k + A_k \nabla_\alpha u^\alpha = 0\,, \qquad k = 1, 2\,, \tag{65}$$

with

$$A_1 = \frac{X_1 \left[X_2 r_3 f_3 \lambda_3^2 (r_1 f_1 \lambda_1^2 - r_2 f_2 \lambda_2^2) + X_3 r_2 f_2 \lambda_2^2 (r_1 f_1 \lambda_1^2 - r_3 f_3 \lambda_3^2) \right]}{X_1 r_2 f_2 \lambda_2^2 r_3 f_3 \lambda_3^2 + X_2 r_1 f_1 \lambda_1^2 r_3 f_3 \lambda_3^2 + X_3 r_1 f_1 \lambda_1^2 r_2 f_2 \lambda_2^2}\,, \tag{66}$$

$$A_2 = \frac{X_2 \left[X_1 r_3 f_3 \lambda_3^2 (r_2 f_2 \lambda_2^2 - r_1 f_1 \lambda_1^2) + X_3 r_1 f_1 \lambda_1^2 (r_2 f_2 \lambda_2^2 - r_3 f_3 \lambda_3^2) \right]}{X_1 r_2 f_2 \lambda_2^2 r_3 f_3 \lambda_3^2 + X_2 r_1 f_1 \lambda_1^2 r_3 f_3 \lambda_3^2 + X_3 r_1 f_1 \lambda_1^2 r_2 f_2 \lambda_2^2}\,. \tag{67}$$

In fact, by virtue of (52) and (58) we obtain

$$dp = f_k \lambda_k^2 dr_k + \left(\frac{\partial p}{\partial S_k} \right)_{r_k} dS_k\,. \tag{68}$$

Using (60), we can write:

$$u^\alpha \partial_\alpha p = f_k \lambda_k^2 u^\alpha \partial_\alpha r_k\,, \qquad k = 1, 2, 3\,, \tag{69}$$

and then, by virtue of $(35)_{1,2}$

$$\frac{X_k}{r_k f_k \lambda_k^2} u^\alpha \partial_\alpha p + \nabla_\alpha (X_k u^\alpha) = 0\,. \tag{70}$$

Finally, summing over k equation (70), we deduce

$$\omega^{-1} u^\alpha \partial_\alpha p + \nabla_\alpha u^\alpha = 0\,, \tag{71}$$

with

$$\omega^{-1} = \frac{X_1}{r_1 f_1 \lambda_1^2} + \frac{X_2}{r_2 f_2 \lambda_2^2} + \frac{X_3}{r_3 f_3 \lambda_3^2}\,. \tag{72}$$

Hence, using equation (71) into (70), evolution equations for volume fractions X_1 and X_2, given by (61), are obtained.

The relaxed model is given by the following system of 9 differential equations in terms of nine variables u^α, p, S_1, S_2, S_3, X_1 and X_2:

$$\begin{cases} \nabla_\alpha (r u^\alpha) = 0\,, \\ u^\alpha \partial_\alpha X_k + A_k \nabla_\alpha u^\alpha = 0\,, \qquad k = 1, 2\,, \\ r f u^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta} \partial_\alpha p = 0\,, \\ u^\alpha \partial_\alpha S_k = 0\,, \qquad k = 1, 2, 3\,, \end{cases} \tag{73}$$

with A_1 and A_2 given by (62) and (67) and $r = X_1 r_1(p, S_1) + X_2 r_2(p, S_2) + X_3 r_3(p, S_3)$.

By applying relations (37) to the previous system, we obtain the following linear homogeneous system in the distributions δu^α, δp, δS_1, δS_2, δS_3, δX_1 and δX_2 given by $(38)_5$ and

$$\begin{aligned}(r_1 - r_3)L\delta X_1 + (r_2 - r_3)L\delta X_2 + \left(\tfrac{X_1}{f_1\lambda_1^2} + \tfrac{X_2}{f_2\lambda_2^2} + \tfrac{X_3}{f_3\lambda_3^2}\right)L\delta p \\ + \textstyle\sum_{k=1}^3 X_k \left(\tfrac{\partial r_k}{\partial S_k}\right)_p L\delta S_k + rN_\alpha \delta u^\alpha = 0\,,\end{aligned} \tag{74}$$

$$L\delta X_k + A_k N_\alpha \delta u^\alpha = 0\,, \qquad k = 1, 2\,, \tag{75}$$

$$rfL\delta u^\alpha - \gamma^{\alpha\beta} N_\beta \delta p = 0\,. \tag{76}$$

From the above system $(38)_5$, (74)-(76), we obtain again the material wave $L = 0$, and for the corresponding discontinuities we find again (50) and, therefore, seven independent eigenvectors in the space of field variables.

For $L \neq 0$, we have that:

$$\delta S_k = 0\,, \qquad k = 1, 2, 3\,, \tag{77}$$

and then, equation (74) becomes

$$(r_1 - r_3)L\delta X_1 + (r_2 - r_3)L\delta X_2 + \left(\frac{X_1}{f_1\lambda_1^2} + \frac{X_2}{f_2\lambda_2^2} + \frac{X_3}{f_3\lambda_3^2}\right)L\delta p + rN_\alpha \delta u^\alpha = 0\,. \tag{78}$$

Equation (76), multiplying by N_α, gives

$$rfLN_\alpha \delta u^\alpha + \ell^2 \delta p = 0\,. \tag{79}$$

Equations (75), (78) and (79) represent a linear homogeneous system in 4 scalar distributions $N_\alpha \delta u^\alpha$, δp, δX_1 and δX_2, which have nontrivial solutions only if

$$\Sigma_2 \equiv rfL^2 - \left(\frac{r_1 f_1 \lambda_1^2 r_2 f_2 \lambda_2^2 r_3 f_3 \lambda_3^2}{X_1 r_2 f_2 \lambda_2^2 r_3 f_3 \lambda_3^2 + X_2 r_1 f_1 \lambda_1^2 r_3 f_3 \lambda_3^2 + X_3 r_1 f_1 \lambda_1^2 r_2 f_2 \lambda_2^2}\right)\ell^2 = 0\,, \tag{80}$$

which corresponds to two hydrodynamical waves propagating in such relaxed model with speeds of propagation λ_{Σ_2} given by

$$\frac{1}{rf\lambda_{\Sigma_2}^2} = \frac{X_1}{r_1 f_1 \lambda_1^2} + \frac{X_2}{r_2 f_2 \lambda_2^2} + \frac{X_3}{r_3 f_3 \lambda_3^2}\,. \tag{81}$$

The associated discontinuities can be written in terms of $\psi_2 = N_\alpha \delta u^\alpha$ as follows:

$$\begin{cases} \delta u^\alpha = -\dfrac{1}{\ell}\psi_2 n^\alpha \,, \\[2ex] \delta p = -\dfrac{rfL}{\ell^2}\psi_2 \,, \\[2ex] \delta X_k = -\dfrac{A_k}{L}\psi_2 \,, \qquad\qquad k = 1,2\,, \\[2ex] \delta S_k = 0\,, \qquad\qquad k = 1,2,3\,. \end{cases} \tag{82}$$

Therefore, we have two independent eigenvectors for $\Sigma_2 = 0$ in the space of field variables.

Now, we observe that, by virtue of (42), the following relation is valid

$$0 \le \lambda^2_{\Sigma_2} \le \lambda^2_{\Sigma_1} \le 1\,. \tag{83}$$

In fact, we have

$$\lambda^{-2}_{\Sigma_2} = \lambda^{-1}_{\Sigma_1}\,[1 + G]\,, \tag{84}$$

where

$$\begin{aligned} G \;=\; & X_1X_2\left[\left(\tfrac{r_1f_1}{r_2f_2}\right)^{1/2}\tfrac{\lambda_1}{\lambda_2} - \left(\tfrac{r_2f_2}{r_1f_1}\right)^{1/2}\tfrac{\lambda_2}{\lambda_1}\right]^2 \\ & + X_1X_3\left[\left(\tfrac{r_1f_1}{r_3f_3}\right)^{1/2}\tfrac{\lambda_1}{\lambda_3} - \left(\tfrac{r_3f_3}{r_1f_1}\right)^{1/2}\tfrac{\lambda_3}{\lambda_1}\right]^2 \\ & + X_2X_3\left[\left(\tfrac{r_2f_2}{r_3f_3}\right)^{1/2}\tfrac{\lambda_2}{\lambda_3} - \left(\tfrac{r_3f_3}{r_2f_2}\right)^{1/2}\tfrac{\lambda_3}{\lambda_2}\right]^2 . \end{aligned} \tag{85}$$

To prove relation (84), following the ideas developed by Lund and Flåtten [39], we note that

$$\lambda^2_{\Sigma_1}\lambda^{-2}_{\Sigma_2} = (X_1r_1f_1\lambda_1^2 + X_2r_2f_2\lambda_2^2 + X_3r_3f_3\lambda_3^2)\left(\frac{X_1}{r_1f_1\lambda_1^2} + \frac{X_2}{r_2f_2\lambda_2^2} + \frac{X_3}{r_3f_3\lambda_3^2}\right). \tag{86}$$

Handling the right–hand side and using $X_1 + X_2 + X_3 = 1$, we can write:

$$\begin{aligned} \lambda^2_{\Sigma_1}\lambda^{-2}_{\Sigma_2} \;=\; & X_1^2 + X_2^2 + X_3^2 + X_1X_2\left(\tfrac{r_1f_1\lambda_1^2}{r_2f_2\lambda_2^2} + \tfrac{r_2f_2\lambda_2^2}{r_1f_1\lambda_1^2}\right) \\ & + X_1X_3\left(\tfrac{r_1f_1\lambda_1^2}{r_3f_3\lambda_3^2} + \tfrac{r_3f_3\lambda_3^2}{r_1f_1\lambda_1^2}\right) + X_2X_3\left(\tfrac{r_2f_2\lambda_2^2}{r_3f_3\lambda_3^2} + \tfrac{r_3f_3\lambda_3^2}{r_2f_2\lambda_2^2}\right) \\ =\; & (X_1 + X_2 + X_3)^2 + X_1X_2\left[\left(\tfrac{r_1f_1}{r_2f_2}\right)^{1/2}\tfrac{\lambda_1}{\lambda_2} - \left(\tfrac{r_2f_2}{r_1f_1}\right)^{1/2}\tfrac{\lambda_2}{\lambda_1}\right]^2 \\ & + X_1X_3\left[\left(\tfrac{r_1f_1}{r_3f_3}\right)^{1/2}\tfrac{\lambda_1}{\lambda_3} - \left(\tfrac{r_3f_3}{r_1f_1}\right)^{1/2}\tfrac{\lambda_3}{\lambda_1}\right]^2 \\ & + X_2X_3\left[\left(\tfrac{r_2f_2}{r_3f_3}\right)^{1/2}\tfrac{\lambda_2}{\lambda_3} - \left(\tfrac{r_3f_3}{r_2f_2}\right)^{1/2}\tfrac{\lambda_3}{\lambda_2}\right]^2 \\ =\; & 1 + G\,, \end{aligned} \tag{87}$$

with G given by (85), that implies (84).

In particular, relation (83) implies that $0 \leq \lambda^2_{\Sigma_2} \leq 1$, *i.e* local causality condition is satisfied. Therefore, relaxed system is hyperbolic.

Moreover, we can conclude that the square of the velocity of propagation of the hydrodynamical waves of relaxed system can never exceed that of relaxation system.

6. CONCLUSIONS

We deduce a relaxation system modeling relativistic single–component three–phase flow. The phases are assumed to flow with the same four–velocities and to be in thermal equilibrium (*i.e.* there is no heat and mass transfer). Mechanical equilibrium is not assumed; instead the pressure non–equilibrium effects are modelled by a relaxation procedure suitably chosen in order to respect first and second laws of thermodynamics.

Furthermore, we study the relaxed limit in which mechanical equilibrium is istantaneously imposed (*i.e.* equality of pressures of three phases). This relaxed limit is sometimes referred to as homogeneous equilibrium model.

The weak discontinuity waves admitted in both regimes are investigated to conclude that the instantaneous equilibrium condition slows down the mixture hydrodynamical velocity with respect to the non–equilibrium case.

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