ROMAI J., v.11, no.1(2015), 165–179

# ON ANALYTIC HERZ-TYPE SPACES IN TUBULAR DOMAINS OVER SYMMETRIC CONES AND PSEUDOCONVEX DOMAINS IN $\mathbb{C}^N$

Romi Shamoyan, Olga Zaytseva

*Department of Mathematics, Bryansk State University, Bryansk, Russia*

rshamoyan@gmail.com, olgazaitseva@gmail.com

**Abstract** We provide complete characterizations of some new analytic spaces of Herz-type in typical Siegel domains namely in tubular domains over symmetric cones and in bounded pseudoconvex domains with smooth boundary.

**Keywords:** analytic functions, Herz-type spaces, polydisk, tubular domains, pseudoconvex domains.
**2010 MSC:** Primary 42B15, Secondary 42B30.

## 1. INTRODUCTION

In this paper we provide complete characterizations of new Herz -type analytic function spaces in typical Siegel domains namely in the unit ball,in the unit polydisk,in tubular domains over symmetric cones and in bounded strongly pseudoconvex domains with smooth boundary. All proofs for these new Herz-type spaces in various domains are completely parallel and we provide a complete detailed simple proof of the unit ball case indicating in this proof all changes needed to pass to other cases. Our proofs are based only on several basic facts ,which are heavily related with the so-called r-lattices and related "dyadic cubes" of these domains and some nice properties of analytic functions on those "dyadic cubes" defined with the help of Kobayashi balls for pseudoconvex domains and with the help of Bergman balls for tubular domains.We alert the reader to avoid arguments duplication and to shorten the exposition, some proofs will be omitted. We first need some definitions.

Let $D_1$ be the unit disk on the complex plane $\mathbb{C}$. Let $H(D_1)$ be the space of all analytic functions in $D_1$.

Let

$$A^{p,q}_{\alpha,\beta}(D_1) = \{f \in H(D_1) : \|f\|^p_{p,q,\alpha} =$$

$$= \int_0^1 (\int_{|z|<r} |f(z)|^q (1-|z|)^\alpha dm_2(z))^{\frac{p}{q}} \cdot (1-r)^\beta dr < \infty\},$$

where $m_2$ is a Lebesque measure on $D_1$ and $0 < p, q < \infty$, $\alpha > -1, \beta > -1$.

Let also $r_k = 1 - \frac{1}{2^k}$; $k \geq 0$ and

$$\widetilde{A}^{p,q}_{\alpha,\beta}(D_1) = \{f \in H(D_1) : \sum_{k\geq o}(\int\limits_{1-2^{-k}}^{1-2^{-k-1}} \int\limits_{T} |f(z)|^q dm(\zeta)(1-r)^\alpha dr)^{\frac{p}{q}} \cdot 2^{-k\beta} < \infty\},$$

where $p, q \in (0, \infty)$, $\alpha > -1$, $\beta > 0$, $dm$ is a Lebesque measure on unit circle $T = \{z : |z| = 1\}$.

It is easy to check spaces we introduced in the unit disk are Banach spaces for $\min(p, q) > 1$ and complete metric spaces for other values of $p$ and $q$. These are simplest versions of Herz - type analytic function spaces. We will consider more general versions of these type spaces replacing $dm_2$ Lebegues measure by general positive Borel measures (see below for definitions).

Note such type analytic spaces were studied by various authors in various papers first in the unit disk and much later in products of unit balls in higher dimension (see, for example, [1], [2], [3], [4],[28], [29], [27], [26] and various references there). In [14] Herz -type analytic spaces were studied in more complicated tubular domains over symmetric cones. In [28] analytic Herz -type spaces were considered in bounded pseudoconvex domains with smooth boundary. In the following we intend to continue the study of such type analytic spaces in tubular domains and pseudoconvex domains. Looking at such type analytic function spaces in $\mathbb{C}^n$ it is natural to find their complete characterizations in those domains. This is the main goal of this paper. In the next section we provide various facts, preliminaries on some domains and also we provide related facts on analytic functions in those domains,and then we formulate our assertions. The case of the unit ball is serving as model case for our research in this paper. For all basic well-known facts we used in proofs of all our main assertions we refer the reader for tubular domains to [11], [12] for the unit ball [22], [7],for the unit polydisk[8], and finnaly for the bounded pseudoconvex domains with smooth boundary we refer the reader to [6], [10], [13]. We denote as usual by $C$ or by $C$ with various indexes various positive constants in inequalities depending on various parameters.

## 2. PRELIMINARIES

The goal of this section to provide various known assertions in the unit ball, in tubular domains over symmetric cones,and in pseudoconvex domains with smooth boundary. Some from these assertions are important for proofs of our main results, these preliminary assertions in some sense are parallel in each domain, and as a consequence proofs of all our main theorems are completely of the same parallel nature. We alert the reader the duplications of arguments in proofs of the main resulls will be omitted by us in this paper. In this section we provide lemmas on r-lattices and their properties, Forelly-Rudin type estimates, properties of weights and analytic (subharmonic) functions on "dyadic cubes" forming r- lattices. Some assertions we provided below will not be used in proofs, we provided them for completeness .

Let $B = \{z \in C^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$ and $S$ be the unit sphere of $C^n$.

Let $d\delta$ be the normalized Lebesque measure on $B$ and $d\sigma$ normalized rotation invariant Lebesque measure on $S$. We denote by $d\delta_\alpha$ and $H(B)$ the standard weighted Lebegues measure in the unit ball and the space of all analytic functions on the unit ball $B$ (we denote by same symbol or by $d\nu$ the Lebesque measure in bounded pseudoconvex domains , in tubular domains we denote the Lebegues measure by $dV$ below).

Let $r > 0$, $z \in B$. The Bergman metric ball at $z$ is defined as

$$D(z, r) = \{\omega \in B : \beta(z, \omega) = \frac{1}{2}(\log \frac{1 + |\varphi_z(\omega)|}{1 - |\varphi_z(\omega)|}) < r\}$$

(the Bergman balls in tubular domains and Kobayashi balls in pseudoconvex domains will be denoted by $B(z, r)$ below,where $z$ is a point from domain). The involutions $\varphi_z$ has form

$$\varphi_z(\omega) = \frac{z - P_z\omega - S_z Q_z \omega}{1 - < \omega, z >}, where S_z = (1 - |z|^2)^{\frac{1}{2}}$$

$$P_z\omega = \frac{< \omega, z > z}{|z|^2}, P_0\omega = 0, Q_z = I - P_z.$$

see [7].

We will need three basic lemmas in the unit ball taken from [7] ,in [8] their complete analogues can be seen in context of the unit polydisk.These type lemmas are also valid in tubular domains and pseudoconvex domains with smooth boundary (see for example [14], [11] and [6]) and they serve as a core of our proofs of all main theorems in all domains.

**Lemma 2.1.** *There exists a positive integer $N$ such that for any $0 < r \leq 1$ we can find a sequence $\{a_k\}$ in $B$ with the following properties*

*(1) $B = \bigcup_k D(a_k, r)$;*

*(2) The sets $D(a_k, \frac{r}{4})$ are mutually disjoint;*

*(3) Each point $z \in B$ belongs to at mast $N$ of the sets $D(a_k, 2r)$.*

Such a sequence will be called r-lattice or a sampling sequence.

**Lemma 2.2.** *For each $r > 0$ there exists a positive constant $C_r$ such that*

$$C_r^{-1} \leq \frac{(1 - |a|)^2}{(1 - |z|)^2} \leq C_r;$$

$$C_r^{-1} \leq \frac{1 - |a|^2}{|1 - < a, z >|} \leq C_r.$$

*for all $a$ and $z$ such that $\beta(a, z) < r$.*

**Lemma 2.3.** *Suppose $r > 0$, $p > 0$, $\alpha > -1$. Then there exists a constant $C > 0$ such that*

$$|f(z)|^p \leq \frac{c}{(1-|z|)^{n+1+\alpha}} \cdot \int\limits_{D(z,r)} |f(\omega)|^p d\delta_\alpha(\omega),$$

*$f \in H(B)$, $z \in B$.*

The important fact here is that this lemma is valid for subharmonic functions in the unit ball (see [7]). Below complete analogues of those three lemmas in tubular domains over symmetric cones $T_\Omega$ and in bounded strictly pseudoconvex domains with smooth boundary will be also formulated. (see for these assertions for tubular domains over symmetric cones [14], [12], for the bounded pseudoconvex domains with smooth boundary we refer the reader to [6], [10], [13] and references there). We do not define all objects in details to shorten the exposition of paper. We refer the reader for $g_0$ and $g_0^\star$ and important determinant $\Delta$ function to [11] and [12]. We also refer the reader to [11] and [12] for definitions and properities of the $B_\alpha$ Bergman kernel in tubular domains over symmetric cones. The following lemma can be seen in [11] and [12].

**Lemma 2.4.** *1) The integral*

$$J_\alpha(y) = \int\limits_{\mathbb{R}^n} \left|\Delta^{-\alpha}\left(\frac{x+iy}{i}\right)\right| dx$$

*converges if and only if $\alpha > 2\frac{n}{r} - 1$. In that case*

$$J_\alpha(y) = \widetilde{C}_\alpha \Delta^{-\alpha+n/r}(y),$$

*$\alpha \in R, y \in \Omega$.*

*2)Let $\alpha \in \mathbb{C}^r$ and $y \in \Omega$. For any multi-indices $s$ and $\beta$ and $t \in \Omega$ the function*

$$y \mapsto \Delta_\beta(y+t)\Delta_s(y)$$

*belongs to $L^1(\Omega, \frac{dy}{\Delta^{n/r}(y)})$ if and only if $\Re s > g_0$ and $\Re(s+\beta) < -g_0^*$. In that case we have*

$$\int\limits_\Omega \Delta_\beta(y+t)\Delta_s(y)\frac{dy}{\Delta^{n/r}(y)} = \widetilde{C}_{\beta,s}\Delta_{s+\beta}(t).$$

In the following lemma (it follows directly from the previous one [11], [12]), we provide so-called classical Forelly-Rudin type estimates for tubular domains over symmetric cones. In [6], [13] it can be seen in context of bounded pseudoconvex domains with smooth boundary, in the polydisk in [8], in [7] in the unit ball. (These type estimates are valid also in bounded symmetric domains and minimal bounded homogeneous domains and in more general Siegel domains of second type).

**Lemma 2.5.** *For all $1 < p < \infty$ and $1 < q < \infty$ and for all $\frac{n}{r} \leq p_1$, where $\frac{1}{p_1} + \frac{1}{p} = 1$ and $\frac{n}{r} - 1 < \nu$ and for all functions $f$ from $A_\nu^{p,q}$ and for all $\frac{n}{r} - 1 < \alpha$ the Bergman representation formula with $\alpha$ index or with the Bergman kernel $B_\alpha(z, w)$ is valid.*

The following lemma is a complete analogue of lemma 1-2 for tubular domains.It shows the existence of r-lattices for tubular domains(see, for example, [11], [14]).

**Lemma 2.6.** *Given $\delta \in (0; 1)$ there exists a sequence of points $\{z_j\}$ in $T_\Omega$ called $\delta$-lattice such that calling $\{B_j\}$ and $\{B'_j\}$ the Bergman balls with center $z_j$ and radius $\delta$ and $\delta/2$ respectively then*

*A) the balls $\{B'_j\}$ are pairwise disjoint;*

*B) the balls $\{B_j\}$ cover $T_\Omega$ with finite overlapping;*

*C)* $\int\limits_{B_j(z_j,\delta)} \Delta^s(y)dV(z) \asymp \int\limits_{B'_j(z_j,\delta)} \Delta^s(y)dV(z) = \widetilde{C}_\delta \Delta^{2\frac{n}{r}+s}(Imz_j);$

$s > \frac{n}{r} - 1, J = |B_\delta(z_j)| \asymp \Delta^{\frac{2n}{r}}(Imz_j), j = 1, ..., m, J \asymp \Delta^{\frac{2n}{r}}(Imw), w \in B_\delta(z_j).$

For our theorem in context of bounded strongly pseudoconvex domains we need also several basic lemmas (on existence of r-lattice, Forelly-Rudin type estimates, and an estimate from below for weighted Bergman type $K_\alpha$ kernels). We refer the reader for these definitions also to [13], [6] and various references there. We denote standard Kobayashi balls in these domains by $B_D(z, r)$ (see [13], [6] for definitions and various properities of these balls ), where $z$ is a point of domain.

**Lemma 2.7.** *Let $D \subset \mathbb{C}^n$ be a strongly pseudoconvex bounded domain. Then there exist $c_1 > 0$ and, for each $r \in (0; 1)$, a $C_{1,r} > 0$ depending on $r$ such that*

$$c_1 r^{2n} dist(z_0, \partial D)^{n+1} \leq \nu(B_D(z_0, r)) \leq C_{1,r} dist(z_0, \partial D)^{n+1}$$

*for every $z_0 \in D$ and $r \in (0, 1)$.*

In the unit ball and more general bounded pseudoconvex domains $dist(z, \partial D)$ is denoted by $\delta(z)$ ,for each $z \in D$. The Lebegues measure is denoted below by $\nu$.

**Lemma 2.8.** *Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, and $r \in (0, 1)$. Then*

$$\nu(B_D(\cdot, r)) \approx \delta^{n+1},$$

*(where the constant depends on $r$).*

**Lemma 2.9.** *Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then there is $C > 0$ such that*

$$\frac{C}{1-r}\delta(z_0) \geq \delta(z) \geq \frac{1-r}{C}\delta(z_0)$$

*for all $r \in (0, 1), z_0 \in D$ and $z \in B_D(z_0, r)$.*

**Definition 2.1.** *Let $D \subset \mathbb{C}^n$ be a bounded domain, and $r > 0$. An r-lattice in D is a sequence $\{a_k\} \subset D$ such that $D = \bigcup_k B_D(a_k, r)$ and there exists $m > 0$ such that any point in D belongs to at most m balls of the form $B_D(a_k, R)$, where $R = \frac{1}{2}(1 + r)$.*

The existence of $r$-lattices in bounded strongly pseudoconvex domains is ensured by the following.

**Lemma 2.10.** *Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then for every $r \in (0, 1)$ there exists an r-lattice in D, that is there exist $m \in N$ and a sequence $\{a_k\} \subset D$ of points such that $D = \bigcup_{k=0}^{\infty} B_D(a_k, r)$ and no point of D belongs to more than m of the balls $B_D(a_k, R)$, where $R = \frac{1}{2}(1 + r)$.*

Note $v_\alpha(B_D(a_k, R)) = (\delta^\alpha(a_k))v(B_D(a_k, R)), \alpha > -1$.

We will call $r$-lattice the family $B_D(a_k, r)$ of balls. Dealing with the positive Bergman $K$ kernel we can always assume $K(z, w) \asymp K(a_k, a_k)$ for any $z \in B_D(a_k, r)$, $r \in (0; 1)$ and for any $w \in B_D(a_k, r)$, $r \in (0; 1)$ (see [6], [13] ).

This vital fact is also valid for weighted Bergman $K_t$ kernels, ($K_{n+1} = K$) as far as $t = (n + 1)l$, $l \in N$, defined in a standard manner with the help of so-called Henkin-Ramirez function (see [18], [13] and references there). The same is valid for any Bergman kernel, if we add the sign of modulus. In this paper we will always assume that the parallel properity is also valid for the Bergman kernel in tubular domains over symmetric cones and for the Bergman balls forming r-lattices on these domains. This assumption hovewer probably can be dropped. We shall use a submean estimate for nonnegative plurisubharmonic functions on Kobayashi balls. This lemma has complete analogues in tubular domain over symmetric cones, and in the unit polydisk [8], [11], [14] and also in other domains (see for example [19]).

**Lemma 2.11.** *Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Given $r \in (0, 1)$, set $R = \frac{1}{2}(1 + r) \in (0, 1)$. Then there exists a $C_r > 0$ depending on r such that*

$$\forall z_0 \in D, \ \ \forall z \in B_D(z_0, r) \ \ \chi(z) \leq \frac{C_r}{v(B_D(z_0, r))} \int\limits_{B_D(z_0,R)} \chi dv$$

*for every nonnegative plurisubharmonic function $\chi : D \to \mathbb{R}^+$.*

**Lemma 2.12.** *Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then*

$$\|K(\cdot, z_0)\|_2 = \sqrt{K(z_0, z_0)} \approx \delta^{-\frac{n+1}{2}}(z_0) \quad \textit{and} \quad \|k_{z_0}\|_2 \equiv 1$$

*for all $z_0 \in D$.*

The next result contains the weighted $L^p$-estimates we shall need:

**Theorem A** (see [6]). *Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, and let $z_0 \in D$ and $1 \le p < \infty$. Then*

$$\int_D |K(\zeta, z_0)|^p \delta^\beta(\zeta) d\nu(\zeta) \le \begin{cases} \delta^{\beta-(n+1)(p-1)}(z_0), & for\ -1 < \beta < (n+1)(p-1); \\ |\log \delta(z_0)|, & for\, \beta = (n+1)(p-1); \\ 1, & for\, \beta > (n+1)(p-1). \end{cases}$$

*In particular:*

*i* $\|K(\cdot, z_0)\|_{p,\beta} \preceq \delta^{\frac{\beta}{p} - \frac{n+1}{q}}(z_0)$ *and* $\|k_{z_0}\|_{p,\beta} \preceq \delta^{\frac{n+1}{2} + \frac{\beta}{p} - \frac{n+1}{q}}(z_0)$ *when* $-1 < \beta < (n+1)(p-1)$*, where* $q > 1$ *is the conjugate exponent of* $p$ *(and* $\frac{n+1}{q} = 0$ *when* $p = 1$*);*

*ii* $\|K(\cdot, z_0)\|_{p,\beta} \preceq 1$ *and* $\|k_{z_0}\|_{p,\beta} \preceq \delta^{\frac{n+1}{2}}(z_0)$ *when* $\beta > (n+1)(p-1)$*;*

*iii* $\|K(\cdot, z_0)\|_{p,(n+1)(p-1)} \preceq \delta^{-\varepsilon}(z_0)$ *and* $\|k_{z_0}\|_{p,(n+1)(p-1)} \preceq \delta^{\frac{n+1}{2} - \varepsilon}(z_0)$ *for all* $\varepsilon > 0$.

*Furthermore,*

*iv* $\|K(\cdot, z_0)\|_{\infty,\beta} \approx \delta^{\beta-(n+1)}(z_0)$ *and* $\|k_{z_0}\|_{\infty,\beta} \approx \delta^{\beta - \frac{n+1}{2}}(z_0)$ *for all* $0 \le \beta < n+1$*; and*
$\|K(\cdot, z_0)\|_{\infty,\beta} \approx 1$ *and* $\|k_{z_0}\|_{\infty,\beta} \approx \delta^{\frac{n+1}{2}}(z_0)$ *for all* $\beta \ge n+1$

A complete analogue of last theorem (Forelly-Rudin type estimates) is valid also for all weighted $K_t$ type kernels $t > 0$ (see[18]). In particular case of the unit ball and also in the unit polydisk this lemma as well as some other our lemmas in context of the unit ball and unit polydisk can be seen in [7] and [8].

The following lemma is also important for this paper. We provide the simplest model of the unit disk case ( the case of more general domains can be covered similarly based on basic results on subharmonic functions in general bounded domains in higher dimension see [16],[17] and references there). We remind the reader we denote by $D_1$ the unit disk in $C$ and by $D_1^2$ the bidisk (a product of two disks). Note even a little bit more general version of this lemma with the same proof is valid for $|F|^p \cdot |G|^q$,where $p$,and $q$ are positive,and where both functions are analytic in bidisk.
**Lemma A.** *Let $F \in H(D_1^2)$, $F = f_1 \cdot f_2$. Then*

$$\psi_\beta^\alpha(z_2) = (\int_{D_1} F(z_1, z_2)|^{p_1} \cdot (1 - |z_1|)^\alpha dm_2(z_1))^\beta;$$

*$\beta \ge 0$, $\alpha > -1$, $z_2 \in D_1$ is subharmonic function in $D_1$, where $p_1$ is an arbitrary positive number.*

*Proof of Lemma A.* For the proof of lemma A we will use basic facts on subharmonic function spaces (see, for example, [17],[16]). Let $D_r = \{z_1 : |z_1| < r\}$, $\delta > 0$.

We show first

$$\psi_r(z_2) = \lg \int_{D_r} (|F(\zeta_1, z_2)| + \delta)^\alpha dm_2(\zeta_1)$$

is subharmonic for all $\alpha \geq 0$, where lg is a logarithm of function. Then we have by known properities of subharmonic functions that the following function $\psi(z_2) = \lim\limits_{r\to 1-0} \psi_r(z_2)$ is also subharmonic.

To show this we note that if $D_r = \bigcup\limits_{k=1}^{m_n} \triangle_k$ where $\triangle_k$ is any decomposition of $D_r$ circle such that $diam(\triangle_k) \leq (\frac{1}{n})$,

$$(u_n)(z_2) = \lg\{\sum_{k=1}^{n}(|F(\zeta_2, z_2)| + \delta)^{\alpha}| \triangle_k |\}$$

$| \triangle_k |$ is a Lebesque measure of $\triangle_k$, then $u_n(z_2)$ is uniformly tending to $\psi_r(z_2)$ on $\overline{D_r}$. So now to show the subharmonicity of $[\psi_r](z_2)$, we show the subharmonicity of $u_n(z_2)$.

We have

$$\lg(|F(\zeta_k, z_2)| + \delta)^{\alpha}| \triangle_k | = \lg((|F(\zeta_k, z_k)| + \delta)^{\alpha} + \lg | \triangle_k | =$$

$$= \alpha \lg(|F(\zeta_k, z_k)| + \delta) + \lg | \triangle_k |.$$

Since both functions $\lg(|F(\zeta_k, z_2)|)$ and $\lg \delta$ are subharmonic

$$\alpha \lg(|F(\zeta_k, z_2)| + \delta) + \lg | \triangle_k |, \alpha \geq 0,$$

is also subharmonic.

Hence $\lg(|F(\zeta_k, z_2)| + \delta)^{\alpha}| \triangle_k |$ is subharmonic.

Hence $u_n(z_2)$ is also subharmonic by known properties of subharmonic functions, [17], [16] $\beta\psi_r(z_2)$ hence and $(\psi_\beta)(z_2) = \lim\limits_{r\to 1-0} \psi_r(z_2)$ are subharmonic and the limit is attianed uniformly in $D_1$.

Hence we have that

$$V_{\alpha,\beta,\delta}(z_2) = \exp(\psi_\beta(z_2)) = (\int\limits_D (|F(z_1, z_2)| + \delta)^{\alpha} dm_2(z_1))^{\beta})$$

is also subharmonic.

Obviously, $V_{\alpha,\beta,\delta} \searrow V_{\alpha,\beta,0}(z_2)$, when $\delta \to 0$ $\mathbf{V}_{\alpha,\beta,\delta_1} \geq V_{\alpha,\beta,\delta_2}$; $\delta_1 < \delta_2$ ($V_{\alpha,\beta,\delta}$ is decreasing to $V_{\alpha,\beta,0}$). Hence using again known properties of subharmonic functions [17], [16]

$$(\psi_{\alpha,\beta})(z_2) = (\int\limits_{D_1} |F(z_1, z_2)|^{\alpha} dm_2(z_1))^{\beta}$$

is subharmonic. So we proved the lemma A.

**Remark.** Note it is enough to only assume in our proof that $\log |F(z)|$ is subharmonic and our assertion is still valid.

At the end of this section we define main objects of this paper Herz-type analytic function spaces in context of pseudoconvex domains in terms of Kobayashi balls. To define this space in tubular domains we have to make obvious changes of r-lattices of pseudoconvex domains to r-lattices of tubular domains which we provided above. We use same notation for both spaces . The polydisk case can be considered similarly.

**Definition 2.2.** *Let $\mu$ be a positive Borel measure in pseudoconvex domain D,* $0 < p, q < \infty$, $s > -1$. *Fix $r \in (0; \infty)$ . Fix an r-lattice $\{a_k\}_{k=1}^{\infty}$. The analytic space $A(p, q, d\mu)(D)$ is the space of all holomorphic functions $f$ such that*

$$\|f\|^q_{A(p,q,d\mu)} = \sum_{k=1}^{\infty}\Big( \int\limits_{B_D(a_k,r)} |f(z)|^p d\mu(z)\Big)^{\frac{q}{p}} < \infty.$$

If $d\mu = \delta^s(z)d\nu(z)$ then we will denote by $A(p, q, s)$ the space $A(p, q, d\mu)$. This is Banach space for $\min(p, q) \geq 1$. It is clear that $A(p, p, s)(D) = A^p_s(D)$.

Let $D^m$ be the unit polydisk (a product of unit disks). Let $H(D^m)$ be the space of all analytic functions in $D^m$, $m \in \mathbb{N}$. We define Herz-type analytic function spaces in the polydisk:

$$H^{p,q}_{\alpha_1 \ldots \alpha_m}(D^m) = \{f \in H(D^m) : \sum_{k_1=0}^{\infty} \ldots \sum_{k_m=0}^{\infty} \sum_{j_1=-2^{k_1}}^{2^{k_1}-1} \ldots \sum_{j_m=-2^{k_m}}^{2^{k_m}-1} \cdot\}$$

$$\{\cdot( \int\limits_{\Delta_{k_1,j_1}} \ldots \int\limits_{\Delta_{k_m,j_m}} |f(z_1 \ldots z_m)|^p \cdot \prod_{j=1}^{m} (1 - |z_j|)^{\alpha_j} dm_2(z_j))^{\frac{q}{p}} < \infty\},$$

where $\alpha_j > -1$, $j = 1, \ldots, m$, $0 < p, q < \infty$, and $\Delta_{k,j}$= $\{z \in D : z = r\zeta :$ $r \in (1 - \frac{1}{2^k}, 1 - \frac{1}{2^{k+1}}], \zeta \in (\frac{-\pi j}{2^k}, \frac{\pi(j+1)}{2^k}]\}$, $k = 0, 1, 2, \ldots$, $j = -2^k, \ldots, 2^k - 1$.

These are analytic Herz-type spaces in the polydisk constructed based on well-known r-lattices in the unit disc $D_1$ (see for these lattices [8] and various referenes there). We can define the same space using directly Bergman balls $D(a_k, r)$ in the unit disk (see for properties of these balls, for example, [7]). Note other two versions in the unit disk can be seen in the start of the paper. The relations between these classes is a separate and interesting problem which will be addressed in a separate paper by authors.

## 3. MAIN RESULTS

This section contains all main results of this paper. We provide complete characterizations of Herz-type analytic function spaces in various domains. First we provide formulations and the proof of the most typical unit ball case in $\mathbb{C}^n$. A carefull analysis of the proof of this model case shows this proof (simply by replacing appropriately similar lemmas addressed hovewer to different domains) can be passed by same arguments also to other domains based on preliminaries of the previous section.

**Theorem 3.1.** *Let $0 < p < q < \infty$, $\alpha > \alpha_0$ for some fixed large enough $\alpha_0$.*

*Let $\{a_k\}_{k\in\mathbb{N}}$ be a r-lattice in the unit ball, let $\mu - a$ be a positive Borel measure on B. Then the following assertions are equivalent*

*(a)* $\int\limits_B (\int\limits_B (\frac{(1-|z|)^n}{|1-<\lambda,z>|^{2n}})^{1+\alpha q} \cdot d\mu(z))^{\frac{q}{q-p}} d\delta_{\alpha qn-1}(\lambda) < \infty$,

*(b)* $\sum\limits_{k=1}^{\infty} (1-|a_k|)^{-(\alpha qn+n)\frac{p}{q-p}} (\mu(D(a_k,r))^{\frac{q}{q-p}}) < \infty$,

*where* $\mu(D(a_k,r)) = (\int\limits_{D(a_k,r)} d\mu(z))$.

*Proof.* $a) \Rightarrow b)$ Using several times Lemma 1 and 2 and restricting the double sum "on diagonal" we have the following chain of estimates.

$$\int\limits_B (\int\limits_B (\frac{(1-|z|)^n}{|1-<\lambda,z>|^{2n}})^{1+\alpha q} d\mu(z))^{\frac{q}{q-p}} d\delta_{\alpha qn-1}(\lambda) \geq$$

$$\geq C \sum_{k=1}^{\infty} \int\limits_{D(a_k,r)} (\sum_{k=1}^{\infty} \int\limits_{D(a_k,r)} (\frac{(1-|z|)^n}{|1-<\lambda,z>|^{2n}})^{1+\alpha q} d\mu(z))^{\frac{q}{q-p}} d\delta_{\alpha qn-1}(\lambda) \geq$$

$$\geq C \sum_{k=1}^{\infty} (1-|a_k|)^{-(\alpha qn+n)\frac{p}{q-p}} [\mu(D(a_k,r))^{\frac{q}{q-p}}].$$

(it can be easily noted these arguments are valid in tube and pseudoconvex domains based on r-lattice properties in these domains and an estimate from below of Bergman kernel). We show the reverse implication. $b) \Rightarrow a)$ . Using Lemma 3 we have

$$|f(z)| = (|f(z)|^{\frac{q}{p}})^{\frac{p}{q}} \leq C(\frac{1}{(1-|a_k|)^{n+1}} \int\limits_{D(a_k,r)} |f(z)^{\frac{q}{p}} d\delta(z))^{\frac{p}{q}}.$$

Using the inequality above and Lemma 1 and 2 we have the following estimates

$$\int\limits_B |f(\omega)| d\mu(\omega) = \sum_{k=1}^{\infty} \int\limits_{D(a_k,r)} |f(\omega)| d\mu(\omega) \leq \sum_{k=1}^{\infty} \max_{z\in D(a_k,r)} |f(z)\mu(D(a_k,r) \leq$$

$$\leq C \sum_{k=1}^{\infty} (\int\limits_{D(a_k,2r)} |f(z)|^{\frac{p}{q}} d\delta_{\alpha qn-1}(z))^{\frac{q}{p}} \cdot (1-|a_k|)^{\frac{-p}{q}(n+qn\alpha)} \cdot \mu(D(a_k,r)).$$

Using Holders inequality we get finally for any subharmonic function $f$

$$\int\limits_B |f(\omega)| d\mu(\omega) \leq C\|f\|_{A^{\frac{q}{p}}_{\alpha qn-1}} [\sum_{k=1}^{\infty} ((1-|a_k|)^{\tau} \cdot \mu(D(a_k,r)))^{\frac{q}{q-p}}]^{\frac{q-p}{q}},$$

where $\tau = -(\frac{np}{q})(1+\alpha q)$.

Let us consider the following function for large enough $t$.

$$S(g)(z) = f(z) = \int_B (\frac{(1-|z|^2)^n}{|1-<z,\omega>|^{2n}})^{1+t} g(\omega)(1-|\omega|)^{tn-1} d\delta(\omega).$$

Then $f$ is subharmonic by lemma A.

But by the classical Bergman type projection theorem in the unit ball (see theorem 2.10 of [7]) we see that

$$\|Sg\|_{L^{\frac{q}{p}}(B,d\delta_{tn-1})} \leq C\|g\|_{L^{\frac{q}{p}}(B,d\delta_{tn-1})}, q > p.$$

So putting this fixed function into the last estimate we will have

$$\int_B \int_B (\frac{(1-|z|^2)^n}{|1-<z,\omega>|^{2n}})^{1+\alpha q} g(\omega)(1-|\omega|)^{\alpha qn-1} d\delta(\omega) d\mu(z) \leq$$

$$\leq C\|g\|_{L^{\frac{q}{p}}_{\alpha qn-1}} (\sum_{k=1}^{\infty} (1-|a_k|)^{\tau} \mu(D(a_k,r))^{\frac{q}{q-p}})^{\frac{q-p}{q}},$$

$$\tau = -(\frac{p}{q}n)(1+q\alpha).$$

Now using standard duality arguments we get the desired result. The proof of the theorem is complete. ∎

We add now some remarks on this proof which serve as model for our other similar type results in other domains.

**Remark 1.** Our theorem can be extended to the case of product domains $B\times...\times B$ (polyball). Analytic spaces in polyball were considered by first author in a series of papers (see for example [25] and various references there.) In this case we have to replace the condition in theorem by following

$$\sum_{k\geq 0}^{\infty} (1-|a_k|)^{\tau} (\mu_j(D(a_k,r))^{\frac{q}{q-p}}),$$

$j = 1,...m, \mu = \prod_{j=1}^{m} \mu_j$,where $\mu_j$ are positive Borel measures in $B$

**Remark 2.** The same passing to product domains can be applied also to other domains we consider in this paper. Note the approach we used in our Theorem 3.1 is based ( the second part) on projection theorems in weighted $L^p_\alpha$ spaces. More general projection results of such type can be found in context of mixed norm Lizorkin-Triebel spaces in [24] and this result can be used similarly for estimates of integrals

of the following type.

$$\int_S (\int_0^1 (\int_B (\frac{(1-|z|^2)^n}{|1-<\lambda,z>|^{2n}})^{1+\alpha q} d\mu(z))^{\frac{q}{q-p}} \cdot (1-|\lambda|)^{\alpha qn-1} d|\lambda|)^{P_1} d\sigma$$

$$\int_0^1 (\int_S (\int_B (\frac{(1-|z|^2)^n}{|1-<\lambda,z>|^{2n}})^{1+\alpha q} d\mu(z))^{\frac{q}{q-p}} d\sigma)^{P_1} \cdot (1-|\lambda|)^{\alpha qn-1} d|\lambda|.$$

Note the unit ball case is the most typical particular case of bounded strongly pseudoconvex domains with smooth boundary in $\mathbb{C}^n$.The general version of Theorem 3.1 with the same proof we formulate below as Theorem 3.2 omitting the proof.

For that reason we just have to repeat all arguments we used above. The main ingredients here are properties of so-called, r- lattices invented recently in [6] and the simple estimate from below of (unweighted)Bergman kernel $K_{n+1}(z,w)$ obtained recently in same paper [6].(see also [13] where it plays a crucial role in the proof of the main result.)

**Theorem 3.2.** *Let D be the bounded strongly pseudoconvex domain with the smooth boundary. Let* $0 < p < q < \infty$, $\alpha > \alpha_0$ *for some fixed large enough* $\alpha_0$. *Let* $\{a_k\}$ *be a sequence forming r- lattice for D(a sampling sequence). Let further* $\mu$ *be a positive Borel measure on D. Then the following two statementis are equivalent.*

$$\int_D (\int_D (\delta(z)^n \cdot |K_{n+1}(z,\lambda)|^{\frac{2n}{n+1}})^{1+\alpha q} d\mu(z))^{\frac{q}{q-p}} (\delta_{\alpha qn-1}(\lambda)) d\nu(\lambda) < \infty;$$

$$\sum_{k=1}^{\infty} (\delta^{\frac{p}{q-p}(-n)(1+\alpha q)}(a_k) \cdot (\mu(B_D(a_k,r))^{\frac{q}{q-p}}) < \infty, n \in \mathbb{N},$$

*where* $\delta_{\alpha qn-1}(\lambda)) d\nu(\lambda) = \delta(\lambda)^{\alpha qn-1} d\nu(\lambda)$

All ingredients of the proof of this theorem we see in previous section in various lemmas. We omit details of this proof. Since to provide it we need to repeat step by step the proof of the unit ball case.We add some remarks for proof. Note the projection theorem on Bergman type integral operators is based on Schur test,(see [7] for unit ball case) and Forelly-Rudin estimates which we formulated in previous section (see theorem A) and can be obtained in both cases similarly in the unit ball and in pseudoconvex domains (see for this theorem also [6]). The proof is based on properities of r-lattices and on lower estimate for the Bergman kernel [6]. Next we formulate a complete analogue of the first theorem in case of typical unbounded Siegel domains, namely for tubular domains over symmetric cones $T_\Omega$.

The main ingredients of the proof of Theorem 3.3 are properties of $r$ lattices on tubular domains over symmetric cones which we provided in previous section (see

also [14], [12], [11]). We will also need a standard Bergman type projection theorem for Bergman type integral operators. That assertion we need (see also the parallel proof of theorem 1) can be seen even in more general setting in recent paper [15]. Hovewer this put additional restriction on parameter(see the formulation of Theorem 3.3 ). Details of this proof will be omitted by us since there is no new idea or new argument in this proof comparing with the proof of Theorem 3.1. The boundedness of Bergman type projectors in tubular domains over symmetric cones is an important part of our proof. This topic was under intensive study in recent years [11], [12]. The vital estimate from below of Bergman kernel $B(z,w)$ in tubular domains is also serve as a partial base of our proof (see for this also previous section, see also [14], [11], [12]). Here is the formulation of our theorem for analytic Herz- type spaces in tubular domains over symmetric cones.

**Theorem 3.3.** *Let $0 < p < q < \infty$, and $\alpha > 0, \alpha \in (\alpha_0, \alpha_1)$, where $\alpha_0$ and $\alpha_1$ depend on $p, q, r, n$. Let $\{a_k\}_{k\in\mathbb{N}}$ be the R-lattice in $T_\Omega$. Let $\mu$ be positive Borel measure on $T_\Omega$. Then the following assertions are equivalent.*

*(a)* $\int\limits_{T_\Omega} (\int\limits_{T_\Omega} (\frac{(\Delta(I_{mz}))^{\frac{n}{r}}}{(\Delta(\frac{z-\overline{w}}{i}))^{\frac{2n}{r}}})^{1+\alpha q} d\mu(z))^{\frac{q}{q-p}} dV\alpha q(\frac{n}{r}) - (\frac{n}{r})(w) < \infty,$

*(b)* $\sum\limits_{k=1}^{\infty} (\Delta^{\tau}(a_k))(\mu(B(a_k,R))^{\frac{q}{q-p}}) < \infty,$

$\tau = -(\frac{p}{q-p})(\alpha q+1)\frac{n}{r}$, *and where* $\mu(B(a_k,R)) = (\int\limits_{B(a_k,R)} d\mu(z))$ *for some* $p \in (p_1, p_2)$, $q \in (q_1, q_2)$, $\alpha \in (\alpha_1, \alpha_2)$ *and for some fixed* $q_j, p_j, \alpha_j$; $j = 1, 2$.

Note finally the complete analogue of our theorem is valid also in case of Herz -type space in the unit polydisk $D^m$ (it is a standard product of finite number of unit disks). Here is the formulation of our theorem. We refer to [8] for machinery we used in polydisk for our proof repeating arguments of proof of Theorem 3.1.

**Theorem 3.4.** *Let $0 < p < q < \infty$, $\alpha > \alpha_0$, and $\alpha_0$ is large enough.*

*Let $\{a_k\}_{k\in\mathbb{N}}$ be a sampling sequence(r-lattice) in $(D^m)$ polydisk, $m \in \mathbb{N}$. Let $\mu$ -be a positive Borel measure on $D^m$ . Then the following assertions are equivalent*

*(a)* $\int\limits_{D^m} (\int\limits_{D^m} \frac{\prod\limits_{j=1}^{m}(1-|z_j|)d\mu(z)}{\prod\limits_{j=1}^{m}|1-\overline{z_j}\lambda_j|^2})^{1+\alpha q} \prod\limits_{j=1}^{m}(d\mu_j(z_j)))^{\frac{q}{q-p}}(d\overline{v}_{\alpha q-1}(\lambda)) < \infty;$

*(b)* $\sum\limits_{k=1}^{\infty}((1-|a_k|)^{-(2q+1)\frac{p}{q-p}})(\mu_j(D(a_k,r))^{\frac{q}{q-p}} < \infty;$

$j = 1, ..., m$; $d\overline{v}_{\alpha q-1} = \prod\limits_{j=1}^{m} dm_{\alpha q-1}(z_j)$, *where $\{D(a_k,r)\}$ is a Bergman ball on the complex plane in the unit disk D, $\mu = \prod \mu_j$, and $\mu_j$ for all $j = 1, ...m$ are positive Borel measures in the unit disk $D_1$.*

Finally some results of this paper can be extended to bounded symmetric domains and even partially to more general minimal bounded homogeneous domains

(see [19]-[21] for some copies of preliminary results we use in our proof of Theorem 3.1 related with r-lattices, for Forelly-Rudin type estimates, and projection theorems). It is also an interesting separate problem to try to extend our theorems to the class of all admissible domains in higher dimensions (see [23] and various references there for various results related with analytic spaces on these type domains).

**Remark 3**. Recently C. Nana and B. Sehba (see [30]) obtained a sharp embedding theorem in tube domains over symmetric cones for Bergman spaces. Their proof hinges heavily on an estimate from below for Bergman kernel on Bergman balls in tube domains. Note that the same estimate, but in a stronger form is the core of our proof of the main result for tube domains over symmetric cones.

## References


[1] P. Ahern, M. Jevtic, *Duality and multipliers for mixed norm spaces*, Mich. Math. Journal 30, 1983, 53-64.

[2] M. Jevtic, *Sur la derive de la function atomigee*, C.R. Acad. Sc. Paris, 292 (1981), 201-203.

[3] M. Jevtic, M. Pavlovic, R. Shamoyan, *A Note on the diagonal mapping in spaces of analytic functions in the unit polydisc*, Publ. Math. Debrechen, 2009, 74,1-14.

[4] M. Jevtic, *Zeros of functions in $B^q$ spaces*, Mat. Vesnik,5,1 981,59-64.

[5] B. Sehba, *Hankel operation on Bergman spaces in tube domains over symmetric cones*, Int. eq. oper theory, E2, 2008, 233-245.

[6] M. Abate, T. Raissy, A. Saracco, *Toeplitz operators and Carleson measures in strongly pseudoconvex domains*, Journal. Funct. Anal, 263, 2012, 3449-3491.

[7] K. Zhu, *Spaces of holomorphic functions in the unit ball*, Springer - Verlag, N-Y, 2005.

[8] F. Shamoyan, A. Djrbashian, *Topics in the theory of $A^p_\alpha$ spaces*, Teubner - Texte zur Math, b.105, Teubner, Leipzig, 1988.

[9] R. Shamoyan, S. Li,*On some properties of analytic spaces related with the Bergman metric ball*,Bull.Iran.Math.Society,2008.

[10] R.Shamoyan, E .Povpritz, *Embedding theorems and multifunctional analytic spaces in products of strongly pseudoconvex domains with smooth boundary*, Kragujevac Math. Journal, 2013.

[11] D.Bekolle, A.Bonami, G. Garrigos, C .Nana, M.Peloso, F.Ricci, *Lecture notes on Bergman projectors in tube domain over cones, an analytic and geometric viewpoint*, Proceeding of the International Workshop on Classical Analysis , Yaounde,2001.

[12] D. Bekolle, A. Bonami, G. Garrigos, F. Ricci, *Littlewood-Paley decomposition and Besov spaces related to symmetric cones*, Preprint, 2008, arxiv.

[13] R. Shamoyan, S. Kurilenko, *On sharp embedding theorems for Bergman type spaces in bounded pseudoconvex domains with smooth boundary*, Journal of Siberian Federal University, 2014.

[14] R. Shamoyan, E. Povpritz, *Sharp trace theorems in Bergman type spaces in tubular domains over symmetric cones*, Journal of Siberian Federal University, 2013.

[15] B.Sehba, *Bergman type integral operators in tubular domains over symmetric cones*, Proc. Edinburg Math. Society, 2012.

[16] W. Hayman, *Subharmonic functions*, Cambridge Univ, 1980.

[17] L. Ronkin, *Introduction to the theory of entire functions of several variables*, Moscow, Nauka, 1971.

[18] F. Beatrous, *$L^p$ estimates in pseudoconvex domains*, Michigan Math. Journal, 1985.

[19] L. G. Gheorghe, *Interpolation of Besov spaces and applications*, Le Matematiche, v. LV, 2000, 29-42.

[20] S. Yamaji, *Composition operators on the Bergman spaces in minimal domains*, Hiroshima Math. Journal, 43, 2014, 107-127.

[21] I. Ish ,S. Yamaji, *Some new estimates of Bergman kernel in minimal bounded homogeneous domains*, arxiv, preprint.

[22] R. F. Shamoyan, H. Li, *On some properties of analytic spaces connected with Bergman metric ball*, Bull. Iran. Math. Soc. 34 (2) (2008), 121-139.

[23] Song-Yong Li, W. Luo, *Analysis on Besov spaces II: Embedding and Duality Theorems*, Journal of Math. Analysis and Applications. 333(2) (2007), 1189-1202.

[24] J. Ortega, J. Fabrega, *Holomoprhic Triebel-Lizorkin spaces*, J. Funct. Anal., Vol. 151, (1997), 177-212.

[25] R. F. Shamoyan, O. Mihic, *On traces of $Q_p$ type spaces and mixed norm analytic function spaces in polyballs*, Siauliau Math.Seminar, 5, (13), 2010,101-119.

[26] R. Shamoyan, O. Mihic, *In search of traces of some holomorphic spaces in polyballs*, Revista notas de matematicas, v.4 (1), 2008, 1-23.

[27] R. Shamoyan, O. Mihic, *A note on traces of some holomorphic classes in polyballs*, Journal of Function Spaces and Applications, 2010.

[28] R. Shamoyan, O. Mihic, *On traces in Bergman spaces in bounded strongly pseudoconvex domains*, Doklady NAN Armenii, 2014.

[29] R. Shamoyan, S. Kurilenko, *On traces of Bergman type spaces in unit polyballs*, Palest. Math. Journal, 2015 (to appear).

[30] C. Nana, B. Sehba, *Carleson embeddings and two operators on Bergman spaces of tube domains over symmetric cones*, arxiv 1408.3072v2, august, 2014.