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A LINEAR DIFFUSION-BASED IMAGE RESTORATION APPROACH

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Abstract A robust linear PDE-based image restoration technique is proposed in this paper. The PDE model provided here is based on a linear second-order diffusion equation. Some nonlinear PDE schemes that can be derived from it are also mentioned. A robust and consistent fast-converging finite-difference based numerical approximation algorithm is then developed for the continuous differential model. Some image denoising experiments method comparisons are also described.

Keywords: image filtering, linear second-order PDE model, hyperbolic diffusion equation, numerical approximation scheme.
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1. INTRODUCTION

The PDE-based models have been widely used in the image processing domains in the last 30 years, since conventional approaches have numerous disadvantages and cannot solve a lot of problems related to this field [1]. One of these issues is the preservation of details during the restoration process. The nonlinear PDE-based restoration schemes, such as Perona-Malik anisotropic diffusion framework [2], TV Denoising [3] and other diffusion and PDE variational schemes derived from these popular algorithms [4], clearly outperform the conventional 2D image filters [1], removing the blurring and preserving the main image details.

Because the second-order nonlinear diffusion solutions may produce the unintended staircase effect, many improved second-order nonlinear PDE techniques alleviating this effect, such as Adaptive TV Denoising [5], TV-$L1$ model [6], anisotropic HDTV regularizer [7] and TV minimization with Split Bregman [8], have been developed recently. We also constructed numerous anisotropic diffusion and variational restoration approaches that reduce considerably the Gaussian noise, preserve the details (edge, corners) and overcome the staircasing [9]–[12]. Although these nonlinear diffusion schemes reduce the undesired effects, they have other drawbacks, like the computational cost and the high running time.

Therefore, some improved linear PDE-based denoising algorithms could represent a solution. The existing linear diffusion-based schemes are affected often by the image blurring effect and the absence of the localization property. Also, they could dislocate the image edges when moving from finer to coarser scales [13]. For this reason, we describe here an improved linear hyperbolic PDE-based denoising method that overcome these disadvantages and also executes quite fast and avoids the staircase effect.

The linear second-order PDE model described in the following section achieves satisfactory restoration results, removes the blurring effect and has the localization property [13]. That means the solution of this second-order equation is propagating with finite speed. Also, our proposed hyperbolic scheme may be further transformed, so that to produce some effective nonlinear PDE restoration models.

A consistent numerical discretization scheme is developed for the continuous filtering model, being described in the third section. The obtained explicit iterative discretization procedure is based on finite differences and converges fast to a solution representing the enhanced image. Some successful denoising tests and method comparisons are discussed in the fourth section. Our article ends with a conclusion section, acknowledgements and references.

2. ROBUST PDE-BASED IMAGE DENOISING MODEL

In this section we propose a diffusion-based image denoising scheme model. Our model is composed of a linear second-order hyperbolic PDE and a set of boundary conditions. Therefore, we have:

$$\begin{cases} \alpha^2 \dfrac{\partial^2 u}{\partial t^2} + \beta \dfrac{\partial u}{\partial t} - \dfrac{\gamma^2}{2} \Delta u + E \cdot \nabla u = 0 \\ u(0,x,y) = u_0(x,y),\ (x,y) \in \Omega \\ u(t,x,y) = 0,\ (x,y) \in \partial\Omega \end{cases} \tag{1}$$

where the domain $\Omega \subseteq (0,\infty) \times R^2$, $\alpha,\beta,\gamma \in (0,1]$, u_0 represents the initial image, which is affected by the Gaussian noise, and the function $E : R^2 \to R^2$ takes the following form:

$$E(x,y) = (e^{-\eta(x^2+y^2)}, e^{-\xi(x^2+y^2)}) \tag{2}$$

where $\eta, \xi > 0$. The second-order PDE provided by (1) represents a non-Fourier model for the heat propagation. It is also well-posed, since it has a unique weak solution u that is propagating with finite speed [14]. That means this well-posed PDE-based model has also the localization property [13].

That unique and weak solution of the model (1) is approximated numerically by applying an explicit finite-difference based discretization scheme that is consistent to the continuous model. The discretization approach is described in the next section.

The linear PDE-based denoising model developed by us can be further modified, so that much more performant nonlinear PDE image restoration schemes be obtained. Thus, both second-order and fourth-order nonlinear diffusion approaches could be derived from this linear hyperbolic equation-based technique. Second-order nonlinear

PDE models that would provide better denoising results can be obtained by replacing the coefficient $\frac{\gamma^2}{2}$ to some function in (1):

$$\alpha^2 \frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} - f(\Delta u)\nabla^2 u + E \cdot \nabla u = 0 \tag{3}$$

The PDE given by (3) can be further transformed into the following nonlinear fourth-order hyperbolic PDE, by applying a Laplacian operator:

$$\alpha^2 \frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} - \nabla^2 (f(\Delta u)\nabla^2 u) + E \cdot \nabla u = 0 \tag{4}$$

where $\nabla^2 u = \Delta u$ and f is a properly selected positive function. The nonlinear PDE models of this type will represents the focus of our future work in this domain.

3. AN EXPLICIT NUMERICAL APPROXIMATION SCHEME

We propose a consistent numerical approximation scheme for the discretization of the continuous PDE model (1). This discretization technique is based on the finite-difference method [15]. Therefore, let us consider a space grid size of h and a time step Δt. The space and time coordinates are quantized as:

$$\begin{gathered} x = ih, \; y = jh, \; t = n\Delta t, \\ \forall\, i \in \{0, 1, ..., I\}, \; j \in \{0, 1, ..., J\}, \; n \in \{0, 1, ..., N\} \end{gathered} \tag{5}$$

The diffusion-based equation given by (1) leads to:

$$\begin{aligned} \alpha^2 \frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} - \frac{\gamma^2}{2} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\ + (e^{-\eta(x^2+y^2)}, e^{-\xi(x^2+y^2)}) \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) = 0 \end{aligned} \tag{6}$$

which is approximated by using finite differences [15], as:

$$\begin{aligned} &\alpha^2 \frac{u^{n+\Delta t}(i, j) + u^{n-\Delta t}(i, j) - 2u^n(i, j)}{\Delta t^2} + \beta \frac{u^{n+\Delta t}(i, j) - u^{n-\Delta t}(i, j)}{2\Delta t} \\ &- \frac{\gamma^2}{2} \; \frac{u^n(i + h, j) + u^n(i - h, j) + u^n(i, j + h) + u^n(i, j - h) - 4u^n(i, j)}{h^2} \\ &+ (e^{-\eta(i^2/h^2 + j^2/h^2)}, e^{-\xi(i^2/h^2 + j^2/h^2)}) \cdot \\ &\cdot \left(\frac{u^n(i + h, j) + u^n(i - h, j)}{2h}, \frac{u^n(i, j + h) + u^n(i, j - h)}{2h} \right) = 0 \end{aligned} \tag{7}$$

We may take $h = 1$ and $\Delta t = 1$ respectively, and (7) will lead to the following explicit numerical approximation scheme of (1):

$$\begin{aligned} &u^{n+1}(i,j) = \frac{4\alpha^2}{2\alpha^2+\beta} u^n(i,j) + \frac{\beta - 2\alpha^2}{2\alpha^2+\beta} u^{n-1}(i,j) \\ &+ \frac{\gamma^2}{2\alpha^2+\beta} (u^n(i+1,j) + u^n(i-1,j) + u^n(i,j+1) \\ &+ u^n(i,j-1) - 4u^n(i,j)) - \frac{2}{2\alpha^2+\beta} (e^{-\eta(i^2+j^2)}, e^{-\xi(i^2+j^2)}) \cdot \\ &\cdot \left(\frac{u^n(i+1,j) + u^n(i-1,j)}{2}, \frac{u^n(i,j+1) + u^n(i,j-1)}{2} \right) \end{aligned} \tag{8}$$

This iterative denoising algorithm starts with the initial $[I \times J]$ noisy image and applies repeatedly (8), for each $n = 1, 2, ..., N$. The number of iterations of this approximating scheme, N, is quite low, since the discretization algorithm converges fast to the solution representing the optimal denoising.

4. EXPERIMENTS AND METHOD COMPARISON

The described linear PDE-based filtering technique has been successfully on hundred images corrupted with various amounts of Gaussian noise. The proposed image restoration scheme produces satisfactory denoising results while preserving the image details, such as boundaries, corners, very well. Although the image blurring is not completely avoided, our approach overcomes successfully other undesired effects, such as staircase [16] or speckle noise.

The following set of empirically detected parameters of this diffusion-based model provides the optimal smoothing results:

$$\alpha = 0.8,\ \beta = 0.5,\ \gamma = 0.7,\ \eta = 2,\ \xi = 3,\ \Delta t = 1,\ h = 1,\ N = 15 \tag{9}$$

The optimal restoration is reached after a low number of iterations, $N = 15$, which means the proposed method is running quite fast. Its execution time is less than one second. We have performed method comparison and found that our linear diffusion technique outperforms not only the 2D conventional filters, but also numerous linear and nonlinear PDE-based techniques. Its restoration performance has been assessed using Peak Signal-to-Noise Ratio (PSNR) measure [1]. Our filtering approach achieves higher PSNR values than some popular 2D classic filters, such as Gaussian, Average, Median and Wiener [1], and influential nonlinear PDE and variational models, such as Perona-Malik algorithm [2] and TV Denoising [3], as one can see in Table 1.

Table 1. PSNR values for the noise reduction techniques

Our model	Gaussian	Average	Median	Wiener	P-M	TV
26.81(dB)	22.38(dB)	23.17(dB)	23.98(dB)	24.7(dB)	25.89(dB)	25.27(dB)

The denoising results obtained by these approaches are displayed in the next figure, where one can see: a) original [512 × 512] *Barbara* image; b) the image affected by a Gaussian noise of μ = 0.21 and variance = 0.023; c) the image enhanced by our PDE approach; d)–g) restoration results achieved by classic [3 × 3] 2D filters (Gaussian, Averaging, Median, Wiener); h) Perona-Malik model; i) TV Denoising.

Fig. 1.: Method comparison: image denoised by various approaches

Therefore, our linear PDE-based restoration technique provides a better image enhancement than classic filters and linear diffusion models. It removes a higher amount of Gaussian noise and produces a better deblurring result. It also performs better than many nonlinear PDE models and variational schemes [2]-[4], operates much faster than them (converges in fewer steps) and overcomes better the undesired effects [16]. Although many improved second-order nonlinear PDE-based approaches [5]-[8] obtain higher PSNRs than our technique and remove totally the blur effect, sometimes our method is preferable to them, since it avoids completely the staircasing and runs much faster (fewer iterations) than those techniques.

5. CONCLUSIONS

A novel linear PDE image restoration model has been proposed in this article. It produces an effective Gaussian noise reduction while preserving image details, such as boundaries, very well.

The second-order linear hyperbolic diffusion-based scheme elaborated by us constitutes a considerable improvement of the existing linear PDE denoising models, because, unlike those models, it has the localization property and do not dislocate the boundaries when moving to coarser scales. It also outperforms clearly the conventional filtering algorithms, not only by providing a better filtering, but also by providing a much better deblurring effect.

Also, our technique executes much faster than many nonlinear PDE-based models and variational approaches for image restorations. The anisotropic diffusion methods have a higher computational cost, therefore they run slower than our restoration technique. The low execution time of our restoration algorithm is due to the developed fast-converging numerical approximation scheme that is based on the finite-difference method and is consistent to the proposed PDE model. Given its low running time, our restoration technique could be applied successfully to voluminous image database denoising and restoration.

Although our model is outperformed by some improved nonlinear diffusion schemes, it can be much improved such that to become an effective nonlinear PDE filtering solution. We have mentioned some second and fourth-order nonlinear PDE-based approaches that can be derived from it. Those models could represent much better filtering approaches that outperform the state-of-the-art nonlinear diffusion-based algorithms. Those nonlinear hyperbolic models will represent the focus of our future research in image denoising field.

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