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THE CONVEXITY IN THE COMPLEX OF MULTI-ARY RELATIONS

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Abstract For a complex of multi-ary relations [12] it is defined the concept of (k,m)-chain which is a generalization of the concept of chain known from the graph theory. Using (k,m)-chains it is introduced the concept of the distance function and it is proved that this function generate a convexity in the complex of multi-ary relations. It is operating with the concepts of convexity and convex hull, axiomatically defined by F.Levi [29] and we describe the iterative procedure to construct a convex hull for a subset of elements from the complex of multi-ary relations.

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1. INTRODUCTION

Introduced initially for the Euclidean space, the notion of convexity generated valuable theoretical results. These results formed the basis for a significant research direction, known today as convex optimization [10], [23]. In the case of discrete structures represented by graphs, hypergraphs, matroids, etc., various types of convexity, used to solve important practical problems have been studied [33], [34]. The complex of multi-ary relations, studied in [11], [14], [16], is a generalization of classical discrete structures, which offers new possibilities for expansion the concept of convexity. This begins from the generalization of the concept of chain and defining, respectively, the distance function. We will define the concept of multi-ary complex relations according to [14].

Let $X = \{x_1, x_2, ..., x_r\}$ be a finite set of elements and $X = X^1, X^2, ..., X^{n+1}$, $n \geq 1$ a sequence of Cartesian products of the set X. Any not empty subset $R^m \subset X^m$, $1 \leq m \leq n+1$, is said to be an m-ary relation of elements from X. If $X = (x_{i_1}, x_{i_2}, ..., x_{i_m})$ is a sequence from R^m, then any subsequence $(x_{j_1}, x_{j_2}, ..., x_{j_l})$, $1 \leq l \leq m$, which preserves the order of elements of $(x_{i_1}, x_{i_2}, ..., x_{i_m})$ is called a hereditary subsequence.

Definition 1.1. *A finite family of relations $\{R^1, R^2, ..., R^{n+1}\}$, which satisfies the conditions:*
I. $R^1 = X^1 = X$,
II. $R^{n+1} \neq \emptyset$,

III. any hereditary subsequence $(x_{j_1}, x_{j_2}, ..., x_{j_l})$, $1 \leq l \leq m \leq n+1$, *of the sequence* $(x_{i_1}, x_{i_2}, ..., x_{i_m}) \in R^m$ *belongs to the l-ary relation* R^l, *is called* ***a generalized complex of multi-ary relations (G-complex)*** *and is denoted by* $\mathfrak{R}^{n+1} = (R^1, R^2, ..., R^{n+1})$.

From this definition we obtain that the set R^m of a generalized complex $\mathfrak{R}^{n+1}$ is not empty for each $1 \leq m \leq n+1$. The elements of relation R^1 are called vertexes, and the elements of relation R^2 - edges.
The study of generalized complex of multi-ary relations is interesting because this notion covers a lot of classical notions like graphs [2], [3], [7], hypergraphs [2], [4], matroids [5], [6], [42], simplicial complexes [9], [32] etc.
In the following, we will consider that the elements of the set R^m, $1 \leq m \leq n+1$, are sequences that don't contain repetitions of elements from X, even if the results exposed below could be extended also for the generalized complex of relations.
The concept of convexity was studied by many mathematicians. In this paper we use some definitions in Levi's variant [29]. Known classical results related to the development of general convexity theory were further expanded by J.W.Elliss [18], P.C.Hammer [20]-[22], D.C.Kay & E.W.Womble [25], G.Sierksma [35], [36]. Different models of convexity are currently known. They have appeared in connection with the need to solve some theoretical-practical problems which can be found in the works of many famous mathematicians: P.Soltan [8], [40], I.Sergienko [33], [34], M.Kovalev [26], [27], V.Soltan [38] etc.
Firstly we will remember the definition of convexity done by F.Levi [29]. Let $\mathcal{P}(X)$ be the family of all subsets of an arbitrary set X. We choose a subfamily $\Phi \subset \mathcal{P}(X)$.

Definition 1.2. *The family of sets* $\Phi \subset \mathcal{P}(X)$ *with the properties:*
a) $X \in \Phi$;
b) if $A_1, A_2 \in \Phi$, *then* $A_1 \cap A_2 \in \Phi$,
is called ***convexity*** *in* X. *The pair* (X, Φ) *is called convex space, and the elements of* Φ *- convex sets.*

According to this definition, for every set of elements X can be indicated a family of subsets which forms a convexity in X. For example, in the linear space the family of all spheres with the center in the origin of the coordinate system is a convexity. Obviously, in the case of linear space, the respective convexity is not unique.
Taking into account the Definition (1.2) we can say that if Φ is a convexity in a space X, then any subset $A \subset X$ belongs to at least one of the convex sets from Φ. In its turn, this means that in X there is a minimal convex set which contains the subset $A \subset X$. The respective set is also called the convex hull of A. We will present here the definition of the convex hull, borrowed from paper [29], even if in the specialized literature there are also equivalent variations thereof [25], [38], [40].

Definition 1.3. *The application* $\varphi : \mathcal{P}(X) \to \mathcal{P}(X)$, *which satisfies the relations:*
a) $A \subseteq \varphi(A)$, *for any subset* $A \subset X$;
b) $\varphi(\varphi(A)) = \varphi(A)$, *for any subset* $A \subset X$;

c) $\varphi(A) \subseteq \varphi(B)$, for any two subsets $A, B \in \mathcal{P}(X)$ so that $A \subset B$,
is called ***convex hull*** *in the space X.*

2. *m*-DIMENSIONAL *k*-CHAIN IN A COMPLEX OF MULTI-ARY RELATIONS

We will now examine the space, which elements are all the sequences of connected complex $\mathfrak{R}^{n+1} = (R^1, R^2, ..., R^{n+1})$. In other words, we will examine the space determined by the union $\bigcup_{i=1}^{n+1} R^i$. To define in this space a convexity we will introduce the notion of chain, different of the notion of linear chain, but seems more natural in the context of those further examined. Let r_i^k, r_j^k be two arbitrary elements of the k-are relation R^k. Obviously, for r_i^k, r_j^k we can always create a sequence ω which consists of the elements of the complex $\mathfrak{R}^{n+1}$, and has the following two properties:
a) the first and the last elements from ω coincide with r_i^k, and, respectively, r_j^k;
b) any two neighboring elements from ω have non-empty intersection.
Such sequences are too general and may not be useful to examine some particularities of the complex of multi-are relations.

Definition 2.1. *The sequence of elements $r_{t_1}^m, r_{t_2}^m, ..., r_{t_s}^m$ of m-ary relation R^m with the properties:*
1) $r_i^k \subset r_{t_1}^m$, $r_j^k \subset r_{t_s}^m$, $1 \le k \le m$;
2) $r_{t_p}^m \cap r_{t_{p+1}}^m \in R^l$, $k \le l < m$, for any $1 \le p \le s-1$,
is called ***m-dimensional k-chain*** *with the extremities in r_i^k, r_j^k and is denoted by ${}^kL^m(r_i^k, r_j^k)$, $1 \le k < m \le n+1$. The number s is called the length of this chain.*

Sometimes we will use the name (k, m)-chain for the chain ${}^kL^m(r_i^k, r_j^k)$. We easily convince ourselves that the (k, m)-chain is a generalization of the concept of chain known from the graph theory [2], [3].

Lemma 2.1. *If $r_{t_1}^m, r_{t_2}^m, ..., r_{t_s}^m$ represents a (k, m)-chain which connects two hereditary subsequences $r_i^k \subset r_{t_1}^m$ and $r_j^k \subset r_{t_s}^m$, then in $\mathfrak{R}^{n+1}$ exists (h, m)-chain which connects any two hereditary subsequences from $r_{t_1}^m$ and $r_{t_s}^m$, each containing h elements, $1 \le k < m$, $1 \le h < k$.*

Proof. Indeed, if in the case of m-dimensional k-chain ${}^kL^m(r_i^k, r_j^k) = (r_{t_1}^m, r_{t_2}^m, ..., r_{t_s}^m)$ we choose the hereditary subsequences $r_1^h \subset r_{t_1}^m$, $r_2^h \subset r_{t_s}^m$, so that $r_1^h \not\subset r_{t_1}^m \cap r_{t_2}^m$ and $r_2^h \not\subset r_{t_{s-1}}^m \cap r_{t_s}^m$, then we may consider ${}^kL^m(r_i^k, r_j^k) = (r_{t_1}^m, r_{t_2}^m, ..., r_{t_s}^m) = {}^hL^m(r_1^h, r_2^h)$. In the case when, $r_1^h \subset r_{t_1}^m \cap r_{t_2}^m$, and $r_2^h \not\subset r_{t_{s-1}}^m \cap r_{t_s}^m$, we may consider the chain ${}^hL^m(r_1^h, r_2^h) = (r_{t_2}^m, ..., r_{t_s}^m)$.
The other two remained cases:

$$1)\ r_1^h \not\subset r_{t_1}^m \cap r_{t_2}^m \text{ and } r_2^h \subset r_{t_{s-1}}^m \cap r_{t_s}^m;$$

$$2)\ r_1^h \subseteq r_{t_1}^m \cap r_{t_2}^m \text{ and } r_2^h \subseteq r_{t_{s-1}}^m \cap r_{t_s}^m$$

are examined similarly. ∎

Definition 2.2. *If for any two elements r_i^k, r_j^k of the complex $\Re^{n+1}$ exists at least one m-dimensional k-chain which connects them, then $\Re^{n+1}$ is called (k,m)-**connected complex**.*

According to the Definition (2.2), any connected complex of multi-ary relations $\Re^{n+1} = (R^1, R^2, ..., R^{n+1})$ can be considered also as an $(1,1)$-connected complex.
Using the m-dimensional k-chains which connect the pairs of elements with the same size k, $0 \leq k \leq m$, of a complex of multi-ary relations $\Re^{n+1}$, we will define the concept of convexity and convex hull for this complex.
Moreover, we will show that a convexity in $\Re^{n+1}$ determines univocally a convex hull, and vice versa - any convex hull univocally determines a convexity in $\Re^{n+1}$.

3. *m*-DIMENSIONAL METRIC *k*-SEGMENT IN A COMPLEX OF MULTI-ARY RELATIONS

The general theory of the convexity in an arbitrary space X was substantiated, beginning with the Definition (1.2), through the efforts of several mathematicians. Such renowned scientists as J.W.Ellis [18], P.C.Hammer [20]-[22], J.Eckhoff [17], D.C.Kay & E.W.Womble [25], K.E.Jamison [24], G.Sierksa [35], [36], M.L.J. van de Vel [43], M.Krein & V.Smulian [28] etc. have essentially contributed to the development of this direction of research in mathematics. Also, various models of convexity have been studied, d-convexity having a special role. For the first time the d-convex sets were examined in the paper [30] by K.Menger. Later this notion was "rediscovered" again in the works of other mathematicians, in connection with their attempt to solve various theoretical and applicative problems: J. de Groot [19], A.Aleksandrov & V.Zalgaller [1], F.Toranzos [41], E.Soetens [37], P.Soltan & K.Prisakaru [39]. Being introduced independently in geometry, topology, functional analysis and graph theory, the d-convexity proved to be a successful model of convexity, which has contributed to solving important applicative problems [15], [31], [40].
In the case of the complex of multi-ary relations $\Re^{n+1} = (R^1, R^2, ..., R^{n+1})$ we will consider that every k-ary relation R^k, $1 \leq k \leq n$, determines a space, its elements being the subsequences from R^k. On each such space we will define the convexity according to the Definition (1.2), using the m -dimensional k-chains introduced by the Definition (2.1), $k < m \leq n+1$. Those exposed below are an extension of the ideas presented in the monograph [13].
For further researches we will need the following two concepts:
1. **Minimum m-dimensional k-chain** which connects two elements of the same size $r_i^k, r_j^k \in R^k$, $1 \leq k < m \leq n+1$. Thus will be called the (k,m)-chain ${}^kL^m(r_i^k, r_j^k)$ with the smallest length, which connects the sequences $r_i^k, r_j^k \in R^k$.

2. **Distance function** $d_k^m : R^k \times R^k \to N$. Thus will be called the function which puts in correspondence for every two subsequences r_i^k, r_j^k with the same size k from $\mathfrak{R}^{n+1}$ a number equal to the length of the minimum m-dimensional k -chain, $1 \le k < m \le n+1$. To prove that the function d_k^m is a distance in the space R^k, we will show that it possesses the properties of a metric defined in R^k. (In the case when between the elements r_i^k, r_j^k does not exist (k,m)-chain, is considered that $d_k^m(r_i^k, r_j^k) = +\infty$).
It is easy to verify the following result:

Lemma 3.1. *The function $d_k^m : R^k \times R^k \to N$, so that $d_k^m(r_i^k, r_j^k)$ is a number equal with the length of the minimum (k,m)-chain with the extremities in $r_i^k, r_j^k \in R^k$, possesses the properties:*
a) $d_k^m(r_i^k, r_j^k) \ge 0$, for any two elements $r_i^k, r_j^k \in R^k$ and $d_k^m(r_i^k, r_j^k) = 0$ if and only if $r_i^k = r_j^k$;
b) $d_k^m(r_i^k, r_j^k) = d_k^m(r_j^k, r_i^k)$, for any two elements $r_i^k, r_j^k \in R^k$;
c) $d_k^m(r_i^k, r_j^k) \le d_k^m(r_i^k, r_t^k) + d_k^m(r_t^k, r_j^k)$, for any three elements $r_i^k, r_j^k, r_t^k \in R^k$.

Being based on this lemma we can consider that on the basis of a complex of multi-ary relations $\mathfrak{R}^{n+1} = (R^1, R^2, ..., R^{n+1})$ we can define metric spaces (R^k, d_k^m), $1 \le k < m \le n+1$.
For the subsequences $r_i^k, r_j^k \in R^k$, we will name m**-dimensional metric** k**-segment** the set:

$$\langle r_i^k, r_j^k \rangle_m = \{r_l^k \in R^k : d_k^m(r_i^k, r_j^k) = d_k^m(r_i^k, r_l^k) + d_k^m(r_l^k, r_j^k)\},$$

where $1 \le k < m \le n+1$. It is also said that the metric segment $\langle r_i^k, r_j^k \rangle_m$ connects the sequences r_i^k, r_j^k, which are also called the extremities of this segment.

4. THE CONVEXITY DETERMINED BY THE DISTANCE FUNCTION d_k^m

Let D^k be the family of all sets $A \subset R^k$ with the property: for every two k-dimensional sequences $r_i^k, r_j^k \in R^k$ the inclusion $\langle r_i^k, r_j^k \rangle_m \subset A$ takes place.

Theorem 4.1. *The family D^k represents a convexity in R^k.*

Proof. We notice first of all that the relation $R^k \in D^k$ is true. Now let us examine the intersection of any two sets $A, B \in D^k$. If we have two elements $r_i^k, r_j^k \in A \cap B$, then because each of sets A and B are from D^k, we obtain:

$$\langle r_i^k, r_j^k \rangle_m \subset A;$$
$$\langle r_i^k, r_j^k \rangle_m \subset B.$$

Therefore, $\langle r_i^k, r_j^k \rangle_m \subset A \cap B$, which means that $A \cap B$ is a set from D^k. From those proved, it results that the family D^k has the properties from the Definition (1.2). Therefore, D^k is a convexity in R^k, $1 \le k < m \le n+1$. ∎

We will name the elements of the family D^k the (k, m)-convex sets of the complex of multi-ary relations $\mathfrak{R}^{n+1}$. Further we will show, that on the base of the convexity D^k, on the set of all subsets $\mathcal{P}(R^k)$ of the space R^k, we can define an application $\varphi : \mathcal{P}(R^k) \to \mathcal{P}(R^k)$ which is a convex hull in R^k.

5. THE CONVEX HULL DETERMINED BY THE CONVEXITY OF THE COMPLEX OF MULTI-ARY RELATIONS

Theorem 5.1. *The application $\varphi : \mathcal{P}(R^k) \to \mathcal{P}(R^k)$, which for each element $A \in \mathcal{P}(R^k)$ puts in correspondence a minimum (k, m)-convex set from D^k, which contains the set A, represents a convex hull.*

Proof. Let be the convexity defined above:

$$D^k = \{B \in \mathcal{P}(R^k) : \langle r_i^k, r_j^k \rangle_m \subset B, \text{ for } \forall r_i^k, r_j^k \in B\} \subset \mathcal{P}(R^k),$$

and $\varphi : \mathcal{P}(R^k) \to \mathcal{P}(R^k)$ - the application which satisfies the requirements of the theorem. We mention that any minimum (k, m)-convex set, which contain an arbitrary set $A \in \mathcal{P}(R^k)$, is obtained by the intersection of all (k, m)-convex sets, which contains the set A. Therefore, we can consider true the equality:

$$\varphi(A) = \cap\{B \in D^k : A \subset B\} \tag{1}$$

for any set $A \in \mathcal{P}(R^k)$.

To prove the theorem it is enough to prove that the application which respects the condition (1) possesses the metric properties (see Definition (1.3)).

a) If A is an arbitrary set from R^k, then the inclusion $A \subseteq \varphi(A)$ is evident, since $\varphi(A)$ respects the condition (1).

b) The equality $\varphi(\varphi(A)) = \varphi(A)$ follows from the fact that $\varphi(A) \in D^k$, representing the intersection of some sets from D^k (see Theorem (4.1)).

c) To prove that application φ satisfies third relation from Definition (1.3), we will consider two arbitrary sets A and B from D^k which respect the condition $A \subset B$. According to the relation (1) we obtain:

$$\varphi(A) = \cap\{E \in D^k : A \subseteq E\}.$$

A set $E \in D^k$ which contains the set A may, in its turn, contains or not the set B. This means that the following inclusion is true:

$$\begin{gathered} \cap\{E \in D^k : A \subset E\} \subset (\cap\{E \in D^k : A \subset E \text{ and } B \not\subset E\})\cup \\ \cup(\cap\{E \in D^k : B \subset E\}) \subset \\ \subset (\cap\{F \in D^k : B \subset F\}) = \varphi(B). \end{gathered}$$

Consequently, $\varphi(A) \subset \varphi(B)$.

From those proved above, it results that, if the application $\varphi : \mathcal{P}(R^k) \to \mathcal{P}(R^k)$ satisfies the relation (1), then it is a convex hull in R^k. ∎

Through Theorem (5.1) it was shown that, knowing the family of convex sets in a complex of multi-ary relations, we can define the convex hull.

6. AN ITERATIVE PROCEDURE TO CONSTRUCT A CONVEX HULL IN THE COMPLEX OF MULTI-ARY RELATIONS

In the following, the set $\varphi(A)$ of an arbitrary subset of sequences $A \subset R^k$ will be called the convex hull of A and will be denoted by $d_k^m - conv(A)$. We describe an iterative procedure to construct a convex hull $d_k^m - conv(A)$.

I. Initially we consider $A_0 = A$.

II. Let the set A_q, $q \geq 0$ was obtained. For every two distinct elements $r_i^k, r_j^k \in A_q$, we determine the m -dimensional metric k-segment $\langle r_i^k, r_j^k \rangle_m$. We obtain the set:

$$A_{q+1} = A_q \cup \Big(\bigcup_{r_i^k, r_j^k \in A_q} \langle r_i^k, r_j^k \rangle_m \Big).$$

III. If $A_{q+1} \neq A_q$, then for A_{q+1} we repeat the operation applied to the set A_q at step II.

IV. If $A_{q+1} = A_q$, then we consider $d_k^m - conv(A) = A_q$.

Through application of the iterative procedure, described by steps I-IV, an extension of the set of sequences $A \in \mathcal{P}(A)$ is obtained, by adding all m-dimensional metric k-segments with the extremities in the elements $r_i^k, r_j^k \in A_q$. In the following, this procedure will be denoted by:

$$P(A) = A \cup \Big(\bigcup_{r_i^k, r_j^k \in A} \langle r_i^k, r_j^k \rangle_m \Big).$$

Because $A \subset \Big(\bigcup_{r_i^k, r_j^k \in A} \langle r_i^k, r_j^k \rangle_m \Big)$, we can write $P(A) = \bigcup_{r_i^k, r_j^k \in A} \langle r_i^k, r_j^k \rangle_m$.

Using this operation of extension of the set A, the construction of the convex hull $d_k^m - conv(A)$ can be associated with the construction of the set:

$$P_0(A) \subset P_1(A) \subset P_2(A) \subset ... \subset P_q(A) = P_{q+1}(A),$$

where $P_0(A) = A$,

$$P_s(A) = P(P_{(s-1}(A)),\ s = 1, 2, 3, ..., q + 1.$$

In this condition $d_k^m - conv(A) = P_q(A)$.

Let us consider the application $\hat{\varphi} : \mathcal{P}(R^k) \to \mathcal{P}(R^k)$. In accordance with this application, for a set $A \in \mathcal{P}(R^k)$ is put in correspondence the convex set $d_k^m - conv(A)$ obtained using the iterative procedure described above.

Lemma 6.1. *The application $\hat{\varphi}$ is a convex hull in R^k.*

Proof. To prove this lemma we will establish that the application $\hat{\varphi}$ verifies the convex hull properties from the Definition (1.3).

a) Let A be an arbitrary set of elements from R^k. If for any two elements $r_i^k, r_j^k \in A$ is respected the relation $\langle r_i^k, r_j^k \rangle_m \subset A$, then, on the bases of the iterative procedure described above, the equality occurs:

$$A = d_k^m - conv(A).$$

Otherwise, it constructs the sequence of sets

$$A = P_0(A) \subset P_1(A) \subset ... \subset P_q(A) = d_k^m - conv(A) \tag{2}$$

hence, the first convex hull property, indicated in the Definition (1.3), is respected.
b) Let's check the second property.
In conformity with the iterative procedure described above, for the set of sequences $A \in R^k$ we form the sequence (2). Thus:

$$\varphi(A) = d_k^m - conv(A) = P_q(A),$$

which means that for any two elements $r_i^k, r_j^k \in P_q(A)$ the following relation take place:

$$\langle r_i^k, r_j^k \rangle_m \subset P_q(A),$$

hence, we conclude:

$$d_k^m - conv(A_q) = d_k^m - conv(A).$$

As a result we obtain:

$$d_k^m - conv(d_k^m - conv(A)) = d_k^m - conv(A).$$

c) Let be A and B two sets from $\mathcal{P}(R^k)$ with the property $A \subset B$.
As any two elements r_i^k, r_j^k from A also belong to B, the construction of the convex hulls for A and B may be represented by one of the schema:

$$\begin{array}{l} P_0(A) \subset ... \subset P_q(A) = d_k^m - conv(A) \\ \ \ \cap \qquad\qquad\quad \cap \\ P_0(B) \subset ... \subset P_q(B) \subset P_{q+1}(B) \subset ... \subset P_t(B) = d_k^m - conv(B) \end{array} \tag{3}$$

or

$$\begin{array}{l} P_0(A) \subset ... \subset P_q(A) \subset P_{q+1}(A) \subset ... \subset P_t(A) = d_k^m - conv(A) \\ \ \ \cap \qquad\qquad\quad \cap \\ P_0(B) \subset ... \subset P_q(B) = d_k^m - conv(B) \end{array} \tag{4}$$

In each of these two schemas, the relations $P_i(A) \subset P_i(B)$, $0 \le i \le q$, are evident, since any two distinct elements from $P_i(A)$ also belong to the set $P_i(B)$.
In the case of schema (3) we obtain the sequence of inclusions

$$P_q(A) \subset P_q(B) \subset P_{q+1}(B) \subset ... \subset P_t(B),$$

from where we obtain:

$$d_k^m - conv(A) \subset d_k^m - conv(B).$$

The realization of the schema (4) means that the following relation is true:

$$d_k^m - conv(A) \subset P_0(B) = B.$$

Indeed, according to the schema (4), we have the situation when the convex hull of the set A does not exceed the set B, because it is supposing that the number of steps applied to construct the convex hull of set A is greater than the number of steps required to construct the convex hull of set B. So the conclusion is that $q = 1$. Therefore, and in the case of the schema (4) the following inclusion takes place:

$$d_k^m - conv(A) \subset d_k^m - conv(B).$$

As a result, the third property from Definition (1.3) is satisfied, which means that $\hat{\varphi}$ is a convex hull. ∎

Theorem 6.1. *The convex hull $\hat{\varphi}$ defines univocally a convexity D^k in the set of k-dimensional sequences of the complex $\Re^{n+1} = (R^1, R^2, ..., R^{n+1})$ as follows: D^k is formed from all the sets $A \subset R^k$, for which the sequence* (2) *consists of a single element $P_0(A)$.*

Proof. To prove the affirmation from the theorem it is enough to show that the family $D^k = \{A \subset R^k : P_0(A) = P_1(A)\}$ satisfies the conditions from the Definition (1.2). We mention that the relation $R^k \in D^k$ is obvious. We will focus to prove the second condition of the Definition (1.2).
We choose two arbitrary sets A and B from D^k. For these sets the following relation is evident:

$$\hat{\varphi}(A \cap B) \subset \hat{\varphi}(A) \cap \hat{\varphi}(B) \tag{5}$$

Since, as we mentioned above,

$$D^k = \{A \subset R^k : P_0(A) = P_1(A)\},$$

we obtain

$$\hat{\varphi}(A) \cap \hat{\varphi}(B) = A \cap B. \tag{6}$$

From the relations (5) and (6) it follows:

$$\hat{\varphi}(A \cap B) \subset A \cap B \tag{7}$$

On the basis of the property a) of the convex hull (see Definition (1.3)) we deduce:

$$A \cap B \subset \hat{\varphi}(A \cap B). \tag{8}$$

In turn, the relations (7) and (8) imply the equality:

$$\hat{\varphi}(A \cap B) = A \cap B.$$

On the basis of the relation $D^k = \{A \subset R^k : P_0(A) = P_1(A)\}$, we deduce that relation $A \cap B \in D^k$ occurs. Therefore D^k is a convexity in the complex of multi-ary relations $\mathfrak{R}^{n+1} = (R^1, R^2, ..., R^{n+1})$.
Now we will prove the uniqueness of the convexity, determined by $\hat{\varphi}$. Assume the contrary. Let us suppose that there is another convexity $\hat{D}^k$, determined by the convex hull $\hat{\varphi}$, in accordance with the relation $\hat{D}^k = \{A \subset R^k : \hat{\varphi}(A) = A\}$. The equality $\hat{D}^k = D^k$ results from the fact that $\hat{\varphi}(A) = A$ if and only if $A \in D^k$. ∎

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