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ON A CLASS OF NONLINEAR INFINITE SEMIPOSITONE PROBLEMS WITH SIGN-CHANGING WEIGHTS

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Abstract We study the existence of positive solution to the nonlinear problem

$$\begin{cases} -\Delta u = \lambda a(x)[f(u) - \frac{1}{u^\alpha}], & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where λ is a positive parameter and $\alpha \in (0,1)$ Ω is a bounded domain with smooth boundary. Here $a(x)$ is C^1 sing-changing function that maybe negative near the boundary and f is C^1 nondecreasing functions such that $f : (0,\infty) \to (0,\infty)$; $f(s) > 0$ for $s > 0$ and $\lim_{s\to\infty} \frac{f(s)}{s} = 0$. We discuss the existence of positive solution when f and $a(x)$ satisfy certain additional conditions. We use the method of sub-supersolution to establish our result.

Keywords: positive solutions, infinite semipositone, sub-supersolutions.
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1. INTRODUCTION

In this paper, we consider the existence of positive solution for the nonlinear problem

$$\begin{cases} -\Delta u = \lambda a(x)[f(u) - \frac{1}{u^\alpha}], & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

where λ is a positive parameter, Ω is a bounded domain with smooth boundary, $\alpha \in (0,1)$. Here $a(x)$ is C^1 sing-changing function that maybe negative near the boundary and f is C^1 nondecreasing functions such that $f : (0,\infty) \to (0,\infty)$; $f(s) > 0$ for $s > 0$.

Let $F(s) = f(s) - \frac{1}{s^\alpha}$, then $\lim_{s\to 0} F(s) = -\infty$ and hence we refer to (1) as an infinite semipositone problem. Recently, such infinite problems have been studied in [1], [3], [4]. Also in [6], the authors have studied the existence results for the problem (1) in the case $a \equiv 1$. Here we focus on further extending the study in [6] to the problem (1). In fact, we study the existence of positive solution to the problem (1) with sign-changing weight functions $a(x)$.n It is well documented that the study of positive solution such semipositone problem is mathematically very challenging

[1, 5]. Our approach is based on the method of sub-super solutions (see [2, 7]).

To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -\Delta\phi = \lambda\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

Let ϕ be the eigenfunction corresponding to the first eigenvalue λ_1 of (2) such that $\phi(x) > 0$ in Ω and $\|\phi\|_\infty = 1$. Let $m, \mu, \delta > 0$ be such that

$$\mu \leq \phi \leq 1, \quad x \in \Omega - \overline{\Omega_\delta}, \tag{4}$$

$$\frac{2}{1+\alpha}(\frac{1-\alpha}{1+\alpha})|\nabla\phi|^2 \geq m, \quad x \in \overline{\Omega_\delta}, \tag{5}$$

with $\overline{\Omega_\delta} := \{x \in \Omega | d(x, \partial\Omega) \leq 1\}$. This is possible since $|\nabla\phi| \neq 0$ on $\partial\Omega$ while $\phi = 0$ on $\partial\Omega$. We will also consider the unique solution $e \in W_0^{1,2}(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta e = 1, & x \in \Omega, \\ e = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result, it is known that $e > 0$ in Ω and $\frac{\partial e}{\partial n} < 0$ on $\partial\Omega$.

Here we assume that the weight functions $a(x)$ takes negative values in $\overline{\Omega_\delta}$, but require $a(x)$ be strictly positive in $\Omega - \overline{\Omega_\delta}$. To be precise we assume that there exist positive constant a_0, a_1 such that $a(x) \geq -a_0$, on $\overline{\Omega_\delta}$ and $a(x) \geq a_1$ on $\Omega - \overline{\Omega_\delta}$.

2. EXISTENCE RESULT

In this section, we shall establish our existence result via the method of sub-supersolution. A function ψ is said to be a subsolution of 1, if it is in $C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta\psi \leq \lambda a(x)[f(\psi) - \frac{1}{\psi^\alpha}], & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega, \end{cases}$$

and z is said supersolution of (1), if it is in $C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta z \geq \lambda a(x)[f(z) - \frac{1}{z^\alpha}], & x \in \Omega, \\ z = 0. & x \in \partial\Omega, \end{cases}$$

Then the following result holds :

Lemma 2.1. *(See [2]) If there exist a subsolution ψ and supersolution z such that $\psi \le z$ in Ω then* (1) *has a solution u such that $\psi \le u \le z$.*

We make the following assumptions :

(H1) $f : (0,\infty) \to (0,\infty)$ is C^1 nondecreasing function such that $f(s) > 0$ for $s > 0$.

(H2) $\lim_{s\to\infty} \frac{f(s)}{s} = 0$.

(H3) Suppose that there exists $\epsilon > 0$ such that :

$$\frac{\lambda_1 f(\epsilon)}{m(1+\alpha)} \le \min\left\{\frac{2^\alpha}{\epsilon^\alpha}, \frac{Na_1}{2a_0}\right\}, \ f(\frac{1}{2}\mu\epsilon) > (\frac{2}{\mu\epsilon})^\alpha,$$

where $N = f(\frac{1}{2}\mu\epsilon) - (\frac{2}{\mu\epsilon})^\alpha$.

We are now ready to give our existence result.

Theorem 2.1. *Let $(H1)-(H3)$ hold. Then there exists a positive solution of* (1) *for every $\lambda \in [\lambda_*(\epsilon), \lambda^*(\epsilon)]$, where*

$$\lambda^* = \frac{m\epsilon}{2a_0 f(\epsilon)} \ \text{ and } \ \lambda_* = \max\left\{\frac{\epsilon^{\alpha+1}\lambda_1}{2^\alpha a_0(1+\alpha)}, \frac{\epsilon\lambda_1}{(1+\alpha)Na_1}\right\}.$$

Proof. We shall verify that $\psi = \frac{1}{2}\epsilon\phi^{\frac{2}{1+\alpha}}$ is a sub-solution of (1). A calculation shows that

$$\nabla\psi = \frac{1}{2}\epsilon(\frac{2}{1+\alpha})\phi^{\frac{1-\alpha}{1+\alpha}}\nabla\phi,$$

and

$$\begin{aligned}
-\Delta\psi &= -\nabla(\nabla\psi) \\
&= -\nabla\Big(\frac{1}{2}\epsilon(\frac{2}{1+\alpha})\phi^{\frac{1-\alpha}{1+\alpha}}\nabla\phi|\Big) \\
&= -\frac{\epsilon}{1+\alpha}\{(\frac{1-\alpha}{1+\alpha})\,\phi^{\frac{-2\alpha}{1+\alpha}}|\nabla\phi|^2 + \phi^{\frac{1-\alpha}{1+\alpha}}\,\Delta\phi\} \\
&= \frac{\epsilon}{1+\alpha}\{\phi^{\frac{1-\alpha}{1+\alpha}}\,(-\Delta\phi) - (\frac{1-\alpha}{1+\alpha})\,\phi^{\frac{-2\alpha}{1+\alpha}}|\nabla\phi|^2\} \\
&= \frac{\epsilon}{1+\alpha}\{\phi^{\frac{1-\alpha}{1+\alpha}}\,(\lambda_1\phi) - (\frac{1-\alpha}{1+\alpha})\,\phi^{\frac{-2\alpha}{1+\alpha}}|\nabla\phi|^2\} \\
&= \frac{\epsilon}{\alpha+1}\,\phi^{\frac{-2\alpha}{1+\alpha}}\{\lambda_1\phi^2 - (\frac{1-\alpha}{1+\alpha})|\nabla\phi|^2\}.
\end{aligned}$$

First we consider the case when $x \in \overline{\Omega_\delta}$. We have

$$-\frac{2}{1+\alpha}(\frac{1-\alpha}{1+\alpha})|\nabla\phi|^2 \leq -m.$$

Hence

$$-\frac{\epsilon}{1+\alpha}\phi^{-\frac{2\alpha}{1+\alpha}}(\frac{1-\alpha}{1+\alpha})|\nabla\phi|^2 \leq -\frac{m\epsilon}{2},$$

and since $\lambda \leq \lambda^*$, then

$$-\frac{m\epsilon}{2} \leq -\lambda a_0 f(\epsilon) \leq -\lambda a_0 f(\frac{\epsilon}{2}\phi^{\frac{2}{1+\alpha}}).$$

Hence

$$-\frac{\epsilon}{1+\alpha}\phi^{-\frac{-2\alpha}{1+\alpha}}(\frac{1-\alpha}{1+\alpha})|\nabla\phi|^2 \leq -\lambda a_0 f(\frac{\epsilon}{2}\phi^{\frac{2}{1+\beta}}), \tag{6}$$

and since $\lambda \geq \lambda_*$ we have

$$\frac{\epsilon}{1+\alpha}\phi^{\frac{2}{1+\alpha}}\lambda_1 \leq \frac{\lambda_1\epsilon}{1+\alpha} \leq \frac{\lambda a_0}{(\frac{2\epsilon}{2})^\alpha} \leq \frac{\lambda a_0}{(\frac{\epsilon}{2}\phi^{\frac{2}{1+\alpha}})^\alpha}. \tag{7}$$

Combining (6) and (7) we see that

$$\begin{aligned} -\Delta\psi &\leq -\lambda a_0[f(\frac{\epsilon}{2}\phi^{\frac{2}{1+\beta}}) - \frac{1}{(\frac{\epsilon}{2}\phi^{\frac{2}{1+\alpha}})^\alpha}] \\ &= -\lambda a_0[f(\psi) - \frac{1}{\psi^\alpha}] \\ &\leq \lambda a(x)[f(\psi) - \frac{1}{\psi^\alpha}]. \end{aligned}$$

On the other hand, on $\Omega - \overline{\Omega_\delta}$, we have $\mu \leq \phi^{\frac{2}{1+\alpha}} \leq 1$, $a(x) \geq a_1$. Since $\lambda \geq \lambda_*$ we have

$$\frac{\lambda_1\epsilon}{(1+\alpha)Na_1} < \lambda.$$

Hence

$$\begin{aligned}
-\Delta\psi_1 &= \frac{\epsilon}{\alpha+1}\,\phi^{\frac{-2\alpha}{1+\alpha}}\{\lambda_1\phi^2 - (\frac{1-\alpha}{1+\alpha})|\nabla\phi|^2\} \\
&\leq \frac{\lambda_1\epsilon\phi^{\frac{2}{1+\alpha}}}{1+\alpha} \\
&\leq \lambda a_1 N \\
&= \lambda\, a_1[f(\frac{\mu\epsilon}{2}) - (\frac{2}{\mu\epsilon})^{\alpha}] \\
&\leq \lambda a_1[f(\frac{\epsilon}{2}\phi^{\frac{2}{1+\beta}}) - \frac{1}{(\frac{\epsilon}{2}\phi^{\frac{2}{1+\alpha}})^{\alpha}}] \\
&= \lambda a_1[f(\psi) - \frac{1}{\psi^{\alpha}}] \\
&\leq \lambda a(x)[f(\psi) - \frac{1}{\psi^{\alpha}}],
\end{aligned}$$

i.e. ψ is a sub-solution of (1).

Now we will prove there exist a c large enough so that $z := ce$ is a supersolution of (1). By (H2) we can choose c large enough so that

$$c > \lambda\|a\|_{\infty} f(c\|e\|_{\infty}).$$

Hence

$$\begin{aligned}
-\Delta z = c &\geq \lambda\|a\|_{\infty} f(c\|e\|_{\infty}) \\
&\geq \lambda a(x) f(ce) \\
&\geq \lambda a(x)[f(ce) - \frac{1}{(ce)^{\alpha}}] \\
&= \lambda a(x)[f(z) - \frac{1}{z^{\alpha}}].
\end{aligned}$$

i.e., z is a supersolution of (1) with $z \geq \psi$ for c large (note $|\nabla e| \neq 0; \partial\Omega$). Thus, there exist a positive solution u of (1) such that $\psi \leq u \leq z$. This completes the proof of Theorem 2.2. ∎

Remark 2.3. Note that $(H3)$ implies $\lambda_* < \lambda^*$.

Example 2.4. Let $A > 1$ and $f(x) = e^{\frac{Ax}{A+x}}$. Then, $f(s) > 0$ for $s > 0$ and f is nondecreasing and

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{e^{\frac{Ax}{A+x}}}{x} = 0.$$

We can choose $\epsilon > 0$ such that f satisfy $(H3)$.

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