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SOME STATISTICALLY CONVERGENT DIFFERENCE SEQUENCE SPACES OF INTERVAL NUMBERS

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Abstract In this paper we have introduced the sequence spaces $c_0^{S(i)}(\triangle)$, $c^{S(i)}(\triangle)$ and $l_\infty^i(\triangle)$ of statistical convergent sequences of interval numbers based on the difference operator $(\triangle)$ and studied some of their algebraic and topological properties. Also we have investigated the relations related to these spaces.

Keywords: interval number, statistical convergence, sequence algebra, difference operator.
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1. INTRODUCTION

The idea of statistical convergence was given by Zygmund [1] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [14] and Fast [11] and then reintroduced by Schoenberg [16] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on, Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of Fridy [17] and Salát [29]. For some very interesting investigations concerning statistical convergence, one may consult the papers of Cakalli [10], Miller [12], Maddox [15] and many others, where more references on this important summability method can be found.

Recently, sequences of interval numbers and usual convergence of sequences of interval numbers were studied by Chiao [18]. Later on, Sengönül and Eryilmaz [19] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric spaces. In the recent days, Esi [2, 3] introduced and studied strongly almost λ- convergence and statistically almost λ-convergence of interval numbers and lacunary sequence spaces of interval numbers, respectively. For more information about interval numbers one may refer to Debnath et. al.[25, 26, 27, 28], Dwyer [20, 21], Fischer [22], Moore [23], Moore and Yang [24], Esi [4, 5].

Throughout the paper w^i, l^i_∞, c^i and c^i_0 denote the spaces of all, bounded, convergent and null sequences of interval numbers $\overline{x} = (\overline{x}_k)$ with complex terms respectively.

Kizmaz [13] studied the notion of difference sequence spaces where l_∞, c and c_0 are the spaces of bounded, convergent and null sequences of real numbers respectively. This notion is defined as follows:

$$Z(\Delta) = \{\text{x} = (\text{x}_k) : (\Delta x_k) \in \text{Z}\}$$

For $Z = \ell_\infty$, c and c_0 , where $\Delta \text{x}_k = x_k - x_{k+1}$.

The idea of Kizmaz [13] was applied to introduce different types of difference sequence spaces and study their different properties by Tripathy [6], Tripathy et al. [8], Tripathy and Mahanta [9], Tripathy and Sen [7] and many others.

2. PRELIMINARIES

We denote the set of all real valued closed intervals by $R(I)$. Any elements of $R(I)$ is called interval number and denoted by $\overline{x} = [x_l, x_r]$. The absolute value (magnitude or interval norm) of an interval number is defined by $|\overline{x}| = \max\{|x_l|, |x_r|\}$. For x_1, x_2, $\in$ R(I), we have $\overline{x}_1 = \overline{x}_2 \Leftrightarrow x_{l1} = x_{l2}$, $x_{r1} = x_{r2}$, $\overline{x}_1 + \overline{x}_2 = \{x \in \mathbb{R} : x_{l1} + x_{l2} \leq x \leq x_{r1} + x_{r2}\}$, and if $\alpha \geq 0$, then $\alpha\overline{x} = \{x \in R : \alpha x_{l1} \leq x \leq \alpha x_{r1}\}$ and if $\alpha < 0$, then $\alpha\overline{x} = \{x \in \mathbb{R} : \alpha x_{r1} \leq x \leq \alpha x_{l1}\}$,

$\overline{x}_1.\overline{x}_2 = \{x \in \mathbb{R} : min\{x_{l1}.x_{l2}, x_{l1}.x_{r2}, x_{r1}.x_{l2}, x_{r1}.x_{r2}\}$ $\leq x \leq max\{x_{l1}.x_{l2}, x_{l1}.x_{r2}, x_{r1}.x_{l2}, x_{r1}.x_{r2}\}\}$.

The set of all interval numbers $R(I)$ is a complete metric space [21] defined by

$$d(\overline{x}_1, \overline{x}_2) = \max\{|x_{l1} - x_{l2}|, |x_{r1} - x_{r2}|\}.$$

In the special case $\overline{x}_1 = [a, a]$ and $\overline{x}_2 = [b, b]$, we obtain usual metric on $\mathbb{R}$ with

$$d(\overline{x}_1, \overline{x}_2) = |a - b|.$$

Let us define the transformation f from $\mathbb{N}$ to $R(I)$ by $k \to f(k) = \overline{x}$, $\overline{x} = (\overline{x}_k)$. Then $(\overline{x}_k)$ is called sequence of interval numbers. The $\overline{x}_k$ is called k^{th} term of sequence $(\overline{x}_k)$.

Definition 2.1. *A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_0$ if for each $\epsilon > 0$ there exists a positive number k_0 such that $d(\overline{x}_k, \overline{x}_0) < \epsilon$ for all $k \geq k_0$ and we denote it by $lim_k\, \overline{x}_k = \overline{x}_0$. Thus $lim_k\, \overline{x}_k = \overline{x}_0 \Leftrightarrow lim_k\, x_{lk} = x_{l0}$ and $lim_k\, x_{rk} = x_{r0}$.*

Definition 2.2. *A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent statistically to the interval number $\overline{x}_0$ if for every $\epsilon > 0$, $\lim_n \frac{1}{n} |\{k \leq n : d(\overline{x}_k, \overline{x}_0) \geq \epsilon\}| = 0$, denote it by writing stat-$lim_k\, \overline{x}_k = \overline{x}_0$.*

Definition 2.3. *An interval valued sequence space* $\overline{E}$ *is said to be solid if* $\overline{y} = (\overline{y}_k) \in \overline{E}$ *whenever* $\|\overline{y}_k\| \leq \|\overline{x}_k\|$ *for all* $k \in N$ *and* $\overline{x} = (\overline{x}_k) \in \overline{E}$.

Definition 2.4. *An interval valued sequence space* $\overline{E}$ *is said to be convergence free if* $\overline{y} = (\overline{y}_k) \in \overline{E}$ *whenever* $\overline{x} = (\overline{x}_k) \in \overline{E}$ *and* $\overline{x}_k = \theta$ *implies* $\overline{y}_k = \theta$, *where* $\theta = [0, 0]$.

Definition 2.5. *An interval valued sequence space* $\overline{E}$ *is said to be sequence algebra if* $(\overline{x}_k \overline{y}_k) \in \overline{E}$ *whenever* $\overline{x} = (\overline{x}_k) \in \overline{E}$, $\overline{y} = (\overline{y}_k) \in \overline{E}$ $(k \in N)$.

The spaces of all statistically null and statistically convergent sequences of interval numbers $c_0^{S(i)}$ and $c^{S(i)}$ respectively have been introduced recently by Debnath and Saha [28]

In this article, for any sequence of interval numbers $\overline{x} = (\overline{x}_k) \in w^i$ we write $\triangle\overline{x} = (\triangle\overline{x}_k) = (\overline{x}_k - \overline{x}_{k+1})$ and we define the new sequence spaces of interval numbers $c_0^{S(i)}(\triangle)$, $c^{S(i)}(\triangle)$ and $l_\infty^i(\triangle)$ respectively are as follows:

$$c_0^{S(i)}(\triangle) = \left\{\overline{x} = (\overline{x}_k) \in w^i : \ stat - lim_k \triangle\overline{x}_k = \theta\right\}, \ where \ \theta = [0, 0].$$

$$c^{S(i)}(\triangle) = \left\{\overline{x} = (\overline{x}_k) \in w^i : \ stat - lim_k \triangle\overline{x}_k = \overline{x}_0\right\}.$$

$$l_\infty^i(\triangle) = \left\{\overline{x} = (\overline{x}_k) \in w^i : \ sup_k (|\triangle x_{lk}|, |\triangle x_{rk}|) < \infty\right\}.$$

Let us also set

$$m_0^{S(i)}(\triangle) = c_0^{S(i)}(\triangle) \cap l_\infty^i(\triangle)$$

and $m^{S(i)}(\triangle) = c^{S(i)}(\triangle) \cap l_\infty^i(\triangle)$.

3. MAIN RESULTS

Theorem 3.1. $\left(m_0^{S(i)}(\triangle), \overline{d}\right), \left(m^{S(i)}(\triangle), \overline{d}\right)$ *are complete metric spaces with the metric defined by*

$$\overline{d}(\overline{x}_k, \overline{y}_k) = sup_k \ max \{|\triangle x_{lk} - \Delta y_{lk}|, |\triangle x_{rk} - \Delta y_{rk}|\}.$$

Proof. We proof the result for the class $m_0^{S(i)}(\triangle)$. The rest can be established similarly.

Let $(\overline{x}_n^k) = (\overline{x}_n^1, \overline{x}_n^2, \overline{x}_n^3) \in m_0^{S(i)}(\triangle)$ for each n, then $lim_{k\to\infty}\overline{x}_n^k = \theta$ for each $n \in N$. Let $(\overline{x}_n)$ be a Cauchy sequence. Then for each $\varepsilon > 0$, there exists a $k_0 \in N$ such that $\overline{d}(\overline{x}_n, \overline{x}_m) < \varepsilon$, whenever $n, m \geq k_0$. Hence we have $sup_{n,m}\{max|\Delta x_{nl}^k - \Delta x_{ml}^k, \Delta x_{nu}^k - \Delta x_{mu}^k|\} < \varepsilon$. Thus we have $|\Delta x_{nl}^k - \Delta x_{ml}^k| < \varepsilon$ and $|\Delta x_{nu}^k - \Delta x_{mu}^k| < \varepsilon$. This means that $(\Delta\overline{x}_n^k)$ is a Cauchy sequence in $R(I)$. Since $R(I)$ is a complete, $(\Delta\overline{x}_n^k)$ is convergent i.e $lim_{nl\to\infty}\Delta x_{nl}^k = \theta$ and $lim_{nu\to\infty}\Delta x_{nu}^k = \theta$.

Now, $|\Delta x_{nl}^{k} - 0| < \varepsilon$ and $|\Delta x_{nu}^{k} - 0| < \varepsilon$, taking$m \to \infty$ gives $sup_{n} max\{|\Delta x_{nl}^{k} - 0|, |\Delta x_{nu}^{k} - 0|\} < \varepsilon$ i.e., $\overline{d}(\overline{x}_n, \theta) < \varepsilon$.
This implies that $(\overline{x}_n)$ is a convergent sequence and converge to $\theta \in m_0^{S(i)}(\triangle)$. ∎

Theorem 3.2. *$m_0^{S(i)}(\triangle)$ and $m^{S(i)}(\triangle)$ are normed interval spaces with the norm*

$$\|\overline{x}\| = max\,(|x_{l1}|, |x_{r1}|) + sup_k\, max\,\{|\triangle x_{lk}|, |\triangle x_{rk}|\}.$$

Proof. Let $\lambda^i = m_0^{S(i)}(\triangle)$ $(or\, m^{S(i)}(\triangle))$ and $\overline{x}, \overline{y} \in \lambda^i$

N_1. Since $\|\overline{x}\|_{\lambda^i} = \max\,(|x_{l1}|, |x_{r1}|) + sup_k\ \max\,\{|\triangle x_{lk}|, |\triangle x_{rk}|\}$

We easily see that $\| \overline{x} \|_{\lambda^i} > 0, \forall\, \overline{x} \in \lambda^i$ - $\{\theta\}$.

N_2. $\| \overline{x} \|_{\lambda^i} = 0 \Longleftrightarrow \max\,(|x_{l1}|, |x_{r1}|) + sup_k \max\,\{| \Delta x_{lk} |, | \Delta x_{rk} |\} = 0 \Longleftrightarrow \overline{x} = \theta$, where $\theta = [0, 0]$.

N_3. $\| \overline{x} + \overline{y} \|_{\lambda^i}$
$= \max\,(|x_{l1} + y_{l1}|, |x_{r1} + y_{r1}|) + sup_k\ \max\,\{| \Delta\,(x_{lk} + y_{lk}) |, | \Delta\,(x_{rk} + y_{rk}) |\}$

$\leq \max(|x_{l1}| + |y_{l1}|, |x_{r1}| + |y_{r1}|) + sup_k\ \max\,\{| \Delta x_{lk} | + | \Delta y_{lk} |, | \Delta x_{rk} | + | \Delta y_{rk} |\}$

$\leq \max(|x_{l1}|, |x_{r1}|) + sup_k \max\,\{| \Delta x_{lk} |, | \Delta x_{rk} |\} + \max\,(|y_{l1}|, |y_{r1}|) + sup_k \max\,\{| \Delta y_{lk} |, | \Delta y_{rk} |\}$

$= \| \overline{x} \|_{\lambda^i} + \| \overline{y} \|_{\lambda^i}$

N_4. $\| \alpha\overline{x} \|_{\lambda^i} = \max\,(|\alpha x_{l1}|, |\alpha x_{r1}|) + sup_k\ \max\,\{| \alpha\Delta x_{lk} |, | \alpha\Delta x_{rk} |\}$

$= \max\,(|\alpha|\,|x_{l1}|, |\alpha|\,|x_{r1}|) + sup_k \max\,\{|\alpha|\,|\Delta x_{lk}|, |\alpha|\,|\Delta x_{rk}|\}$
$= |\alpha| \max\,(|x_{l1}|, |x_{r1}|) + | \alpha |\, sup_k \max\,\{| \Delta x_{lk} |, | \Delta x_{rk} |\}$

$= | \alpha |\,\| \overline{x} \|_{\lambda^i}$

So, $\| \overline{x} \|_{\lambda^i}$ is a norm on λ^i. ∎

Theorem 3.3. *The spaces $m_0^{S(i)}(\triangle)$ and $m^{S(i)}(\triangle)$ are solid.*

Proof. We consider only $m_0^{S(i)}(\triangle)$.

Now, let $\left\|\overline{y}_k\right\| \leq \|\overline{x}_k\|$, for all $k \in \mathbb{N}$ and for some $\overline{x} \in m_0^{S(i)}(\triangle)$. Then we have, $\overline{d}(\overline{y}_k, \theta) \leq \overline{d}(\overline{x}_k, \theta)$, that is $\{| \Delta y_{lk} - 0 |, | \Delta y_{rk} - 0 |\} \leq \{| \Delta x_{lk} - 0 |, | \Delta x_{rk} - 0 |\}$.

Thus we have $\Delta y_{lk} \leq \Delta x_{lk}$ and $\Delta y_{rk} \leq \Delta x_{rk}$, i.e., $\Delta\overline{y} \leq \Delta\overline{x}$.

So, clearly $\overline{y} \in m_0^{S(i)}(\triangle)$. Hence $m_0^{S(i)}(\triangle)$ is solid. ∎

Theorem 3.4. *The spaces $c^{S(i)}(\triangle)$ and $c_0^{S(i)}(\triangle)$ are not solid.*

Proof. We consider only $c^{S(i)}(\triangle)$.

Let $\overline{x} = (\overline{x}_k) \in c^{S(i)}(\triangle)$, where $\overline{x}_k = [k,\ k+1]$ and $k \in \mathbb{N}$ and

let $\alpha_k = \begin{cases} [1,1] & \text{for } k = 2n, \ and\ n \in \mathbb{N} \\ 0, \text{otherwise} \end{cases}$

Then, $(\alpha_k \overline{x}_k) \notin c^{S(i)}(\triangle)$ and so $c^{S(i)}(\triangle)$ is not solid.

For the space $c_0^{S(i)}(\triangle)$ the result can be proved similarly. ∎

Theorem 3.5. *The spaces $m_0^{S(i)}(\triangle)$ and $m^{S(i)}(\triangle)$ are sequence algebra.*

Proof. We prove that $m_0^{S(i)}(\triangle)$ is a sequence algebra.

Let $(\overline{x}_k), (\overline{y}_k) \in m_0^{S(i)}(\triangle)$.

Then, $\underset{k}{stat-lim}\ \Delta\overline{x}_k = \theta$ and $\underset{k}{stat-lim}\ \Delta\overline{y}_k = \theta$, where $\theta = [0,\ 0]$.

Then we have, $\underset{k}{stat-lim}\ (\Delta\overline{x}_k \Delta\overline{y}_k) = \theta$.

Thus $(\overline{x}_k \overline{y}_k) \in m_0^{S(i)}(\triangle)$. Hence $m_0^{S(i)}(\triangle)$ is a sequence algebra.

For the space $m^{S(i)}(\triangle)$, the result can be proved similarly. ∎

Theorem 3.6. *The spaces $c^{S(i)}(\triangle)$ and $c_0^{S(i)}(\triangle)$ are not convergence free .*

Proof. Here, we give a counter example.

Let, $\overline{x} = (\overline{x}_k)$ and $\overline{y} = (\overline{y}_k)$ be two sequences of interval numbers.

Now let, $\overline{x}_k = [k, k+1]$

and $\overline{y}_k = \left[k^2, \frac{1}{k}\right]$ for all $k \in \mathbb{N}$.

Then $(\overline{x}_k) \in c^{S(i)}(\triangle)$ but $(\overline{y}_k) \notin c^{S(i)}(\triangle)$.

Hence the space $c^{S(i)}(\triangle)$ is not convergence free in general.

Similarly, it can be shown that the space $c_0^{S(i)}(\triangle)$ is not convergence free. ∎

Theorem 3.7. *The inclusion* $c_0^{S(i)}(\triangle) \subset c^{S(i)}(\triangle)$ *holds.*

Proof. If we take $\overline{x} = (\overline{x}_k) \in c_0^{S(i)}(\triangle)$ then clearly $(\overline{x}_k) \in c^{S(i)}(\triangle)$. Now we will prove the inclusion is strict.

Consider, the interval sequence $\overline{x} = (\overline{x}_k)$ is defined as $\overline{x}_k = [k,\ k+2]$, where $k \in \mathbb{N}$.

Then, clearly $(\overline{x}_k) \in c^{S(i)}(\triangle)$ but $(\overline{x}_k) \notin c_0^{S(i)}(\triangle)$. ∎

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