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STRATEGIC GAMES ON DIGRAPHS

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Abstract We investigate strategic form games on digraphs, and examine maximin solution concepts based on different types of digraph substructures [45]. Necessary and sufficient conditions for maximin solution existence in digraph matrix games with pure strategies are formulated and proved. Some particular games are considered. Algorithms for finding maximin substructures are suggested. Multi-player simultaneous games and dynamical/hierarchical games on digraphs are considered too.

Keywords: network games, maximin solutions, matrix games on digraphs, polymatrix games on digraphs.

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1. INTRODUCTION

We regard games which can appear in real situations when several companies manage the activity of a big network. Decision-making subjects may have antagonistic interests. In such circumstances, well-known extremal network/digraph problems [10, 33] and problems of constructing various structures on networks/digraphs [5, 10, 33] become mono or multi criteria strategic network game problems. Systems of human, information, hardware (servers, routers, etc.) or other types, controled by different subjects, involve their interactions [42]. As a consequence, many traditional network problems have to be treated from the perspective of game theory [42], including problems of routing [36], load balancing [41], facility location [47], network design [13], etc.

A series of related problems have been investigated and described in scientific literature [19, 42] in the context of cyclic games solving. That approach has used a special type of strategy definition [19]. This work is based on paper [45] which introduced some types of games on digraphs by defining originally the notions of pure strategies, outcome and payoff functions, and may be seen as a survey of related works.

The paper is divided into six sections, including introduction and conclusions. Section 2 introduces the notion of zero-sum matrix games on digraphs. Some properties giving a general tool for matrix games investigations are proved. Section 3 presents some particular solvable games. A special investigation is provided on flow games. It is proved that the problem of maximin cost flow finding is NP-hard. Section 4 generalises the notion of digraph matrix game for an arbitrary finite number of players. Section 5 introduces the notion of dynamic games.

2. MATRIX GAMES ON DIGRAPHS

In this section we investigate three types of matrix games on directed graphs:

- basic (root) matrix games or simply **matrix games**,
- **matrix games with admissible strategies**,
- and **matrix games with feasible strategies and profiles**.

The games are defined by means of two related matrices: an outcome matrix, and a payoff matrix.

2.1. CONCEPTS

Let us consider a digraph $G = (V, E)$, $|V| = n$, $|E| = m$, further called simply graph. Every directed edge $e \in E$ has the length (weight) $c(e) \in Z$. The vertex set V is partitioned into two disjoint subsets

$$V_1,\ V_2 \quad (V_1 \cup V_2 = V,\ V_1 \cap V_2 = \emptyset)\,,$$

being positions of two players. The edge set E is partitioned into two disjoint subsets too, as

$$E_1 = \{(u, v) \in E | u \in V_1\}, \quad E_2 = \{(u, v) \in E | u \in V_2\}.$$

Any subset $S_1 \subseteq E_1$ or $S_2 \subseteq E_2$ is called a strategy of the corresponding player. The pair of strategies $(S_1, S_2) \in 2^{E_1} \times 2^{E_2}$ is called a game profile. Any game profile generates a subgraph $G_S = (V, S_1 \cup S_2)$, called the graph of the profile (S_1, S_2), where $S = S_1 \cup S_2$.

Let us introduce some notation.

- $2^G = \{G' = (V, E') \mid E' \subseteq E\}$ denotes the set of all subgraphs of the graph G;
- $\mathcal{D} = \{G' \in 2^G \mid \mathcal{P}\}$ denotes the set of all subgraphs of G, verifying a set of properties $\mathcal{P}$, that is, $\mathcal{D}$ is the set of feasible subgraphs;
- M: $2^{E_1} \times 2^{E_2} \multimap \mathcal{D}$, $M(S_1, S_2) = 2^{G_S} \cap \mathcal{D}$ denotes the set-valued choice function which maps the graph of the profile $(S_1, S_2) \in 2^{E_1} \times 2^{E_2}$ into the set of all feasible subgraphs of G_S, that is into the subgraphs which verify the set of properties $\mathcal{P}$;
- C: $\mathcal{D} \to \mathcal{R}$ denotes the choice criterion.

Let

$$k(E_1, E_2, M) = \max_{(S_1, S_2) \in 2^{E_1} \times 2^{E_2}} |M(S_1, S_2)|.$$

be the cardinality of the choice function. There exists four alternatives, for given E_1, E_2, and M:

0^0. $k(E_1, E_2, M) = 0$;

1^0. $k(E_1, E_2, M) = 1$;

2^0. $k(E_1, E_2, M) > 1$ and for any $M(S_1, S_2) \neq \emptyset, G', G'' \in M(S_1, S_2)$, the equality $C(G') = C(G'')$ is true;

3^0. $k(E_1, E_2, M) > 1$ and there exists $M(S_1, S_2) \neq \emptyset$, and the subgraphs G', $G'' \in M(S_1, S_2)$, such that the relation $C(G') \neq C(G'')$ holds.

The case 0^0 doesn't make sense.

The matrix game can be defined when the choice function $M(S_1, S_2)$ verifies either the property 1^0 or 2^0. The case 3^0 (as well as the case 2^0) can be reduced to 1^0 by introducing the choice function

$$\overline{M}(S_1, S_2) = \underset{G' \in M(S_1, S_2)}{\operatorname{argmax}} C(G').$$

It is a mapping assigning to each profile $(S_1, S_2) \in 2^{E_1} \times 2^{E_2}$ an element of $M(S_1, S_2)$, optimal by criterion C. The value $\overline{M}(S_1, S_2) \neq \emptyset$ is a feasible subgraph of the profile (S_1, S_2). We has to remark that the choice function $\overline{M}(S_1, S_2)$ reduces both the cases 2^0 and 3^0 to 1^0.

Now, we can define the (root) matrix game on digraph G via the means of two related matrices: **the outcome matrix** $\overline{M}(S_1, S_2)$ and **the payoff matrix** $\overline{C}(S_1, S_2)$.

The outcome matrix of the matrix game is defined by the function $\overline{M}(S_1, S_2)$ and has the same notation. Its elements are either feasible subgraphs or the empty set. The lines of the matrix are identified by the strategies $S_1 \in 2^{E_1}$ of the first player and the columns are identified by the strategies $S_2 \in 2^{E_2}$ of the second player.

The payoff matrix of the matrix game with the same dimensions as the dimensions of the outcome matrix is defined by the function

$$\overline{C}(S_1, S_2) = \begin{cases} C(\overline{M}(S_1, S_2)), & \text{if } \overline{M}(S_1, S_2) \neq \emptyset, \\ -\infty, & \text{if } \overline{M}(S_1, S'_2) = \emptyset \quad \text{for all } S'_2 \in 2^{E_2}, \\ +\infty, & \text{otherwise,} \end{cases}$$

and has the same notation.

Depending on the set of properties $\mathcal{P}$, various types of games may be investigated.

Remark 2.1. *The set $\mathcal{P}$ of properties can induce in a particular game various feasible subgraphs: trees; paths between two fixed vertices v_s and v_t; flows between output* *vertex v_s and input vertex v_t; matchings; medians; cliques; cycles; Hamiltonian cycles, etc. Any feasible subgraph $\overline{M}(S_1, S_2)$ satisfies both the set of properties $\mathcal{P}$ and the optimality criterion C.*

First, let us consider **the root strategic game**

$$\Gamma = \langle 2^{E_1},\ 2^{E_2},\ \overline{C}(S_1, S_2)\rangle$$

which is a (zero-sum) **matrix game** defined on the graph G. The first player has $2^{|E_1|}$ strategies, the second — $2^{|E_2|}$. The players choose their strategies simultaneously and independently. The first player chooses his strategy S_1 from E_1, the second — S_2 from E_2. Every profile $(S_1, S_2) \in 2^{E_1} \times 2^{E_2}$ has a numerical value $\overline{C}(S_1, S_2)$. For the first player it means the gain $\overline{C}(S_1, S_2)$ if $\overline{C}(S_1, S_2) > 0$ and the loss $\overline{C}(S_1, S_2)$ if $\overline{C}(S_1, S_2) < 0$. For the second player is valid vice versa — it means the loss $\overline{C}(S_1, S_2)$ if $\overline{C}(S_1, S_2) > 0$ and the gain $|\overline{C}(S_1, S_2)|$ if $\overline{C}(S_1, S_2) < 0$. Let us recall that in a zero-sum game the gain $\overline{C}(S_1, S_2)$ of one of the players means the loss $\overline{C}(S_1, S_2)$ of the other.

To introduce two other types of games we need some additional notation and concepts. The sets

$$\mathcal{B}_1 = \left\{S_1 \in 2^{E_1} | \exists S_2 \in 2^{E_2} \colon \overline{M}(S_1, S_2) \neq \emptyset\right\}, \tag{1}$$

$$\mathcal{B}_2 = \left\{S_2 \in 2^{E_2} | \exists S_1 \in 2^{E_1} \colon \overline{M}(S_1, S_2) \neq \emptyset\right\}, \tag{2}$$

are sets of admissible strategies. The sets

$$\mathcal{B}_1(S_2) = \left\{S_1 \in \mathcal{B}_1 | \overline{M}(S_1, S_2) \neq \emptyset\right\},$$

$$\mathcal{B}_2(S_1) = \left\{S_2 \in \mathcal{B}_2 | \overline{M}(S_1, S_2) \neq \emptyset\right\},$$

are sets of admissible strategies, connected with S_1 and S_2 correspondingly.

In such notation, we may consider the game

$$\Gamma^+ = \langle \mathcal{B}_1, \mathcal{B}_2, \overline{C}(S_1, S_2)\rangle$$

which is **a matrix game with admissible strategies**. All the profiles of the game Γ^+ are admissible.

Let us introduce a generic notation

$$\Gamma^+_* = \langle \mathcal{B}_1(S_2), \mathcal{B}_2(S_1), C(\overline{M}(S_1, S_2))\rangle$$

for a game based on two Stackelberg games [37]:

the game

$$\Gamma^+_1 = \langle \mathcal{B}_1, \mathcal{B}_2(S_1), C(\overline{M}(S_1, S_2))\rangle,$$

and the game

$$\Gamma_2^+ = \langle \mathcal{B}_2, \mathcal{B}_1(S_2), C(\overline{M}(S_1, S_2)) \rangle.$$

The players select their strategies consequently on two stages in these three games.

1 In the game Γ_1^+ the first player moves at the first stage and the second player moves at the second stage.

2 In the game Γ_2^+ the second player moves at the first stage and the first player moves at the second stage.

3 In the game Γ_*^+ we distinguish two stages as well as for Γ_1^+ and Γ_2^+. At the first stage the first player selects his strategy as he plays the game Γ_1^+, and the second player selects his strategy as he plays the game Γ_2^+. At the second stage, one of them, chosen aleatory, may change his strategy, knowing the choice of his opponent at the first stage.

Remark 2.2. *It is obvious that the Stackelberg games Γ_1^+ and Γ_2^+ are* **matrix games with feasible strategies**. *As all the profiles of the game Γ_*^+ are feasible, we will call it **matrix game with feasible profiles**. Clearly, when the game Γ_*^+ is referred, it means implicitly that the games Γ_1^+ and Γ_2^+ are referred too.*

Remark 2.3. *The game Γ_*^+ may be seen as a special "matrix" game, for which the outcome and payoff matrices are obtained from the matrices of the game Γ^+ by deleting the elements with non-finite payoff values. These "special" or pseudo matrices may be associated with two dimensional lists in the Wolfram Language [50, 51].*

2.2. PROPERTIES OF DIGRAPH MATRIX GAMES

The games Γ, Γ^+, and Γ_*^+, have some interesting and important properties. Let us investigate and highlight them.

Lemma 2.1. *In any games Γ and Γ^+ the following relations between lower and upper values of the games hold:*

$$\max_{S_1 \in 2^{E_1}} \min_{S_2 \in 2^{E_2}} \overline{C}(S_1, S_2) \le \min_{S_2 \in 2^{E_2}} \max_{S_1 \in 2^{E_1}} \overline{C}(S_1, S_2),$$

$$\max_{S_1 \in \mathcal{B}_1} \min_{S_2 \in \mathcal{B}_2} \overline{C}(S_1, S_2) \le \min_{S_2 \in \mathcal{B}_2} \max_{S_1 \in \mathcal{B}_1} \overline{C}(S_1, S_2).$$

Lemma 2.1 exposes a well-known property of the matrix games. The concept of upper and lower values of the games Γ and Γ^+ are imposed by the right and left members of the inequalities in Lemma 2.1.

Definition 2.1. *The matrix game has a solution (is solvable) if its upper and lower values are equal. The corresponding profile is called an* ***equilibrium*** *(equilibrium* *solution, equilibrium profile, equilibrium outcome) of the game and its value is called the value of the game.*

In the game with feasible profiles Γ_*^+, the opposite inequality may occur when the payoff function C satisfies some special properties, that is, it is possible that the value $\min\limits_{S_2\in\mathcal{B}_2}\max\limits_{S_1\in\mathcal{B}_1(S_2)}\overline{C}(S_1,S_2)$ of the game Γ_2^+ do not surpass the value $\max\limits_{S_1\in\mathcal{B}_1}\min\limits_{S_2\in\mathcal{B}_2(S_1)}\overline{C}(S_1,S_2)$ of the game Γ_1^+.

Lemma 2.2. *If $C(\overline{M}(S_1,S_2)) = C'(S_1) + C''(S_2)$ and $\mathcal{B}_1 \neq \emptyset$, then*

$$\max_{S_1\in\mathcal{B}_1}\min_{S_2\in\mathcal{B}_2(S_1)}\overline{C}(S_1,S_2) \geq \min_{S_2\in\mathcal{B}_2}\max_{S_1\in\mathcal{B}_1(S_2)}\overline{C}(S_1,S_2)$$

Proof. It is obvious that if $\mathcal{B}_1 \neq \emptyset$, then $\mathcal{B}_2 \neq \emptyset$ and vice versa. Thus, we have

$$\max_{S_1\in\mathcal{B}_1}\min_{S_2\in\mathcal{B}_2(S_1)} C(\overline{M}(S_1,S_2)) =$$

$$= \max_{S_1\in\mathcal{B}_1}[C'(S_1) + \min_{S_2\in\mathcal{B}_2(S_1)} C''(S_2)] \geq$$

$$\geq \max_{S_1\in\mathcal{B}_1} C'(S_1) + \min_{S_2\in\mathcal{B}_2} C''(S_2) \geq$$

$$\geq \min_{S_2\in\mathcal{B}_2}[C''(S_2) + \max_{S_1\in\mathcal{B}_1(S_2)} C'(S_1)] =$$

$$= \min_{S_2\in\mathcal{B}_2}\max_{S_1\in\mathcal{B}_1(S_2)} C(\overline{M}(S_1,S_2)).$$

The truth of the lemma follows from the above chain of the equalities and inequalities. ∎

Definition 2.2. *The game with feasible profiles Γ_*^+ has an equilibrium if the values of the Stackelberg games Γ_1^+ and Γ_2^+ are equal.*

This solution concept may have an integrative power for all the precedent ones. The following results have to prove this.

Lemma 2.3. *If $\mathcal{B}_1 \neq \emptyset$, then*

$$\max_{S_1\in 2^{E_1}}\min_{S_2\in 2^{E_2}}\overline{C}(S_1,S_2) = \max_{S_1\in\mathcal{B}_1}\min_{S_2\in\mathcal{B}_2}\overline{C}(S_1,S_2) =$$

$$= \max_{S_1\in\mathcal{B}_1}\min_{S_2\in\mathcal{B}_2(S_1)} C(\overline{M}(S_1,S_2)).$$

Proof. The payoff function is so defined that $\min\limits_{S_2 \in 2^{E_2}} \overline{C}(S_1, S_2) < \infty$ for any strategy $S_1 \in 2^{E_1}$. As $\mathcal{B}_1 \neq \emptyset$, then

$$-\infty < \max_{S_1 \in 2^{E_1}} \min_{S_2 \in 2^{E_2}} \overline{C}(S_1, S_2) < +\infty.$$

Therefore, the maximin profile is feasible and

$$\max_{S_1 \in 2^{E_1}} \min_{S_2 \in 2^{E_2}} \overline{C}(S_1, S_2) = \max_{S_1 \in \mathcal{B}_1} \min_{S_2 \in \mathcal{B}_2} \overline{C}(S_1, S_2). \tag{3}$$

For some admissible strategy $S_1 \in \mathcal{B}_1$ we have

$$\min_{S_2 \in \mathcal{B}_2} \overline{C}(S_1, S_2) = \min_{S_2 \in \mathcal{B}_2(S_1)} \overline{C}(S_1, S_2),$$

as $\overline{C}(S_1, S_2) = +\infty$ for any strategy $S_2 \notin \mathcal{B}_2(S_1)$. Then

$$\max_{S_1 \in \mathcal{B}_1} \min_{S_2 \in \mathcal{B}_2} \overline{C}(S_1, S_2) = \max_{S_1 \in \mathcal{B}_1} \min_{S_2 \in \mathcal{B}_2(S_1)} \overline{C}(S_1, S_2). \tag{4}$$

Relations (3) – (4) prove the lemma. ∎

Considering Lemma 2.3 and the equality $\max\limits_{S_1 \in 2^{E_1}} \overline{C}(S_1, S_2) = +\infty$ for all $S_2 \notin \mathcal{B}_2$, the following theorem becomes obvious.

Theorem 2.1. *Let $\mathcal{B}_1 \neq \emptyset$. The upper (lower) values of the games Γ and Γ^+ are equal.*

For a sufficiently large set of edges E, an exhaustive search of equilibria in the games $\Gamma, \Gamma^+, \Gamma^+_*$ is a hard task. Theorem 2.1 suggests how to narrow sets of admissible strategies by taking into account properties of G and the set of properties $\mathcal{P}$ of feasible subgraphs.

Supposition 2.1. *Further on, we assume that $\mathcal{B}_1 \neq \emptyset$.*
Supposition 2.2. *We may define/consider the sets of admissible strategies $\mathcal{B}_1$ and $\mathcal{B}_2$ being subsets of 2^{E_1} and 2^{E_2} less powerful than (1) – (2) too.*

Thus, we have for the fixed game Γ several possible games Γ^+, and Γ^+_*, by defining, e.g., the game Γ^+ constrained by the condition

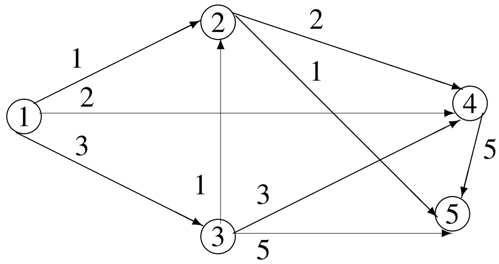
$$|M(S_1, S_2)| \leq 1,$$

for all $(S_1, S_2) \in \mathcal{B}_1 \times \mathcal{B}_2$.

It is easy to observe that the payoff function is defined in such a way, that if there exists only one feasible profile in Γ^+_*, then for both players it is advantageous the same profile, namely the equilibrium profile. However, in some games Γ^+_* several feasible profiles exist, but the equilibrium profile itself does not exist in all the games Γ, Γ^+ and Γ^+_*.

The following example has to illustrate the above exposition.

Example 2.1. Let the games Γ, Γ^+, and Γ^+_*, be formulated on the next acyclic digraph $G = (V, E)$:



If $V_1 = \{1\}$ and $V_2 = \{2;\ 3;\ 4;\ 5\}$ are the positions of the players, then

$$\begin{aligned} E_1 &= \{(1,2);\ (1,3);\ (1,4)\}, \\ E_2 &= \{(2,4);\ (2,5);\ (3,2);\ (3,4);\ (3,5);\ (4,5)\}, \end{aligned}$$

are their sets of edges.

We have a purpose to construct the matrix games based on three sets of paths from the input vertex $v_s = 1$ to the output vertex $v_t = 5$.

First, let us consider the set $\mathcal{D}_3$ of feasible graphs as the set of all paths from $v_s = 1$ to $v_t = 5$ that **contain exactly** 3 **edges**.

It is easy to observe that in the game Γ the first player has $2^3 = 8$ strategies and the second — $2^6 = 64$.

It is a difficult task to find the equilibrium by an exhaustive search because of 512 profiles of the game Γ. But taking into account the essence of the feasible graphs, admissible strategies are defined as the sets with cardinality $|E_1| = |V_1| = 1$ for the first player, and $|E_2| = |V_2| - 1 = 3$ for the second player, with an additional property that exactly one edge exits from every vertex, except $v_t = 5$.

As the graph G is acyclic, the subgraph G_S is a directed tree entering the vertex $v_t = 5$ for any profile S, that is, it has a path from $v_s = 1$ to $v_t = 5$, not obligatory with exactly 3 edges.

Observe that for $S_1 = \{(1,4)\}$ a 3 – path from $v_s = 1$ to $v_t = 5$ does not exist for any strategy of the second player. Besides that, for strategy $S_2 = \{(2,5); (3,5); (4,5)\}$ a 3 – path from $v_s = 1$ to $v_t = 5$ does not exist for any strategy of the first player.

If the payoff function C is defined as the length of the path from $v_s = 1$ to $v_t = 5$, and the first player has the purpose to maximise the length of the path (the second trying to minimise it), then it is easy to find the payoff matrices of the games Γ^+ and Γ^+_* :

$\mathcal{B}_1$	$\mathcal{B}_2$: (3,2) (2,4) (4,5)	(2,4) (3,4) (4,5)	(2,4) (4,5) (3,5)	(3,2) (2,5) (4,5)	(2,5) (3,4) (4,5)	(2,5) (3,5) (4,5)	Γ^+: $\min\limits_{S_2\in\mathcal{B}_2}$	Γ_1^+: $\min\limits_{S_2\in\mathcal{B}_2(S_1)}$
(1,2)	8	8	8	$+\infty$	$+\infty$	$+\infty$	8	8
(1,3)	$+\infty$	11	$+\infty$	5	11	$+\infty$	5	5
(1,4)	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	
Γ^+: $\max\limits_{S_1\in\mathcal{B}_1}$	$+\infty$	11	$+\infty$	$+\infty$	$+\infty$	$+\infty$	11\ 8	
Γ_2^+: $\max\limits_{S_1\in\mathcal{B}_1(S_2)}$	8	11	8	5	11			5\ 8

In the game Γ^+ (consequently in Γ)

$$\max_{S_1\in\mathcal{B}_1}\min_{S_2\in\mathcal{B}_2}\overline{C}(S_1,S_2)=8\leq\min_{S_2\in\mathcal{B}_2}\max_{S_1\in\mathcal{B}_1}\overline{C}(S_1,S_2)=11.$$

But in the game Γ_*^+ we have

$$\max_{S_1\in\mathcal{B}_1}\min_{S_2\in\mathcal{B}_2(S_1)}\overline{C}(S_1,S_2)=8\geq\min_{S_2\in\mathcal{B}_2}\max_{S_1\in\mathcal{B}_1(S_2)}\overline{C}(S_1,S_2)=5.$$

Second, let $\mathcal{D}_2$ be the set of all paths from $v_s=1$ to $v_t=5$, **having exactly** 2 **edges**.

As in the first case, the games Γ,Γ^+ and Γ_*^+, with their payoff matrices are considered:

$\mathcal{B}_1$	$\mathcal{B}_2$: (3,2) (2,4) (4,5)	(2,4) (3,4) (4,5)	(2,4) (4,5) (3,5)	(3,2) (2,5) (4,5)	(2,5) (3,4) (4,5)	(2,5) (3,5) (4,5)	Γ^+: $\min\limits_{S_2\in\mathcal{B}_2}$	Γ_1^+: $\min\limits_{S_2\in\mathcal{B}_2(S_1)}$
(1,2)	$+\infty$	$+\infty$	$+\infty$	2	2	2	2	2
(1,3)	$+\infty$	$+\infty$	8	$+\infty$	$+\infty$	$\boxed{8}$	8	8
(1,4)	7	7	7	7	7	7	7	7
Γ^+: $\max\limits_{S_1\in\mathcal{B}_1}$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	11	8	8\ 8	
Γ_2^+: $\max\limits_{S_1\in\mathcal{B}_1(S_2)}$	7	7	7	7	11	8		7\ 8

In the game Γ^+ (consequently in Γ)

$$\max_{S_1\in\mathcal{B}_1}\min_{S_2\in\mathcal{B}_2}\overline{C}(S_1,S_2)=8=\min_{S_2\in\mathcal{B}_2}\max_{S_1\in\mathcal{B}_1}\overline{C}(S_1,S_2)=8.$$

But in the game Γ_*^+

$$\max_{S_1 \in \mathcal{B}_1} \min_{S_2 \in \mathcal{B}_2(S_1)} \overline{C}(S_1, S_2) = 8 \geq \min_{S_2 \in \mathcal{B}_2} \max_{S_1 \in \mathcal{B}_1(S_2} \overline{C}(S_1, S_2) = 7.$$

Remark, there exists an equilibrium only in the games Γ, Γ^+. The game Γ_*^+ doesn't have equilibrium.

Third, let us consider the set $\mathcal{D}$ of **all paths from** $v_s = 1$ **to** $v_t = 5$ as the set of feasible graphs.

$\mathcal{B}_1$	$\mathcal{B}_2$: (3,2) (2,4) (4,5)	(2,4) (3,4) (4,5)	(2,4) (4,5) (3,5)	(3,2) (2,5) (4,5)	(2,5) (3,4) (4,5)	(2,5) (3,5) (4,5)	Γ^+: $\min_{S_2 \in \mathcal{B}_2}$	Γ_1^+: $\min_{S_2 \in \mathcal{B}_2(S_1)}$
(1,2)	8	8	8	2	2	2	2	2
(1,3)	11	11	8	5	11	8	5	5
(1,4)	7	7	7	$\boxed{\boxed{7}}$	7	7	7	7
Γ^+: $\max_{S_1 \in \mathcal{B}_1}$	11	11	8	7	11	8	7\ 7	
Γ_2^+: $\max_{S_1 \in \mathcal{B}_1(S_2)}$	11	11	8	7	11	8		7\ 7

Let us remark, that all the games $\Gamma, \Gamma^+, \Gamma_*^+$ have the equilibrium

$$(S_1, S_2) = (\{(1,4)\},\ \{(3,2); (2,5); (4,5)\}),$$

with the 2-edge feasible path $P = \{(1,4); (4,5)\}$ and the length $C(P) = 7$.

Theorem 2.2. *If*

$$C(\overline{M}(S_1, S_2)) = C'(S_1) + C''(S_2)$$

for all $\overline{M}(S_1, S_2) \neq \emptyset$ *and if all the profiles in the game* Γ^+ *are feasible, then* $\Gamma, \Gamma^+, \Gamma_*^+$ *have an equilibrium, moreover, it is the same in all three games* $\Gamma, \Gamma^+, \Gamma_*^+$.

Proof. All the profiles in the game Γ^+ are feasible. Then $\mathcal{B}_2(S_1) = \mathcal{B}_2$ for all $S_1 \in \mathcal{B}_1$, and $\mathcal{B}_1(S_2) = \mathcal{B}_1$ for all $S_2 \in \mathcal{B}_2$. Then, taking into account Lemma 2.1, we have

$$\max_{S_1 \in \mathcal{B}_1} \min_{S_2 \in \mathcal{B}_2(S_1)} C(\overline{M}(S_1, S_2)) = \max_{S_1 \in \mathcal{B}_1} \min_{S_2 \in \mathcal{B}_2} C(\overline{M}(S_1, S_2)) \leq$$

$$\leq \min_{S_2 \in \mathcal{B}_2} \max_{S_1 \in \mathcal{B}_1} C(\overline{M}(S_1, S_2)) = \min_{S_2 \in \mathcal{B}_2} \max_{S_1 \in \mathcal{B}_1(S_2)} C(\overline{M}(S_1, S_2)),$$

and, taking into account Lemma 2.2, we have

$$\max_{S_1\in\mathcal{B}_1}\min_{S_2\in\mathcal{B}_2} C(\overline{M}(S_1,S_2)) = \max_{S_1\in\mathcal{B}_1}\min_{S_2\in\mathcal{B}_2(S_1)} C(\overline{M}(S_1,S_2)) \geq$$

$$\geq \min_{S_2\in\mathcal{B}_2}\max_{S_1\in\mathcal{B}_1(S_2)} C(\overline{M}(S_1,S_2)) = \min_{S_2\in\mathcal{B}_2}\max_{S_1\in\mathcal{B}_1} C(\overline{M}(S_1,S_2)).$$

From these inequalities follows that

$$\max_{S_1\in\mathcal{B}_1}\min_{S_2\in\mathcal{B}_2} C(\overline{M}(S_1,S_2)) = \max_{S_1\in\mathcal{B}_1}\min_{S_2\in\mathcal{B}_2(S_1)} C(\overline{M}(S_1,S_2)) =$$

$$= \min_{S_2\in\mathcal{B}_2}\max_{S_1\in\mathcal{B}_1(S_2)} C(\overline{M}(S_1,S_2)) = \min_{S_2\in\mathcal{B}_2}\max_{S_1\in\mathcal{B}_1} C(\overline{M}(S_1,S_2)).$$

Therefore Γ^+ and Γ^+_* have the equilibrium profile. Finally, it follows from Theorem 2.1 that all the games $\Gamma, \Gamma^+, \Gamma^+_*$ have the same equilibrium profile. ∎

Remark 2.4. *Lemma 2.2 and Theorem 2.1 may be extended for other types of functions. One of such types is, for example:* $C(\overline{M}(S_1,S_2)) = C'(S_1)\cdot C''(S_2)$, *where* $C'\colon \mathcal{B}_1 \to N^*$, $C''\colon \mathcal{B}_2 \to N^*$.

Remark 2.5. *Example 2.1 shows that if in the game* Γ^+ *there is a profile that is not feasible, then the equilibrium profile may be absent from any game* Γ, Γ^+, *or* Γ^+_*.

Remark 2.6. *Example 2.1 illustrates also that the equilibrium profile may exist in the games* Γ, Γ^+, *but in the corresponding game* Γ^+_* *it may be absent. Inverse is possible: in the game* Γ^+_* *there is an equilibrium profile, but in the games* Γ, Γ^+ *it is absent.*

Theorem 2.2 formulates only sufficient condition for the existence of the equilibrium profile in the games $\Gamma, \Gamma^+, \Gamma^+_*$. The following theorem formulates necessary and sufficient conditions.

Theorem 2.3. *Let*

$$C(\overline{M}(S_1,S_2)) = C'(S_1) + C''(S_2)$$

for all $\overline{M}(S_1,S_2) \neq \emptyset$. *The profile* (S_1^*, S_2^*) *forms an equilibrium in all the games* $\Gamma, \Gamma^+, \Gamma^+_*$ *if and only if the profile* (S_1^*, S_2^*) *is an equilibrium in the game* Γ^+ *and*

$$\max_{S_1\in\mathcal{B}_1(S_2)} C(\overline{M}(S_1,S_2)) \geq C(\overline{M}(S_1^*,S_2^*)),$$

for all $S_2 \in \mathcal{B}_2$.

Proof. Necessity is obvious.

Sufficiency follows from the following relations:

$$\begin{aligned} & \min\max\Gamma^+ \overset{\text{L. 7.2.1}}{\geq} \max\min\Gamma^+ = \\ \overset{\text{L. 7.2.3}}{=} \quad & \max\min\Gamma_1^+ \overset{\text{L. 7.2.2}}{\geq} \min\max\Gamma_2^+, \end{aligned} \tag{5}$$

and Theorem 2.1. ∎

If the game Γ^+ has an equilibrium profile (S_1^*, S_2^*), then, taking into account Lemma 2.3 and Theorem 2.3, we deduce that $\min\max\Gamma^+ < +\infty$. This means that, for strategy $S_2^* \in \mathcal{B}_2$,

$$\overline{C}(S_1, S_2^*) = C(\overline{M}(S_1, S_2^*)) < +\infty$$

for all $S_1 \in \mathcal{B}_1$.

Definition 2.3. *A strategy that may have only feasible profiles, $\mathcal{B}_1(S_2) = \mathcal{B}_1$ for the first player and $\mathcal{B}_2(S_1) = \mathcal{B}_2$ for the second one, is called an **essential feasible strategy**.*

From Theorem 2.3 follows the next statement.

Corollary 2.1. *If the second player does not have at least one essential feasible strategy in the game Γ^+, then both the games Γ and Γ^+ do not have equilibrium profiles.*

3. SOLVABLE MATRIX GAMES ON DIGRAPHS

An investigation of digraph matrix games implies solving of three important problems:

1 the problem of determining maximin and minimax profiles;

2 the problem of determining feasible subgraphs in graphs of maximin and minimax profiles;

3 the problem of determining an equilibrium profile.

From relations (5) it follows that for an equilibrium profile computing in Γ, Γ^+, Γ_*^+, it is sufficiently to determine and to compare minimax profiles in the games Γ^+ and Γ_2^+.

If $\min\max\Gamma^+ = \min\max\Gamma_2^+$, **then** all three games Γ, Γ^+, Γ_*^+ are solvable and have the same equilibrium profile,

else it is necessary to find $\max\min\Gamma^+$ that is equal to $\max\min\Gamma_2^+$ and to compare it with $\min\max\Gamma^+$ and $\min\max\Gamma_1^+$:

- **if** $\max\min\Gamma^+ = \min\max\Gamma^+$, **then** Γ^+ is solvable;
- **if** $\max\min\Gamma^+ = \min\max\Gamma_1^+$, **then** Γ_*^+ is solvable;
- **if** $\max\min\Gamma^+ \neq \min\max\Gamma^+$ and $\max\min\Gamma^+ \neq \min\max\Gamma_1^+$, **then** all three games Γ, Γ^+, Γ_*^+ are unsolvable.

Consequently, in order to investigate the games Γ^+ and Γ_*^+, the problem of determining maximin and minimax profiles with the corresponding maximin and minimax feasible graphs becomes very important. As the games have limited numbers of profiles, maximin and minimax profiles hypothetically may be found by an exhaustive search, which for large m becomes a hard computational problem. It is obvious, that a game has polynomial complexity if both maximin and minimax profiles may be found in polynomial time on n and m. If the problem of a feasible graph $\overline{M}(S_1, S_2) \neq \emptyset$ construction in G_S for some profile (S_1, S_2) is NP – complete, then the game is at least NP – hard.

The exhaustive search method for solving the game Γ has an exponential complexity, supposing it is necessary to examine $2^{|E|}$ profiles. If the algorithm for construction a feasible subgraph in G has the polynomial complexity $O(n^{k_0}m^{l_0})$ and $|\mathcal{B}_1| = O(n^{k_1}m^{l_1}), |\mathcal{B}_2| = O(n^{k_2}m^{l_2})$, where $k_0, k_1, k_2, l_0, l_1, l_2$ are numbers independent of m and n, then the straightforward method in Γ^+ and Γ_*^+ has the polynomial complexity

$$O(n^{k_0+k_1+k_2}m^{l_0+l_1+l_2}).$$

Thus, depending on properties of G and elements of $\mathcal{D}$, this problems may be essentially simplified in particular games. Further on, we illustrate this for some particular games.

3.1. MAXIMIN DIRECTED TREE

Let G be an acyclic digraph and assume that there exist paths in G from every vertex $v \in V$ to vertex v_0. Let $\mathcal{D}$ be the set of all directed trees of G going to v_0; $C : \mathcal{D} \to \mathcal{R}$ be the length of tree (sum of edge lengths). The first player has the aim to maximize the length of tree, the second tries to minimize it.

Take into consideration that feasible graphs are directed trees, we will define admissible strategies so that from every vertex except v_0 exactly one edge is going out. In this case every element belonging to $\mathcal{D}$ at least once is feasible subgraph in Γ^+. Remark, that inadmissible strategies are not advantageous to players because they either not ensure tree construction or they lead to adversary possibility to choice from several alternatives. Therefore, either $\mathcal{B}_1$ and $\mathcal{B}_2$ contain the optimal strategies of the players and the game Γ^+ is right defined, that ensure equality of costs of the games Γ and Γ^+.

Remark further, for all $(S_1, S_2) \in \mathcal{B}_1 \times \mathcal{B}_2$ we have $\overline{M}(S_1, S_2) = (V, S_1 \cup S_2)$. This means, that all profiles of the game Γ^+ are feasible. From theorem 2 follows that $\Gamma, \Gamma^+, \Gamma_*^+$ have the same equilibrium profile.

To determine the maximin input in v_0 tree we propose the following application of the dynamical programming method [45]:

$M = \{v_0\},\ T = \emptyset$
while $|M| < n$ do
begin $(u^*, v^*) = \underset{(u,v)\in((V\setminus M)\times M)\cap E}{\arg\max} C(u, v)$
If $(u^* \in V_1)$ or $((u^* \in V_2)$ and $|((V\setminus M)\times M)\cap E| = 1)$
then $M = M \cup u^*,\ T = T \cup (u^*, v^*)$
else $E = E\setminus(u^*, v^*)$.
end
T – maximin tree.

Remark 3.1. *Algorithm determines maximin tree in arbitrary digraph. But, because maximin may be not equal to minimax in general case, for determining of minimax tree, we must found on every iteration*

$$(u^*, v^*) = \underset{(u,v)\in((V-M)\times M)\cap E}{\arg\min} C(u, v).$$

3.2. MAXIMIN DIRECTED PATH

Let G be an acyclic digraph and assume that there exists a path from every vertex $v \in V$ to vertex v_t. Let $\mathcal{D}$ be the set of all directed paths from v_0 to v_t; $C : \mathcal{D} \to \mathcal{R}$ be the length of path (sum of edge lengths). The first player has the aim to maximize the length of path, the second has the aim to minimize it.

We will define admissible strategies so that from every vertex except v_0 exactly one edge is going out. In this case every G_S is an input in v_0 tree, containing a path from v_s to v_t. The set of all feasible profiles of the maximin path game Γ^+ is equivalent to the set of all feasible profiles of the maximin tree game Γ^+. Therefore all three games $\Gamma, \Gamma^+, \Gamma^+_*$ have the same equilibrium profile.

To determine maximin path we may use an adaptation of Dijkstra algorithm. An example of such adaptation is presented in [7].

3.3. MAXIMIN TRAVELING SALESMAN PROBLEM WITH TRANSPORTATION

Generaly, a Traveling Salesman Problem (TSP) includes diverse mathematical models of distinct real practical problems. Its history may be traced back to the Irish mathematician Sir William Rowan Hamilton and British mathematician Thomas Penyngton Kirkman, who treated it incipiently in 1800s [6, 38]. In the 1930s Karl Menger studied TSP in general form and Hassler Whitney and Merrill Floodlater promoted promoted TSP later [38].

We consider and investigate an original model of TSP motivated by applications — a synthesis of classical TSP and classical Transportation Problem. Algorithms based on Integer Programming cutting-plane methods and Branch and Bound Techniques are obvious. A maximin traveling salesman problem and the correspondent maximin Hamiltonian cycle problem may be formulated similar to the problem of maximin directed tree. So, in this subsection we expose only the specific features of the traveling salesman problem with transportation. But, why is important this problem in context of considered games? The answer is rather obvious: it suggests an example of games which are simple formulated but are hard to solve.

3.3.1 Introduction. The TSP gained notoriety over the past century as the prototype of problem that is easy to state and hard to solve practically. It is simply formulated: *a traveling salesman has to visit exactly once each of n cities and to return to the start city but in a order that minimizes the total cost (it is supposed that the cost c_{ij} of traveling from every city i to every city j is known).* There are other related formulations of this problem and a lot of methods for solving [26, 35, 14, 17, 3]. It is well known equivalence of TSP with Hamiltonian circuit problem [16].

The TSP is representative for a large class of discrete problems known as NP-complete combinatorial optimization problems [16]. NP-complete problems have an important property that all of them have or don't have simultaneously polynomial-time algorithms for its solving [16]. To date, no one has found efficient (polynomial-time) algorithms for the TSP. But over the past few years many practical problems of really large size are solved [1]. Thus, at present, the largest solved Norwegian instance of TSP has 24 978 cities (D. Applegate, R. Bixby, V. Chvátal, W. Cook, and K. Helsgaun — 2004). But, the largest solved instance of TSP includes 85,900 cities (points) in an application on chips (2005-2006) [1].

The Transportation Problem is a well known classical problem [20]. There are several efficient methods for its solving [33, 21, 34]. Note that there exists also an impressive extension of Transportation Problem in functional spaces [48, 49], that have to highlight once and more a transportation problem significance.

The TSP with Transportation and Fixed Additional Payments (TSPT) generalizes these two problems: TSP and Transportation Problem [46]. The TSPT has some affinities with a Vehicle Routing Problem [9, 43].

3.3.2 TSPT Formulations. Let a digraph $G = (V, E), |V| = n, |E| = m$ be given. Each node $j \in V$ has its own capacity δ_j (demand, if $\delta_j < 0$, supply, if $\delta_j > 0$), such that $\sum_{j=1}^{n} \delta_j = 0$. A salesman starts his traveling from the node $k \in V$ with $\delta_k > 0$. The unit cost of transportation throw arc $(i, j) \in E$ is equal to c_{ij}. If the arc (i, j) is active, then the additional payment d_{ij} is demanded. If $(i, j) \notin E$, then $c_{ij} = d_{ij} = \infty$. We must find a Hamiltonian circle and a starting node $k \in V$ with a property that the respective salesman travel satisfies all the demands $\delta_j, j = 1, ..., n$, and minimizes the total cost.

The TSPT may be formulated as an integer programming problem. Let x_{ij} be a quantity of product which is transported via (i, j). Let $y_{ij} \in \{0; 1\}$ be equal to 1 if $x_{ij} > 0$, and let x_{ij} be equal to 0 if $y_{ij} = 0$. In such notation the TSPT, as stated above, is equivalent to the following problem:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left(c_{ij}x_{ij} + d_{ij}y_{ij}\right) \to \min, \tag{6}$$

$$\sum_{i=1}^{n} y_{ij} = 1, \quad j = 1, ..., n, \tag{7}$$

$$\sum_{j=1}^{n} y_{ij} = 1, \quad i = 2, ..., n, \tag{8}$$

$$\sum_{k=1}^{n} x_{jk} - \sum_{i=1}^{n} x_{ij} = \delta_j, \quad j = 1, ..., n, \tag{9}$$

$$u_i - u_j + ny_{ij} \leq n - 1, \quad i, j = 2, ..., n, \ i \neq j, \tag{10}$$

$$x_{ij} \leq My_{ij}, \quad i, j = 1, ..., n, \tag{11}$$

$$x_{ij} \geq 0, \ y_{ij} \in \{0; 1\}, \ u_j \geq 0, \quad i, j = 1, ..., n, \tag{12}$$

where $M = \sum_{j=1}^{n} |\delta_j|$.

If $c_{ij} = 0$ for all $i, j = 1, ..., n$, then the problem (6)–(12) becomes a classical TSP. If $d_{ij} = 0$ for all $i, j = 1, ..., n$, then the problem (6)–(12) is simplified to a classical Transportation Problem.

Theorem 3.1. *The TSPT and the problem (6)–(12) are equivalent.*

Proof. The components (6)–(8), (10) and (12) of the problem (6)–(12) define a Hamiltonian circuit [33]. The components (6), (9) and (12) state the transportation problem [20]. The constraint (11) realizes a connection between these two *"facets"* of the TSPT. The starting node $k \in V$ may determined by an elementary sequential search. ∎

A capacity mathematical model of the problem is obtained when any arc $(i, j) \in E$ has an upper bound capacity $u_{ij} > 0$. Constraints

$$x_{ij} \leq u_{ij}y_{ij}, \ (i, j) \in E,$$

substitute (11).

The inequalities

$$l_{ij}y_{ij} \leq x_{ij} \leq u_{ij}y_{ij}, \ (i, j) \in E,$$

substitute (11) when any arc $(i, j) \in E$ has also a lower bound capacity $l_{ij} > 0$.

Kuhn [33] restrictions (10) may be substituted by equivalent restrictions:

$$\sum_{i \in K} \sum_{j \in K} y_{ij} = |K| - 1, \ \forall K \subset V,$$

where K is any proper subset of V.

3.3.3 Algorithms for TSPT. It is obvious that the solution of the classical TSP does not solve TSPT.

The branch-and-bound algorithm may be constructed on back-tracking technique for a branch generation and lower bound estimation of the sum of a 1-tree value and T_0, where T_0 is calculated at the first step. T_0 represents the value of a minimal cost flow problem obtained in a relaxed problem without Hamiltonian circuit requirement. For an efficient bounding, T_0 may be substituted, at every step of the algorithm, by the exactly cost of transportation throw the respective fragment of the circuit.

A direct solving of (6)–(12) with an Gomory type cutting-plane algorithms is rational for a problem with modest size. In recent vogue opinion, the branch-and-cut super-algorithm [21, 34] may be much more recommended for the TSPT.

Finally, note that a dynamic programming approach [4] to solve the TSPT implies some difficulties as the TSPT optimal value depends on first node choosing from which the travel starts. This fact may be simply taken into consideration in previous methods, but not in dynamic programming method.

3.3.4 TSP Matrix Games. Finally, let us only remark once again that the TSP and TSPT suggest us an example of matrix games that are simple formulated, but are difficult to solve because of its computation complexity.

3.4. MAXIMIN COST FLOW

Let us consider a flow network on a digraph $G = (V, E)$, $|V| = n$, $|E| = m$ with an output (source) vertex $v_s \in V$ and a input vertex (sink) $v_t \in V$, $v_s \neq v_t$, where any edge $(u, v) \in E$ has a capacity $b(u, v) \in Z^+$ and a unit transportation cost $c(u, v) \in Z$. The set V is partitioned into two disjoint sets of player positions:

$$V_1, \ V_2, \ (V_1 \cup V_2 = V, \ V_1 \cap V_2 = \emptyset).$$

Without loss of generality let us assume that $v_s \in V_1$. Thus, the player edge sets are

$$E_1 = \{(u, v) \in E \,|\, u \in V_1\}, \quad E_2 = \{(u, v) \in E \,|\, u \in V_2\}.$$

Subsets $S_1 \subseteq E_1$, $S_2 \subseteq E_2$ are strategies of the first and second players, correspondingly. Any profile $(S_1, S_2) \in 2^{E_1} \times 2^{E_2}$ generates a net $G_S = (V, S_1 \cup S_2)$. In the net G_S, a flow f of a fixed value φ_0 is defined as the vector $f \in \mathbb{R}^{|S_1 \cup S_2|}$ which satisfies the following properties:

$$1^0. \qquad 0 \le f(u, v) \le b(u,v), \qquad (u, v) \in S_1 \cup S_2,$$

$$2^0. \sum_{(u,v)\in S_1 \cup S_2} f(u, v) - \sum_{(v,u)\in S_1 \cup S_2} f(v, u) = \begin{cases} 0, & u \notin \{v_s, v_t\}, \\ \varphi_0, & u = v_s, \\ -\varphi_0, & u = v_t. \end{cases}$$

The cost of the flow f is equal to $\sum\limits_{(u,v)\in S_1 \cup S_2} c(u, v) f(u, v)$.

Let us suppose that there exists at least one flow with the value φ_0 in the considered net.

For any pair of strategies (S_1, S_2) there is a polyhedron of solutions of system $1^0 - 2^0$, denoted by $F_S = F(G_S)$. Generally, the polyhedron F_S may be an empty set for some pair of strategies if system $1^0 - 2^0$ does not have solutions, but, due to our supposition, there exists at least one pair of strategies for which $F_S \neq \emptyset$. It is known that if the capacity $b(u, v)$ is integer for any $(u, v) \in E$, then all the vertices of the polyhedron F_S have integer components. Thus, as F_S is bounded, the set of all flows, corresponding to (S_1, S_2), is a linear convex combination of a finite number of integer flows. The cardinality of the set F_S may be equal to 0, when system $1^0 - 2^0$ does not have solutions, may be equal to 1, when $1^0 - 2^0$ has one solution, or may be equal to $\aleph$, when $1^0 - 2^0$ has an infinite number of solutions.

Let $\mathcal{D}$ be a set of all subgraphs (subnets) of G that has a flow of a value φ_0 from v_s to v_t. Let

$$M\colon 2^G \to \mathcal{D},\ M(S_1, S_2) = G_s \cap \mathcal{D}$$

be the choice function,

$$C\colon \mathcal{D} \to R,\ C(\Gamma) = \max_{f\in F(\Gamma)} \sum c(e) f(e)$$

be the choice criterion and

$$\overline{M}(S_1, S_2) = \begin{cases} \operatorname*{argmax}_{\Gamma\in M(S_1,S_2)} C(\Gamma), & \text{if } M(S_1, S_2) \neq \emptyset, \\ \emptyset, & \text{otherwise}, \end{cases}$$

be a mono-valued choice function that chooses the flow of the value φ_0 with a minimal cost in the net G_S. Then, we have the following cost function (payoff matrix):

$$\overline{C}(S_1, S_2) = \begin{cases} C(\overline{M}(S_1, S_2)), & \text{if } \overline{M}(S_1, S_2) \neq \emptyset, \\ -\infty, & \text{if } \overline{M}(S_1, S_2') = \emptyset,\ \forall S_2' \in 2^{E_2}, \\ +\infty, & \text{otherwise}. \end{cases}$$

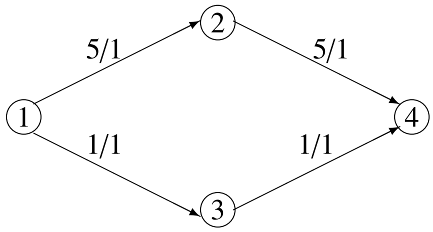
So, the matrix game Γ is defined. By analogy with the general case, we define games Γ^+ and Γ^+_* too. A strategy is called admissible if there exists an adversary strategy for which the corresponding profile is feasible (has a φ_0 flow).

Lemma 3.1. *Let the net G has at least one φ_0 flow. Then both the players have at least one feasible strategy in* Γ^+.

Proof. Let us order rows and columns of the matrices of Γ^+ in a non-decreasing order of cardinalities of corresponding strategies. It is obvious that the pair of strategies, which are equal to a union of all admissible strategies of a correspondent player, are feasible. ∎

The following example shows that for a solvable game Γ there are several non-identical approaches to define games Γ^+, Γ^+_* so that the game Γ^+_* may be both solvable and unsolvable.

Example 3.1. Consider the flow games Γ, Γ^+, Γ^+_*, defined on the following graph



where $\varphi_0 = 1;\ v_s = 1;\ v_t = 4;$

$$V_1 = \{1\}; \quad V_2 = \{2;\ 3;\ 4\};$$

$$E_1 = \{(1,\ 2);\ (1,\ 3)\}; \quad E_2 = \{(2,\ 4);\ (3,\ 4)\}.$$

The following table contains the payoff matrices of the considered games.

1	2				
	(2, 4)	(3, 4)	(2, 4) (3, 4)	$\min\limits_{\mathcal{B}_2}$	$\min\limits_{\mathcal{B}_2(S_1)}$
(1, 2)	10	$+\infty$	$\boxed{10}$	10	10
(1, 3)	$+\infty$	2	2	2	2
(1, 2) (1, 3)	10	2	2	2	2
$\max\limits_{\mathcal{B}_1}$	$+\infty$	$+\infty$	10	10\10	
$\max\limits_{\mathcal{B}_1(S_2)}$	10	2	10		2\10

The games Γ, Γ^+ have the equilibrium profile. The game Γ^+_* does not have.

If the sets of admissible strategies are narrowed so that from every vertex, except the fourth, at least one edge is going out, then the payoff matrices are modified

	2		
1	(2, 4) (3, 4)	$\min\limits_{\mathcal{B}_2}$	$\min\limits_{\mathcal{B}_2(S_1)}$
(1, 2)	$\boxed{\boxed{10}}$	10	10
(1, 3)	2	2	2
(1, 2) (1, 3)	2	2	2
$\max\limits_{\mathcal{B}_1}$	10	$10\backslash 10$	
$\max\limits_{\mathcal{B}_1(S_2)}$	10		$10\backslash 10$

and all the games Γ, Γ^+, Γ^+_* have the same equilibrium profile.

Thus, for a game Γ there are two pairs of games Γ^+, Γ^+_*. For one pair, the game Γ^+_* does not have an equilibrium profile. For another one, all the games Γ, Γ^+, Γ^+_* have the same equilibrium profile.

It is well known that the problem of minimal cost flow may be represented as a linear programming problem [33]. By numbering the vertices and edges of G in a such way that vertices and edges of the first player are the first in the order list, we can define elements of an incidence matrix $A = [a_{ij}]$ of the graph G as

$$a_{ij} = \begin{cases} +1, & \text{if } e_j \text{ exits from vertex } i, \\ -1, & \text{if } e_j \text{ enters in vertex } i, \\ \ \ 0, & \text{otherwise,} \end{cases}$$

$i = 1, ..., n;\ j = 1, ..., m$. By notation $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = (f_1, \ldots, f_m)^T \in \mathbb{R}^m$, f_j — the flow through edge e_j; $b, c \in \mathbb{R}^m$, b_j — the capacity of edge e_j, c_j — the unit cost of edge e_j flow;

$$d \in \mathbb{R}^m, \qquad d_i = \begin{cases} -1, & \text{if } v_i = v_s, \\ +1, & \text{if } v_i = v_t, \\ \ \ 0, & \text{otherwise,} \end{cases}$$

the following minimal cost flow problem in the net G may be formulated

$$c^T f \to \min, \tag{13}$$

$$\begin{cases} Af &= d\varphi_0, \\ f &\le b, \\ f &\ge 0. \end{cases} \tag{14}$$

Let us associate with the first n constraints, corresponding to the safety law of the φ_0 flow, dual variables π_i, and with the remains m constraints — dual variables γ_k.

Then, the problem (13) – (14) has the following dual problem

$$\varphi_0(\pi_s - \pi_k) + b^T\gamma \to \max, \tag{15}$$

$$\begin{cases} \pi_i - \pi_j + \gamma_k \le c_k, & \text{for } e_k = (v_i,\, v_j) \in E, \\ \qquad \gamma_k \le 0, & k = 1, ..., m. \end{cases} \tag{16}$$

According to the Strong Duality Theorem of Linear Programming Theory, problems (13) – (14) and (15) – (16) have optimal solutions if and only if the system

$$\begin{cases} c^T f &=& \varphi_0\pi_s - \varphi_0\pi_t + b^T\gamma, \\ Af &=& d\varphi_0, \\ f &\le& b, \\ f &\ge& 0, \\ \pi_i - \pi_j + \gamma_k &\le& c_k, \quad \text{for } e_k = (v_i,\, v_j) \in E, \\ \gamma_k &\le& 0, \qquad k = 1, ..., m, \end{cases} \tag{17}$$

has a solution (remark, the first equality is the binding one).

It is important to observe that for any profile $(S_1,\, S_2) \in \mathcal{B}_1 \times \mathcal{B}_2$ there is a system of (17) type. Let $\Phi(S_1, S_2)$ be the set of all the solutions of the corresponding system (17). Then, the cost function may be defined as

$$\overline{C}(S_1,\, S_2) = \begin{cases} c^T f, & \text{if } \Phi(S_1, S_2) \ne \emptyset \ \text{ where } (f, \pi, \gamma)^T \in \Phi(S_1, S_2), \\ -\infty, & \text{if } \Phi(S_1, S_2') = \emptyset \ \text{ for all } \ S_2' \in \mathcal{B}_2, \\ +\infty, & \text{otherwise.} \end{cases}$$

By applying linear programming concepts and results, let us show now that problems of finding maximin and minimax profiles in the flow game are equivalent to maximin and minimax linear problems.

Clearly, the set of feasible solutions of problem (13) – (14) is an polyhedron in $\mathbb{R}^m$, and the minimum is attained on its vertex. Then, the first player purpose is to maximize the flow cost by rational choice of the net G_S structure. This is equivalent to maximisation of the flow cost by optimal choice of the basic columns of the matrix A that corresponds to a choice of edges of E_1. The second player purpose is to minimize the cost of the flow by rational choice of the net G_S structure. This is equivalent to minimisation of the flow cost by optimal choice of the basic columns of the matrix A. Therefore, the first player has to choice a feasible solution for which at least columns, that correspond to edges from E_1, have non-negative dual estimations. The second player has to choice a feasible solution for which at least columns, that correspond to edges from E_2, have non-positive dual estimations. Consequently, a feasible solution is both saddle point and equilibrium profile. So, the problem of an equilibrium profile computing in the flow game is equivalent to a maximin linear problem

$$\max_{f_1} \min_{f_2} c^T \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{18}$$

$$\begin{cases} A\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} & = & d\varphi_0, \\ 0 \leq & \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} & \leq b. \end{cases} \tag{19}$$

Problem (18) – (19) is equivalent to

$$\varphi(f_1) = \min_{f_2}(c_1 f_1 + c_2 f_2) = c_1 f_1 + \min_{f_2} c_2 f_2 \to \max. \tag{20}$$

$$\begin{cases} A_2 f_2 & = & d\varphi_0 - A_1 f_1, \\ 0 \leq & f_1 & \leq b_1, \\ 0 \leq & f_2 & \leq b_2. \end{cases} \tag{21}$$

In (21), the function $\varphi(f_1)$ is determined as a solution of a linear parametric program with restrictions (21). It is known that solutions of such problems are piecewise linear convex functions [18]. Therefore, $\varphi(f_1)$ is a piecewise-linear convex function on on (21).

Analogically, the function

$$\psi(f_2) = \max_{f_1}(c_1 f_1 + c_2 f_2) = \max_{f_1}\ c_1 f_1 + c_2 f_2$$

is a piecewise-linear concave function on (21).

Theorem 3.2. *The function $\varphi(f_1)$ is a piecewise-linear convex function on (21). The function $\psi(f_2)$ is a piecewise-linear concave function on (21).*

The problems of maximizing $\varphi(f_1)$ and minimizing $\psi(f_2)$ are the problems of concave programming, which, as it is well known, are NP – hard even on a unit hypercube [44]. Consequently, taking into the consideration that (20) – (21) may be represented as the problem of maximizing a piecewise-linear convex function over a hyper-parallelepiped the following result becomes obvious.

Theorem 3.3. *The maximin (minimax) cost flow problem is an NP – hard problem.*

4. POLYMATRIX GAMES ON DIGRAPHS

Matrix games may be generalized on the case of arbitrary number of players $p \leq n$. The vertex set V is partitioned into disjoint subsets of player positions

$$V_1, V_2, \ldots, V_p \quad \left(\bigcup_{i=1}^{p} V_i = V,\ \ V_i \cap V_j = \emptyset,\ \text{ for }\ i \neq j\right),$$

which define evidently the corresponding sets of player edges

$$E_i = \{(u, v) \in E \mid u \in V_i\},\ \ i = 1, ..., p.$$

All the players independently choose their strategies

$$(S_1, S_2, \ldots, S_p) \in 2^{E_1} \times \cdots \times 2^{E_p}.$$

After that, $\overline{M}(S)$ (defined analogically as above) is determined, $S = (S_1, S_2, \ldots, S_p)$. Each player determines his gain

$$c_i(S) = \begin{cases} c_i(\overline{M}(S)), & \text{if } \overline{M}(S) \neq \emptyset, \\ -\infty, & \text{if } \overline{M}(S') = \emptyset,\ \forall S'_k \in 2^{E_k},\ k \neq i, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $i = 1, \ldots, p$. Thus, the vector payoff function is defined as the mapping

$$c: \ 2^{E_1} \times \cdots \times 2^{E_p} \to \mathbb{R}^p,$$

which sets the correspondence between every profile of player strategies and their gains.

Analogically with the case of matrix games, polymatrix games with feasible strategies can be defined, requiring of course the ordering of player vertices ans edges.

The solution of the polymatrix game may be defined, e.g., as a Nash equilibrium profile [52, 22, 12]. If the characteristics of some players have similar tendencies to increase or decrease, coalition games may be considered. Evidently, if $p = 2$ and $c_2(S_1, S_2) = -c(S_1, S_2)$, a zero-sum matrix game is defined.

5. DYNAMIC GAMES ON DIGRAPHS

In this section, we consider games [45] which are closely related to extensive form games [23, 24, 25], network and algorithmic games [42, 30, 39, 11, 28]. They extend *simultaneous single stage* games considered above to *multi-stage* games.

Consider the above digraph G. Denote by Γ a *digraph polymatrix game* with p players defined on G. It is evident that the digraph matrix game is a particular case of the digraph polymatrix game when $p = 2$ and the gain of one of the players is a loss of his opponent.

The game Γ is a *single stage/single shot* game. Players choose their strategies simultaneously, at the same *single stage/single time moment*. As a result a feasible graph G^*_S is set, the cost of the game is determined and the gain or loss is distributed to players.

Let $\mathbb{G}$ be the set of all possible polymatrix games on a digraph G.

Generally, a *dynamic/multi-stage* game may be seen as a sequence of single stage games. It is denoted by $\Gamma(t)$, and it is defined equivalently both as a mapping

$$\Gamma : \mathbb{N} \to \mathbb{G},$$

and a sequence

$$(\Gamma_t)_{t\in\mathbb{N}} = (\Gamma(t))_{t\in\mathbb{N}} = (\Gamma(1), \Gamma(2), \ldots, \Gamma(t), \ldots).$$

The definition of the dynamic game $\Gamma(t)$ may be completed by a terminus criterion — the criterion which defines conditions for which the game ends/stops/finishes. According to considered types of future horizons, the dynamic games may be divided into two classes:

- the class of *dynamic games with finite time horizon* θ denoted by $\Gamma_\theta(t)$ and defined equivalently both as a mapping
$$\Gamma : \{1, 2, \ldots, \theta\} \to \mathbb{G},$$
and a finite sequence
$$(\Gamma_{\theta 1}, \Gamma_{\theta 2}, \ldots, \Gamma_{\theta\theta}) = (\Gamma(1), \Gamma(2), \ldots, \Gamma(\theta)),$$
- the class of *dynamic games with infinite time horizon* denoted by $\Gamma_\infty(t)$ or simply $\Gamma(t)$, and defined equivalently both as a mapping
$$\Gamma : \mathbb{N} \to \mathbb{G},$$
and an infinite sequence
$$(\Gamma_1, \Gamma_2, \ldots) = (\Gamma(1), \Gamma(2), \ldots).$$

Remark, the infinite and finite dynamic games $\Gamma_\infty(t)$ and $\Gamma_\theta(t)$ are *repeated games* (supergames or iterated games) [29, 2, 15, 32, 27, 40, 31] if the set of all single-stage games $\mathbb{G}$ consists only of one element. A game of a single fixed type is played at every stage t of a repeated game. Evidently, the class of repeated games may be enlarged with dynamic games in which a subsequence of games is repeated.

Games considered in *theory of moves* present an alternative point of view on digraph matrix and polymatrix games [8] based on *dynamics* of player moves.

The strategies of the player $i \in \{1, 2, \ldots, p\}$ in the dynamic game $\Gamma(t)$ are defined as sequences of stage game strategies

$$(S_i(t))_{t\in\mathbb{N}} = (S_i(1), S_i(2), \ldots, S_i(t), \ldots) \in 2^{E_i} \times 2^{E_i} \times \cdots \times 2^{E_i} \times \cdots$$

The game $\Gamma(\tau)$ of any particular stage τ generates a stage profile digraph $G^*_S(\tau)$. Denote by $S(\tau)$ the profile of player strategies at the stage τ, i.e.

$$S(\tau) = (S_1(\tau), \ldots, S_p(\tau)).$$

The payoff of the stage game $\Gamma(\tau)$ is a vector denoted equivalently both by $c(\Gamma(\tau))$ and $c(S(\tau))$, with components

$$c_i(S(\tau)) = \begin{cases} c_i(\overline{M}(S(\tau))), & \text{if } \overline{M}(S(\tau)) \neq \emptyset, \\ -\infty, & \text{if } \overline{M}(S'(\tau)) = \emptyset,\ \forall S'_k(\tau) \in 2^{E_k}, k \neq i, \\ +\infty, & \text{otherwise,} \end{cases}$$

$i = 1, \ldots, p$.

If the game $\Gamma(t)$ is considered on the finite discrete time interval defined as $\Theta = \{1, 2, \dots, \theta\}$, then it is obvious that the player strategies are finite sequences of the form $(S_i(1), S_i(2), \dots, S_i(\theta))$, where $S_i(t),\ t = 1, 2, \dots \theta$, are the strategies of the i^{th} player on the corresponding stages.

The cost/payoff of the dynamic game $\Gamma_\theta(t)$ is determined on the base of the cost/payoff of stage games on the stages $1, 2, \dots, \theta$. In the simplest case, the cost function may be defined by

$$1.\ c(\Gamma_\theta(t)) = \sum_{t=1}^{\theta} c(\Gamma(t));\quad 2.\ \overline{c}(\Gamma_\theta(t)) = \frac{1}{\theta}\sum_{t=1}^{\theta} c(\Gamma(t)),$$

interpreted as vector expressions for polymatrix games and scalar expressions for matrix games, or by

$$3.\ c_i(\Gamma_\theta(t)) = \sum_{t=1}^{\theta} c_i(\Gamma(t));\quad 4.\ \overline{c}_i(\Gamma_\theta(t)) = \frac{1}{\theta}\sum_{t=1}^{\theta} c_i(\Gamma(t)),$$

$i = 1, \dots, p$, interpreted as components of the payoff vector function in polymatrix dynamic games.

If we suppose that the type of stage games is fixed, the result of the t-stage game does not depend on results of previous stages and the dynamic game ends at the step θ, then it is obvious that for solvable stage game Γ with optimal strategies (S_1^*, S_2^*) the corresponding dynamic game $\Gamma_\theta(t)$ has optimal strategies

$$(S_1(t))_{t\in\Theta} = \underbrace{(S_1^*, \dots, S_1^*)}_{\theta}$$

and

$$(S_2(t))_{t\in\Theta} = \underbrace{(S_2^*, \dots, S_2^*)}_{\theta}$$

with cost $c(\Gamma_\theta(t)) = \theta c(S_1^*, S_2^*)$ or $\overline{c}(\Gamma_\theta(t)) = c(S_1^*, S_2^*)$. Therefore, such dynamic games are generally identical with matrix and polymatrix games.

A dynamic game model acquires more valuable features if the *terminus criterion implies to finish the game* when a feasible subgraph with prescribed structure is constructed in some stage τ graph $G_S(\tau)$. In such case the time horizon may be infinite, but the game nevertheless stops if the terminus criterion is satisfied.

Next, we associate with every stage t a dynamic game state

$$G(t) = (W(t), E(t))$$

as a couple formed by a vertex set and an edge set which depend on t. Evidently, sets of player strategies depend on game states.

If the initial state is

$$G(0) = (W(0), E(0)) \subseteq (V, E)$$

at the initial time moment, the set of all possible strategies of the i^{th} player at the moment t may be defined as:

$$E_i(t) = \{(u, v) \in E | u \in V_i \cap W(t-1)\},\ t = 1, 2, \ldots;\ i = 1, \ldots, p,$$

where $W(t-1)$ is the set of vertices in the graph $G(t-1)$ at the stage $t-1$, V_i — the set of positions of the i^{th} player. Every subset $S_i(t) \subseteq E_i(t)$ is called a strategy of player i at the stage t.

The state of the dynamic game at the stage t is determined after examination of the corresponding game at the previous stage $t-1$ according to formula

$$W(t) = \{v \in V | \exists (u, v) \in \bigcup_{i=1}^{p} S_i(t)\},\ \ t = 1, \ldots, \tau.$$

As the player strategies at the stage t depend both on their positions and the game state at the moment t, we have to solve games of the same type at consecutive time moments t and $t+1$, but generally with different sets of strategies for every player. After that, as the player strategies are the edges which determine the mapping of $W(t-1)$ in $W(t)$, then players, in antagonistic interests, endeavour to increase (to decrease) the set of their own advantageous positions of the stage $W(t)$, and to decrease (to increase) the set of advantageous (non-advantageous), positions of adversaries on the same state $W(t)$. Therefore, G is the graph of all possible passages (strategies), their number being limited by 2^m. The set V defines the set of all possible states, which cardinality is limited by 2^n.

If the dynamical game is examined on infinite time interval $\{1, 2, \ldots\}$, then it follows from the limited number of states that some states of the game and corresponding strategies of players will be repeated in some sequences. It is obvious, that the payoff function $c(\Gamma(t))$, having the form of the number sequences, will be unlimited on $\tau \to +\infty$. Therefore, the definition of such game must be completed with special ending criterion: or the value cost function is larger then determined limit, or at some moment the graph of determined structure is constructed, etc. In the case of cost function $\overline{c}(\Gamma(t))$ we can examine the limit

$$\overline{c}(\Gamma(t)) = \lim_{\theta \to +\infty} \Big(\frac{1}{\theta} \sum_{t=0}^{\theta} c(\Gamma(t)) \Big),$$

for which there exists (as is mentioned above) repeated sequences of game states with limited value of the cost function, such that $\overline{c}(\Gamma(t)$ is equal to fixed number. In this case, the problem to find cycle of the states may be considered.

Next, lengths of edges can be functions depending on t and the cost of a dynamic game is calculated using static game costs only at some stages.

As mentioned, it is clear that the contents and type of dynamic games are depending on:

- static game;
- initial state and restriction of cardinality of the game states;
- cost function;
- edge length function;
- time interval on which the game is examined;
- terminus criterion,
- etc.

In investigation of dynamic games $\Gamma(t)$ it is useful sometimes to use the property that every dynamic game $\Gamma(t)$ can be represented as a matrix game.

6. CONCLUDING REMARKS

It is necessary to mention that strategies of players may be defined also as subsets of vertices, or the pair of subsets of vertices and edges. The investigation of such games, the determination of solvable games and the elaboration of corresponding algorithms are problems for future work.

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