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THE VANDERMONDE-TYPE SEQUENCES IN GROUPS

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Abstract In this work, we define the Vandermonde-type sequence and then we obtain the relations among the elements of the sequence and generating matrix of the sequence. Also, we study the Vandermonde-type sequence modulo m and we obtain the cyclic groups from the generating matrix of the sequence when read modulo m. Then we derive the relationships among the orders of the obtained cyclic groups and the periods of the Vandermonde-type sequence modulo m. Finally, we redefine the adjacency-type sequences by means of the elements of the groups which have two or more generators and then we obtain the periods of the Vandermonde-type sequence in the polyhedral groups $(n, 2, 2)$, $(2, n, 2)$ and $(2, 2, n)$ for $n \geq 3$ as applications of the results produced.

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1. INTRODUCTION AND PRELIMINARIES

It is well-known that the Vandermonde matrix V of order n is defined by

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

or

$$V_{i,j} = x_i^{j-1}$$

for all indices i and j.

It is clear that a Vandermonde matrix presents a geometric sequence in every row with the first element being 1. Note that some authors use the transpose of the above matrix.

Suppose that the $(n+k)$th term of a sequence is defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [11], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

$$A = \left[a_{i,j}\right]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument he obtained that

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geqslant 0$.

Let G be a finite j-generator group and let

$$X = \left\{ \left(x_1, x_2, \ldots, x_j\right) \in \underbrace{G \times G \times \cdots \times G}_{j} \mid \langle\left\{x_1, x_2, \ldots, x_j\right\}\rangle = G \right\}$$

we call $\left(x_1, x_2, \ldots, x_j\right)$ a generating j-tuple for G.

In Section 2, we define the Vandermonde-type sequence and then we give the relationship among the elements of the sequence and the generating matrix of the sequence. In [4, 5, 6, 7, 8, 12, 14], the authors obtained the cyclic groups and semigroups via some special matrices. In Section 3, we consider the multiplicative orders of the generating matrix of the Vandermonde-type sequence according to modulo α and then, we obtain the cyclic groups and semigroups. The study of recurrence sequences in groups began with the earlier work of Wall [15] where the ordinary Fibonacci sequences in cyclic groups were investigated. Recently, many authors have studied some special recurrence sequences in groups; see for example, [1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14]. In Section 4, we extend the theory to the Vandermonde-type sequence and then, we study this sequences in finite groups. Finally, we obtain the lengths of the periods of the Vandermonde-type sequences in the polyhedral groups $(n, 2, 2)$, $(2, n, 2)$ and $(2, 2, n)$, $(n \geq 3)$ in the 3-generator cases.

2. MAIN RESULTS AND PROOFS

Define the Vandermonde-type sequence as shown:

$$x_n^k = x_{n-1}^k + kx_{n-2}^k + k^2x_{n-3}^k + \cdots + k^{k-1}x_{n-k}^k \text{ for } k \geq 3 \text{ and } n \geq k+1,$$

where $x_1^k = x_2^k = \cdots = x_{k-1}^k = 0$ and $x_k^k = 1$.

Letting

$$M_k = \left[m_{i,j}\right]_{k\times k} = \begin{bmatrix} 1 & k & k^2 & \cdots & k^{k-1} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}, \ (k \geq 3)\,.$$

The matrix M_k is said to be the Vandermonde-type matrix. It can be readily established by mathematical induction that

$$(M_3)^n = \begin{bmatrix} x_{n+3}^3 & 9x_{n+1}^3 + 3x_{n+2}^3 & 9x_{n+2}^3 \\ x_{n+2}^3 & 9x_n^3 + 3x_{n+1}^3 & 9x_{n+1}^3 \\ x_{n+1}^3 & 9x_{n-1}^3 + 3x_n^3 & 9x_n^3 \end{bmatrix} \text{ for } n \geq 2, \tag{1}$$

$$(M_4)^n = \begin{bmatrix} x_{n+4}^4 & x_{n+5}^4 - x_{n+4}^4 & 64x_{n+2}^4 + 16x_{n+3}^4 & 64x_{n+3}^4 \\ x_{n+3}^4 & x_{n+4}^4 - x_{n+3}^4 & 64x_{n+1}^4 + 16x_{n+2}^4 & 64x_{n+2}^4 \\ x_{n+2}^4 & x_{n+3}^4 - x_{n+2}^4 & 64x_n^4 + 16x_{n+1}^4 & 64x_{n+1}^4 \\ x_{n+1}^4 & x_{n+2}^4 - x_{n+1}^4 & 64x_{n-1}^4 + 16x_n^4 & 64x_n^4 \end{bmatrix} \text{ for } n \geq 3, \tag{2}$$

$$(M_k)^n = \begin{bmatrix} x_{n+k}^k & x_{n+k+1}^k - x_{n+k}^k & k^{k-1}x_{n+2}^k + k^{k-2}x_{n+3}^k + \cdots + k^2x_{n+k-1}^k & & \\ x_{n+k-1}^k & x_{n+k}^k - x_{n+k-1}^k & k^{k-1}x_{n+1}^k + k^{k-2}x_{n+2}^k + \cdots + k^2x_{n+k-2}^k & & \\ \vdots & \vdots & \vdots & M^1 & M^2 \\ x_{n+1}^k & x_{n+2}^k - x_{n+1}^k & k^{k-1}x_{n-k+3}^k + k^{k-2}x_{n-k+4}^k + \cdots + k^2x_n^k & & \end{bmatrix} \tag{3}$$

for $n \geq k-1$ and $k \geq 5$ where

$$M^1 = \begin{bmatrix} k^{k-1}x_{n+3}^k + k^{k-2}x_{n+4}^k + \cdots + k^3x_{n+k-1}^k & \cdots & k^{k-1}x_{n+k-3}^k + k^{k-2}x_{n+k-2}^k + \cdots + k^{k-3}x_{n+k-1}^k \\ k^{k-1}x_{n+2}^k + k^{k-2}x_{n+3}^k + \cdots + k^3x_{n+k-2}^k & \cdots & k^{k-1}x_{n+k-4}^k + k^{k-2}x_{n+k-3}^k + \cdots + k^{k-3}x_{n+k-2}^k \\ \vdots & \cdots & \vdots \\ k^{k-1}x_{n-k+4}^k + k^{k-2}x_{n-k+5}^k + \cdots + k^3x_n^k & \cdots & k^{k-1}x_{n-2}^k + k^{k-2}x_{n-1}^k + \cdots + k^{k-3}x_n^k \end{bmatrix}_{(k)\times(k-5)}$$

and

$$M^2 = \begin{bmatrix} k^{k-1}x_{n+k-2}^k + k^{k-2}x_{n+k-1}^k & k^{k-1}x_{n+k-1}^k \\ k^{k-1}x_{n+k-3}^k + k^{k-2}x_{n+k-2}^k & k^{k-1}x_{n+k-2}^k \\ \vdots & \vdots \\ k^{k-1}x_{n-1}^k + k^{k-2}x_n^k & k^{k-1}x_n^k \end{bmatrix}_{k\times 2}.$$

Since $\det M_k = (-k)^{k-1}$, we easily derive that $\det (M_k)^n = (-k)^{(k-1)\cdot n}$. It is well-known that the Simson formula for a recurrence sequence can be obtained from the determinant of its generating matrix. For example, the Simpon formula of the Vandermonde-type sequence for $k = 3$ and $n \geq 2$ is as follows:

$$9\left(x_{n+1}^{3}\right)^{3}-18x_{n+1}^{3}x_{n+2}^{3}x_{n}^{3}+2x_{n+3}^{3}x_{n+1}^{3}x_{n}^{3}-9x_{n+3}^{3}x_{n-1}^{3}x_{n+1}^{3}-2\left(x_{n+2}^{3}\right)^{2}x_{n}^{3}+9x_{n+3}^{3}\left(x_{n}^{3}\right)^{2}=9^{n-1}$$

For an integer matrix $A = \left[a_{i,j}\right]$ with $a_{i,j}$'s integers, $A \,(\mathrm{mod}\, m)$ means that all entries of A are modulo m, that is, $A \,(\mathrm{mod}\, m) = \left(a_{i,j} \,(\mathrm{mod}\, m)\right)$. Let us consider the set $\langle A\rangle_m = \left\{A^i \,(\mathrm{mod}\, m) \mid i \geq 0\right\}$. If $\gcd(m, \det A) = 1$, then $\langle A\rangle_m$ is a cyclic group; if $\gcd(m, \det A) \neq 1$, then $\langle A\rangle_m$ is a semigroup. We denote the order of the set $\langle A\rangle_m$ by $|\langle A\rangle_m|$. Since $\det M_k = (-k)^{k-1}$, we easily see that the set $\langle M_k\rangle_m$ is a cyclic group if $\gcd(m, k) = 1$. Similary, the set$\langle M_k\rangle_m$ is a semigroup if $\gcd(m, k) \neq 1$.

Now we consider the order of the cyclic groups which are generated by the matrices M_k, $(k \geq 3)$.

Theorem 2.1. *Let p be a prime such that $\gcd(p, k) = 1$ with $k \geq 3$. If u is the largest positive integer such that $\left|\langle M_k\rangle_{p^u}\right| = \left|\langle M_k\rangle_p\right|$, then $\left|\langle M_k\rangle_{p^v}\right| = p^{v-u} \cdot \left|\langle M_k\rangle_p\right|$ for every $v \geq u$. In particular, if $\left|\langle M_k\rangle_{p^2}\right| \neq \left|\langle M_k\rangle_p\right|$, then $\left|\langle M_k\rangle_{p^v}\right| = p^{v-1} \cdot \left|\langle M_k\rangle_p\right|$.*

Proof. Since $\gcd(p, k) = 1$, the sets $\langle M_k\rangle_{p^\alpha}$ are cyclic groups for every positive integer α. Let λ be a positive integer such that $(M_k)^{\left|\langle M_k\rangle_{p^{\lambda+1}}\right|} \equiv I\left(\mathrm{mod}\, p^{\lambda+1}\right)$,where I is the $k \times k$ identity matrix. Then it is clear that $(M_k)^{\left|\langle M_k\rangle_{p^{\lambda+1}}\right|} \equiv I\left(\mathrm{mod}\, p^{\lambda}\right)$, which implies that $\left|\langle M_k\rangle_{p^\lambda}\right|$ divides $\left|\langle M_k\rangle_{p^{\lambda+1}}\right|$.

Furthermore, if we denote $(M_k)^{\left|\langle M_k\rangle_{p^\lambda}\right|} = I + \left(m_{i,j}^{(\lambda)} \cdot p^\lambda\right)$, then by the binomial expansion, we can write

$$(M_k)^{\left|\langle M_k\rangle_{p^\lambda}\right| \cdot p} = \left(I + \left(m_{i,j} \cdot p^\lambda\right)\right)^p = \sum_{i=0}^{p} \binom{p}{i} \left(m_{i,j} \cdot p^\lambda\right)^i \equiv I\left(\mathrm{mod}\, p^{\lambda+1}\right).$$

Then we get that $\left|\langle M_k\rangle_{p^{\lambda+1}}\right|$ divides $\left|\langle M_k\rangle_{p^\lambda}\right| \cdot p$. Thus, it is clear that $\left|\langle M_k\rangle_{p^{\lambda+1}}\right| = \left|\langle M_k\rangle_{p^\lambda}\right|$ or $\left|\langle M_k\rangle_{p^{\lambda+1}}\right| = \left|\langle M_k\rangle_{p^\lambda}\right| \cdot p$. It is easy to see that the latter holds if and only if there is a $m_{i,j}^{(\lambda)}$ which is not divisible by p. Since u is the largest positive integer such that $\left|\langle M_k\rangle_{p^u}\right| = \left|\langle M_k\rangle_p\right|$, $\left|\langle M_k\rangle_{p^{u+1}}\right| \neq \left|\langle M_k\rangle_{p^u}\right|$. Thus, it is readily seen that there is an $m_{i,j}^{(u+1)}$ which is not divisible by p. Then, we obtain $\left|\langle M_k\rangle_{p^{u+2}}\right| \neq \left|\langle M_k\rangle_{p^{u+1}}\right|$. The proof is finished by induction on u. ∎

Reducing the Vandermonde-type sequence by a modulo m, we can write the following recuurence sequence:

$$\left\{x_n^k(m)\right\} = \left\{x_1^k(m),\ x_2^k(m),\ \ldots,\ x_i^k(m),\ \ldots\right\},$$

where $x_i^k(m) = x_i^k \,(\mathrm{mod}\, m)$ and $k \geq 3$. It has the same recurrence relation as in the Vandermonde-type sequence.

It is well known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the period of the sequence.

Theorem 2.2. *For every positive integer m, the Vandermonde-type sequence modulo m,* $\left\{x_n^k(m)\right\}$ *is periodic.*

Proof. Let us consider the set

$$S = \{(s_1, s_2, \ldots, s_k) \mid s_i\text{'s are interegers such that } 0 \leq s_i \leq m-1\}.$$

Since $|Q| = m^k$, there are m^k distinct k-tuples of elements of $\mathbb{Z}_m$. Then, it is easy to see that at least one of the k-tuples appears twice in the sequence $\left\{x_n^k(m)\right\}$ for $k \geq 3$. Therefore, the subsequence following this k-tuple repeats; hence, the sequence $\left\{x_n^k(m)\right\}$ is periodic for $k \geq 3$. ∎

We denote the length of the period of the sequence $\left\{x_n^k(m)\right\}$ by $P_k(m)$.

By (1), (2) and (3), it is readily seen that $P_k(m) = |\langle M_k \rangle_m|$ when $\gcd(m,k) = 1$.

Theorem 2.3. *If* $\gcd(m,k) = 1$ *and m has the prime factorization* $m = \prod_{i=1}^{\tau} p_i^{u_i}$, $(\tau \geq 1)$*, then* $P_k(m)$ *equals the least common multiple of the* $P_k\left(p_i^{u_i}\right)$*'s.*

Proof. $P_k\left(p_i^{u_i}\right)$ is the period of the sequence $\left\{x_n^k\left(p_i^{u_i}\right)\right\}$, the sequence $\left\{x_n^k\left(p_i^{u_i}\right)\right\}$ repeats only after blocks of length $\lambda \cdot P_k\left(p_i^{u_i}\right)$, $(\lambda \in \mathbb{N})$. Since also $P_k(m)$ is the period of the sequence $\left\{x_n^k(m)\right\}$, the sequence $\left\{x_n^k\left(p_i^{u_i}\right)\right\}$ repeats after $P_k(m)$ terms for all values i. Then, it is clear that $P_k(m)$ is the form $\lambda \cdot P_k\left(p_i^{u_i}\right)$ for all values i, and since any such number gives a period of $P_k(m)$, we obtain that $P_k(m)$ equals the least common multiple of the $P_k\left(p_i^{u_i}\right)$'s. ∎

We will now consider the Vandermonde-type sequences in groups.

Definition 2.1. *Let G be a k-generator group and let* $(x_1, x_2, \ldots, x_k)$ *be a generating k-tuple of G, where* $k \geq 3$*. Then the Vandermonde-type orbit of the group G,* $V^G_{(x_1,x_2,\ldots,x_k)}$ *is defined as shown:*

$$a_n^k = \left(a_{n-k}^k\right)^{k^{k-1}} \left(a_{n-k+1}^k\right)^{k^{k-2}} \cdots \left(a_{n-2}^k\right)^{k} a_{n-1}^k \quad \text{for } k \geq 3 \text{ and } n \geq k+1,$$

where $a_i^k = x_i$ *for* $1 \leq i \leq k$.

Theorem 2.4. *If G is a finite group, then the sequence* $V^G_{(x_1,x_2,\ldots,x_k)}$ *is periodic.*

Proof. Assume that n is the order of G. Since there n^k distinct k-tuples of elements of G, at least one of the k-tuples appears twice in the sequence $V^G_{(x_1,x_2,\ldots,x_k)}$. Therefore, the subsequence following this k-tuple repeats. On account of the repetition, the orbit $V^{G}_{(x_1,x_2,\ldots,x_k)}$ is periodic. ∎

We denote the period of the orbit $V^{G}_{(x_1,x_2,\ldots,x_k)}$ by $PV^{G}_{(x_1,x_2,\ldots,x_k)}$.

Now we consider the Vandermonde-type orbits of the polyhedral groups $(n,2,2)$, $(2,n,2)$ and $(2,2,n)$ for $n \geq 3$.

The polyhedral group (l,m,n) for $l,m,n > 1$, is defined by the presentation

$$\left\langle x,y,z : x^l = y^m = z^n = xyz = e \right\rangle$$

or

$$\left\langle x,y : x^l = y^m = (xy)^n = e \right\rangle.$$

The polyhedral group (l,m,n) is finite if and only if the number

$$\mu = lmn\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right) = mn + nl + lm - lmn$$

is positive, i.e., in the case $(2,2,n)$, $(2,3,3)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$. Its order is $\frac{2lmn}{\mu}$. Using Tietze transformations we may show that $(l,m,n) \cong (m,n,l) \cong (n,l,m)$.

For detail information about the polyhedral groups, see [3].

Conjecture 1. *Let* $\gcd(n,3) = 1$*, let G be any the polyhedral groups* $(n,2,2)$*,* $(2,n,2)$ *and* $(2,2,n)$*, then*

$$\gcd\left(PV^{(G)}_{(x,y,z)}, P_3(n)\right) \neq 1.$$

Conjecture 2. *The periods of the Vandermonde-type orbits* $PV^{(n,2,2)}_{(x,y,z)}$*,* $PV^{(2,n,2)}_{(x,y,z)}$ *and* $PV^{(2,2,n)}_{(x,y,z)}$ *are odd integers.*

Now we give the periods of the Vandermonde-type orbits of the polyhedral groups $(n,2,2)$, $(2,n,2)$ and $(2,2,n)$ for some integers n by the following table:

n	$PV^{(2,n,2)}_{(x,y,z)}$	$PV^{(n,2,2)}_{(x,y,z)}$	$PV^{(2,2,n)}_{(x,y,z)}$
2	4	4	4
3	6	4	4
4	8	8	8
5	24	12	12
6	12	4	4
7	48	96	48
8	16	16	16
9	18	4	4
10	24	12	12
15	24	12	12

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